Parameter Estimation of Continuous Processes Using Financial High-Frequency Data

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Financial High-Frequency Data

Financial High-Frequency Data

- Ultra-high-frequency data are irregularly spaced time series recorded at highest possible frequency corresponding to each transaction or change in bid/ask prices.
  - *Engle, R. F. (2000). The Econometrics of Ultra-High-Frequency Data. Econometrica, 68(1), 1–22.*
- Financial high-frequency time series include
  - exchange rates,
  - stock prices,
  - commodity prices.
- The price process is often modeled with continuous values and continuous time.
- However, there are some crucial market microstructure specifics such as
  - rounding error (prices have discrete values),
  - discreteness of price changes (transactions can occur only at discrete times),
  - bid-ask spread (transactions can happen either on bid or ask side),
  - informational effects (agents do not behave according to the economic theory).

Market Microstructure Noise

- Market microstructure specifics can be captured by the model with additive noise

$$X_i = P_{T_i} + E_i, \qquad i = 1, \ldots, n,$$

where

  - X_i is the observable price with discrete time,
  - P_{T_i} is the efficient price with continuous time sampled at discrete times T_i,
  - E_i is the market microstructure noise with discrete time.

- We assume the market microstructure noise to be independent white noise and

$$E_i \sim (0, \omega^2).$$

- Generally, the market microstructure noise can be dependent in time and dependent on efficient price.
  - *Hansen, P. R., & Lunde, A. (2006). Realized Variance and Market Microstructure Noise. Journal of Business & Economic Statistics, 24(2), 127–161.*

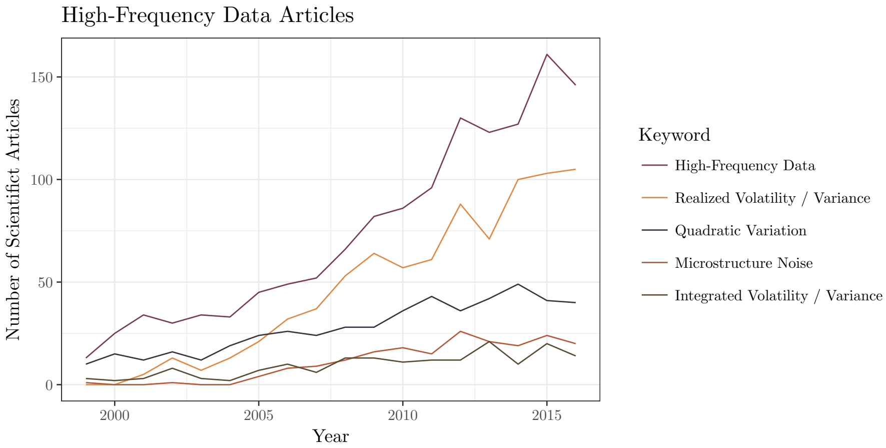
Non-Parametric Approach

- Estimation of the quadratic variation and the integrated variance.
  - *Zhang, L., Mykland, P. A., & Aït-Sahalia, Y. (2005). A Tale of Two Time Scales: Determining Integrated Volatility with Noisy High-Frequency Data. Journal of the American Statistical Association, 100(472), 1394–1411.*
  - *Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., & Shephard, N. (2008). Designing Realized Kernels to Measure the ex post Variation of Equity Prices in the Presence of Noise. Econometrica, 76(6), 1481–1536.*
  - *Jacod, J., Li, Y., Mykland, P. A., Podolskij, M., & Vetter, M. (2009). Microstructure Noise in the Continuous Case: The Pre-Averaging Approach. Stochastic Processes and Their Applications, 119(7), 2249–2276.*
  - *Nolte, I., & Voev, V. (2012). Least Squares Inference on Integrated Volatility and the Relationship Between Efficient Prices and Noise. Journal of Business & Economic Statistics, 30(1), 94–108.*

