Ergodic Control for stochastic evolution equations with Lévy noise.

Mgr. Karel Kadlec, prof. RNDr. Bohdan Maslowski, DrSc

KPMS MFF UK

25. 1. 2018

Contents.

1. Functional analysis notions
2. Lévy process
3. Controlled SEE
4. Ergodic control
5. Results

Functional analysis notions.

1. $\mathbb{H}$, $\mathbb{Y}$ real separable Hilbert spaces.
2. A^* the adjoint operator of $A \in \mathcal{L}(\mathbb{H}, \mathbb{Y})$ iff "$\langle Ax, y\rangle_{\mathbb{Y}} = \langle x, A^*y\rangle_{\mathbb{H}}$"
    1. ($\mathbb{D}(A^*)$ (the domain of A^*) is the set of $y \in \mathbb{Y}$ for which exists $z \in \mathbb{H}$ such that for all $x \in \mathbb{D}(A)$: $\langle Ax, y\rangle_{\mathbb{Y}} = \langle x, z\rangle_{\mathbb{H}}$ and then $A^*y = z$),
3. The operator $A : \mathbb{H} \to \mathbb{H}$ selfadjoint iff $A^* = A$.

1. $S : \mathbb{R}_+ \to \mathcal{L}(\mathbb{H})$ the C_0-semi-group:
   1. $S(t+s) = S(t)S(s)$, $t, s \geq 0$, $S(0) = I$,
   2. $S(t)x \to x$ strongly, $t \to 0_+$, $x \in \mathbb{H}$,
2. A is the infinitesimal generator of the C_0-semi-group S iff

$$\frac{S(t)x - x}{t} \to Ax, \quad t \to 0_+,$$

   for $x \in \mathbb{D}(A)$, where $\mathbb{D}(A)$ is the set of all $x \in \mathbb{H}$ for which the limit exists.
   1. $S(t)x$ is the solution to the equation $\dot{y} = Ay$, $y(0) = x \in \mathbb{D}(A)$.
   2. S analytic if S can be extended to $\{z \in \mathbb{C}, |\arg(z)| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2})$.

Lévy process.

1. Stochastic basis $(\Omega, \mathcal{A}, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$,
2. L Lévy process (in $\mathbb{H}$):
    1. indexed $\mathbb{R}_+$,
    2. stationary independent increments,
    3. $L(0) = 0_{\mathbb{H}}$,
    4. stochastically continuous.
3. In particular L has cádlág version.
4. In particular, we work with L square integrable martingale in the form

$$L_t = \int_{[0,t]} \int_{\mathbb{H}} x \tilde{N}(ds, dx), \quad t \in \mathbb{R}_+,$$

1. $N(t, A) = \#\{s \in [0, t]; \Delta L_s \in A\}$, $A \in \mathcal{B}(\mathbb{H} - \{0_{\mathbb{H}}\})$,
2. $\nu(A) = \mathbb{E}N(1, A)$, $A \in \mathcal{B}(\mathbb{H} - \{0_{\mathbb{H}}\})$,
3. $\tilde{N}(t, A) = N(t, A) - t\nu(A)$ ($\mathcal{F}$-martingale),
4. $cov(L_1) = \mathcal{Q}$

Controlled SEE.

1. $$dX^U(t) = (AX^U(t) + BU(t))dt + dL_t, \quad X^U(0) = x, \qquad (1)$$

   where
   1. A the infinitesimal generator of the exponentially stable analytic semi-group S.
   2. $B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_A^{\epsilon-1} = \mathcal{D}((-A + \beta \mathbf{I})^{\epsilon-1}))$, where $\epsilon \in (0, 1)$.
   3. U $\mathcal{F}$-progressively measurable control in $\mathbb{L}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$ for fixed $p > \max\{2, \frac{1}{\epsilon}\}$ ($\mathcal{U}$ denotes the space of controls).
2. X^U Strong solution:
   1. $\int_0^T |AX^U(s)|_{\mathbb{H}} ds < \infty$ **P**-s.j.,
   2. $X^U(t) \in \mathbb{D}(A)$ **P**-s.j.,
   3. $$X^U(t) = x + \int_0^t (AX^U(s) + BU(s))ds + L_t, \quad t > 0.$$
   4. Assumptions for A too restrictive. Less strict concepts of solutions (avoiding A) needed.

Controlled SEE.