Parametric Approach

- Estimation of Wiener process parameters.
  - Discrete process contaminated by the independent white noise follow ARIMA(0,1,1).
  - Parameters are estimated by maximum likelihood of reparametrization.
  - *Aït-Sahalia, Y., Mykland, P. A., & Zhang, L. (2005). How Often to Sample a Continuous-Time Process in the Presence of Market Microstructure Noise. The Review of Financial Studie, 18(2), 351–416.*
- Estimation of Ornstein-Uhlenbeck process parameters.
  - Discrete process contaminated by the independent white noise follow ARIMA(1,0,1).
  - Parameters are estimated by method of moments, maximum likelihood of reparametrization and direct maximum likelihood.
  - *Holý, V., & Tomanová, P. (2017). Ornstein-Uhlenbeck Process Contaminated by the White Noise: Effects, Estimation and Application. In review.*

Academic Literature



Wiener Process

Wiener Process

- The standard Wiener process is a random variable W_t that satisfies
  - $W_t = 0$,
  - for $0 \leq s < t < u < v$, $W_t - W_s$ and $W_v - W_u$ are independent,
  - for $0 \leq s < t$, $W_t - W_s \sim \mathrm{N}(0, t-s)$.
- We consider the price process given by

$$P_t = \sigma W_t,$$

  where $\sigma > 0$ is the instantaneous volatility.
- When the process is contaminated, the estimate of σ is biased.

ARIMA(0,1,1) Reparametrization

- We consider equidistant sampling $\Delta = T_i - T_{i-1}$, $i = 1, \ldots, n$.
- The first difference of process X_i is

$$X_i - X_{i-1} = P_{T_i} - P_{T_{i-1}} + E_i - E_{i-1}.$$

- We can rewrite the process X_i as

$$X_i = \underbrace{X_{i-1}}_{X_i^{DF}} + \underbrace{R_{T_{i-1},T_i} + E_i - E_{i-1}}_{X_i^{MA}}, \qquad R_{T_{i-1},T_i} = P_{T_i} - P_{T_{i-1}} \sim \mathrm{N}(0, \sigma^2 \Delta).$$

- This is ARIMA(0,1,1) process, which can be reparametrized as

$$X_i = \underbrace{X_{i-1}}_{X_i^{DF}} + \underbrace{\theta V_{i-1} + V_i}_{X_i^{MA}}, \qquad V_i \sim \mathrm{N}(0, \gamma^2).$$

ARIMA(0,1,1) Estimation

- Parameters θ and γ^2 can be estimated by maximizing the likelihood function

$$L(\theta, \gamma^2) = f_{X_0}(x_0) f_{X_1}(x_1 | X_0 = x_0) \cdots f_{X_n}(x_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}).$$

- To identify σ^2 and ω^2, we solve the equations

$$\begin{aligned} \gamma^2(1+\theta^2) &= \mathrm{var}[X_i^{MA}] = \sigma^2 \Delta + 2\omega^2, \\ \theta\gamma^2 &= \mathrm{cov}[X_i^{MA}, X_{i-1}^{MA}] = -\omega^2. \end{aligned}$$

- Finally, we get the original estimates as

$$\begin{aligned} \hat{\sigma}^2 &= \Delta^{-1}\hat{\gamma}^2(1+\hat{\theta})^2, \\ \hat{\omega}^2 &= -\hat{\gamma}^2\hat{\theta}. \end{aligned}$$

Ornstein-Uhlenbeck Process

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process P_t satisfy

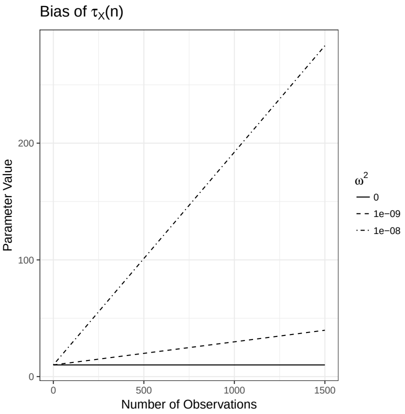
$$\mathrm{d}P_t = \tau(\mu - P_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

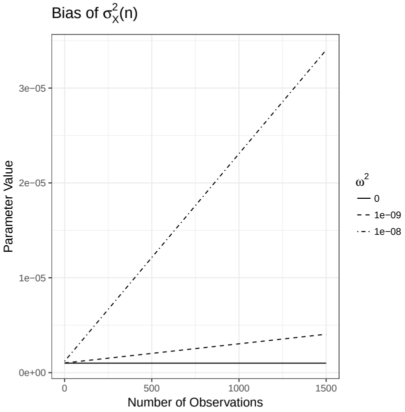
  where
  - W_t is a Wiener process,
  - μ is the long term mean level,
  - $\tau > 0$ is the speed of reversion,
  - $\sigma > 0$ is the instantaneous volatility.
- This stochastic differential equation has solution

$$P_t = P_0 e^{-\tau t} + \mu(1 - e^{-\tau t}) + \sigma \int_0^t e^{-\tau(t-s)} \mathrm{d}W_s.$$

- When the process is contaminated, the estimates of τ and σ are biased.

Bias of Ornstein-Uhlenebeck Parameters





ARIMA(1,0,1) Reparametrization

- When considering equidistant sampling, the process X_i can be decomposed as

$$X_i = P_{T_i} + E_i = P_{T_{i-1}} e^{-\tau\Delta} + \mu(1 - e^{-\tau\Delta}) + \sigma \int_0^\Delta e^{-\tau(\Delta - s)} \mathrm{d}W_s + E_i.$$

- Furthermore, from relation $P_{T_{i-1}} = X_{i-1} - E_{i-1}$ we obtain

$$X_i = \underbrace{\mu(1 - e^{-\tau\Delta})}_{X_i^{IC}} + \underbrace{e^{-\tau\Delta} X_{i-1}}_{X_i^{AR}} + \underbrace{\sigma \int_0^\Delta e^{-\tau(\Delta - s)} \mathrm{d}W_s - e^{-\tau\Delta} E_{i-1} + E_i}_{X_i^{MA}}.$$

- This is ARIMA(1,0,1) process, which can be reparametrized as

$$X_i = \underbrace{\alpha}_{X_i^{IC}} + \underbrace{\varphi X_{i-1}}_{X_i^{AR}} + \underbrace{\theta V_{i-1} + V_i}_{X_i^{MA}}, \qquad V_i \sim \mathrm{N}(0, \gamma^2).$$

ARIMA(1,0,1) Estimation

- Parameters α, φ, θ and γ^2 can be estimated by maximizing the likelihood function

$$L(\alpha, \varphi, \theta, \gamma^2) = f_{X_0}(x_0) f_{X_1}(x_1 | X_0 = x_0) \cdots f_{X_n}(x_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}).$$

- Finally, we get the original estimates by solving equations

$$\begin{aligned}
\alpha = X_i^{IC} &= \mu(1 - e^{-\tau\Delta}), \\
\varphi X_{i-1} = X_i^{AR} &= e^{-\tau\Delta} X_{i-1}, \\
\gamma^2(1+\theta^2) = \mathrm{var}[X_i^{MA}] &= \frac{\sigma^2}{2\tau}(1 - e^{-2\tau\Delta}) + \omega^2(1 + e^{-2\tau\Delta}), \\
\theta\gamma^2 = \mathrm{cov}[X_i^{MA}, X_{i-1}^{MA}] &= -\omega^2 e^{-\tau\Delta}.
\end{aligned}$$

Direct Method of Moments

- We use four unconditional moments

$$m_1 = \mathrm{E}[X_i] = \mu, \qquad m_2 = \mathrm{var}[X_i] = \frac{\sigma^2}{2\tau} + \omega^2,$$
$$m_3 = \mathrm{cov}[X_i, X_{i-1}] = \frac{\sigma^2}{2\tau} e^{-\tau\Delta}, \quad m_4 = \mathrm{cov}[X_i, X_{i-2}] = \frac{\sigma^2}{2\tau} e^{-2\tau\Delta}.$$