1. X^U Mild solution:

$$X^U(t) = x + \int_0^t S(t-s)BU(s)ds + \int_0^t S(t-s)dL_s, \quad t \geq 0. \tag{2}$$

2. X^U Weak solution:

$$\langle a, X^U(t)\rangle_{\mathbb{H}}$$
$$= \langle a, x\rangle_{\mathbb{H}} + \int_0^t \langle A^* a, X^U(s)\rangle_{\mathbb{H}} ds + \int_0^t \langle B^* a, U(r)\rangle_{\mathbb{H}} dr + \langle a, L_t\rangle_{\mathbb{H}},$$

   1. $a \in \mathcal{D}(A^*)$, $t \in \mathbb{R}_+$.

3. In our case, X^U weak solution iff X^U mild solution and both exist and are unique in the space $\mathbb{L}^{2,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{H})$.

Controlled SEE, example,
point control.

1.
$$w_{tt}(t,x) - \Delta w_t(t,x) + \Delta^2 w(t,x) = \mathbb{I}_{x=x_0} u(t) + l(t,x), \quad (t,x) \in \mathbb{R}_+ \tag{3}$$

    1. $$w(0,x) = w_0, \quad w_t(0,x) = w_1, \quad x \in G,$$
    2. $$w(t,x) = w_t(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial G,$$

    where $G \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, $x_0 \in G$, l formally represents Lévy noise.

Ergodic control

1. $$J(U,T) = \int_0^T (\langle QX^U(s), X^U(s)\rangle_{\mathbb{H}} + \langle RU(s), U(s)\rangle_{\mathbb{Y}})ds, \tag{4}$$
   1. $Q \in \mathcal{L}(\mathbb{H})$ symmetric positive semi-definite operator,
   2. $R \in \mathcal{L}(\mathbb{Y})$ symmetric positive definite operator.
2. Find $C_{\mathcal{U}} \in \mathbb{R}$ (optimal cost) s. t. for all $U \in \mathcal{L}_{\mathcal{U}}$ **P**-a.s.
   $$\liminf_{t\to\infty} \frac{J(U,t)}{t} \geq C_{\mathcal{U}},$$
   and $U_{\mathcal{U}} \in \mathcal{L}$ (optimal control) s. t. **P**-a.s.
   $$\lim_{t\to\infty} \frac{J(U_{\mathcal{U}},t)}{t} = C_{\mathcal{U}}.$$
3. Denote V solution to the stationary Riccati equation
   $$VA + A^*V + Q - VBR^{-1}B^*V = 0.$$
   and $h(\cdot)$ the continuous extension of $\langle A^*V\cdot, \cdot\rangle_{\mathbb{H}}$ on $\mathbb{H}$

Results.

1. We have a.s.

$$\left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}} - \langle x, Vx\rangle_{\mathbb{H}}$$

$$= 2\int_0^t h(X^U(s))ds + \int_0^t 2\left\langle B^* VX^U(s), U(s)\right\rangle_{\mathbb{H}} ds$$

$$+2\int_{\mathbb{H}} \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}} + \sum_{s\le t} \left|\Delta V^{\frac{1}{2}} L(s)\right|^2_{\mathbb{H}}.$$

2. Optimal control and optimal cost. $C_{\mathcal{U}_p} = Tr(V\mathcal{Q})$ and $U_{\mathcal{U}_p} = -R^{-1}B^* VX$, where $\mathcal{U}_p$ is the set of all $U \in \mathcal{U}$ such that

$$\frac{\langle VX_t^U, X_t^U\rangle_{\mathbb{H}}}{t} \to 0, \quad t\to\infty, \quad a.s., \tag{5}$$

$$\lim_{t\to\infty} \sup \frac{\int_0^t |X_s^U|^2_{\mathbb{H}} ds}{t} < \infty \quad a.s. \tag{6}$$

References.

[1] Chen G., Triggiani R.: *Proof of extensions of two conjectures on structural damping for elastic systems.* Pacific J. Math., 136, 15–55, 1989.

[2] Duncan T. E., Maslowski B., Pasik-Duncan B.: Adaptive Boundary and Point Control of Linear Stochastic Distributed Parameter Systems. *SIAM Journal on Control and Optimization*, 33, 648–672, 1994.

[3] Hausenblas E.: Maximal Inequalities of the Itô Integral with Respect to Poisson Random Measures or Lévy Processes on Banach Spaces. *Potential Anal*, 223–251, 2011.

[4] Kadlec K., Maslowski B.: *Ergodic Control for Lévy-driven linear stochastic equations in Hilbert spaces. Applied Mathematics & Optimization, 2017.*

[5] *Peszat S., Zabczyk J.: Stochastic Partial Differential Equations Driven by Lévy Processes. Cambridge University Press, Cambridge, 2006.*

Thank you for your attention.