- By replacing theoretical moments with their estimated counterparts, we get

$$\hat{\mu}_{MOM} = \hat{m}_1, \qquad \hat{\tau}_{MOM} = \frac{1}{\Delta} \log \frac{\hat{m}_3}{\hat{m}_4},$$
$$\hat{\sigma}^2_{MOM} = 2\frac{1}{\Delta}\frac{\hat{m}_3^2}{\hat{m}_4} \log \frac{\hat{m}_3}{\hat{m}_4}, \quad \hat{\omega}^2_{MOM} = \hat{m}_2 - \frac{\hat{m}_3^2}{\hat{m}_4}.$$

Direct Maximum Likelihood Estimation

- Observed variables X_i are normally distributed with conditional density functions $f_{X_{T_i}}(x_{T_i}|X_{T_{i-1}})$ and conditional moments

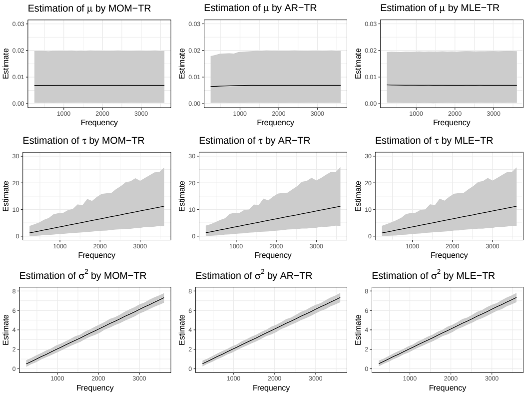
$$\mathrm{E}[X_{T_i}|X_{T_{i-1}} = x_{T_{i-1}}] = \frac{x_{T_{i-1}}\sigma^2 + 2\tau\mu\omega^2}{\sigma^2 + 2\tau\omega^2} e^{-\tau\Delta_i} + \mu\left(1 - e^{-\tau\Delta_i}\right),$$

$$\mathrm{var}[X_{T_i}|X_{T_{i-1}} = x_{T_{i-1}}] = \frac{\sigma^2\omega^2}{\sigma^2 + 2\tau\omega^2} e^{-2\tau\Delta_i} + \frac{\sigma^2}{2\tau}\left(1 - e^{-2\tau\Delta_i}\right).$$

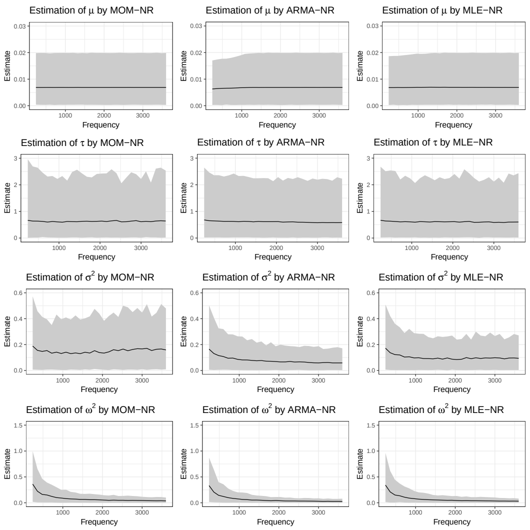
- We get the estimates by maximizing the logarithmic likelihood function

$$l(\mu, \tau, \sigma^2, \omega^2) = \sum_{i=1}^{n} \log f_{X_{T_i}}(x_{T_i}|X_{T_{i-1}} = x_{T_{i-1}}).$$

Bias of Traditional Parameter Estimates



Comparison of Proposed Estimators



Conclusion

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- Ignoring the market microstructure noise leads to significant bias and inconsistency of estimates of the process parameters.
- For the Ornstein-Uhlenbeck process, finite-sample simulations show that the maximum likelihood of ARIMA(1,0,1) reparametrization gives the most accurate estimates.
- We assume the standard Ornstein-Uhlenbeck process and the independent Gaussian white noise. However, both of these assumptions are too restrictive for financial data as many papers suggest. Relaxation of Gaussian assumptions is a topic for the future research.
- High-frequency Ornstein-Uhlenbeck process can be utilized e.g. in pairs trading strategies and stochastic volatility models.

Thank you for your attention!

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