Non-unbiased two-sample nonparametric tests.
Numerical example.

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- Lehmann E.L. 1959 **Testing statistical hypotheses**
- Sugiura et al. 1965, 2006 **Biased and unbiased two-sided Wilcoxon tests**
- Liu and Singh 1993 **Proposed a two-sample test of Wilcoxon type, based on the ranks of depths of the data**
- Amrhein, P. 1995 **An example of a two-sided Wilcoxon signed rank test which is not unbiased**
- Jurečková et al. 2002, 2012 **The two-sample multivariate testing problem**

- Two-sample Wilcoxon test
- The power function and the significance level. Basic definitions
- Distribution
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    - Simulations
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Let $X_1, \dots, X_n \sim F(x)$ and $Y_1, \dots, Y_m \sim G(x)$ be two random samples from the absolutely continuous distributions.

Let us assume that $G(x) = F(x - \Delta)$.

We want to test the hypothesis

$$H_0 : \Delta = 0,$$

against the two-sided alternative:

$$H_1 : \Delta \neq 0,$$

using the following two-sample Wilcoxon test:

$$\phi(X_1, \dots, X_n, Y_1, \dots, Y_m) = \begin{cases} 1, & \text{if } X_{(n)} < Y_{(1)} \text{ or } X_{(1)} > Y_{(m)}, \\ 0, & \text{otherwise.} \end{cases}$$

The power function of ϕ is given by:

$$\beta_{n,m}(\Delta) = P(X_{(n)} < Y_{(1)}) + P(X_{(1)} > Y_{(m)}) = \int_{-\infty}^{+\infty} F^n(x) d(1-(1-G(x))^m) +$$

$$+ \int_{-\infty}^{+\infty} G^m(x) d(1-(1-F(x))^n) = \int_{-\infty}^{+\infty} F^n(X) d(1-(1-F(x-\Delta))^m) +$$

$$+ \int_{-\infty}^{+\infty} F^m(x-\Delta) d(1-(1-F(x))^n).$$

The significance level α of this test is:

$$\alpha = P_{H_0}(X_{(n)} < Y_{(1)}) + P_{H_0}(X_{(1)} > Y_{(m)}) = \beta_{n,m}(0) = \frac{2m!n!}{(m+n)!}.$$

We have a property of the power function $\beta_{n,m}(\Delta) = \beta_{m,n}(-\Delta)$. Therefore, everywhere below we will assume that $n \geq m$.

The test ϕ is unbiased if it satisfies $\inf_{\Delta} \beta_{n,m}(\Delta) \geq \alpha$.

Suppose that F is a Skew Logistic Distribution with a scale parameter $b > 0$ and a skew parameter $\lambda > 0$

$$f(x) = \frac{2\lambda}{1+\lambda^2} \begin{cases} \dfrac{e^{-x/b\lambda}}{b(1+e^{-x/b\lambda})^2}, & \text{if } x < 0, \\ \dfrac{e^{-x\lambda/b}}{b(1+e^{-x\lambda/b})^2}, & \text{if } x \geq 0. \end{cases}$$

$$F(x) = \begin{cases} \dfrac{2\lambda^2}{1+\lambda^2} \dfrac{1}{1+e^{-x/b\lambda}}, & \text{if } x < 0, \\ \dfrac{\lambda^2}{1+\lambda^2} + \dfrac{1}{1+\lambda^2} \dfrac{1-e^{-x\lambda/b}}{1+e^{-x\lambda/b}}, & \text{if } x \geq 0. \end{cases}$$

where $f(x)$ is a density function of the X.

All graphics were built in **scilab-5.5.0**

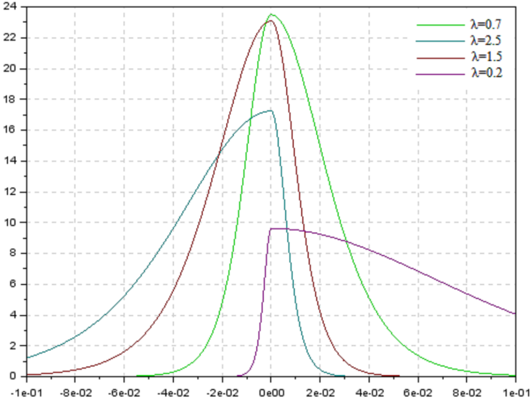


Figure: PDF of a skew logistic distribution for different values of λ, for $b = 0.01$

Unless otherwise stipulated, throughout $b = 0.01$ and $\lambda = 2.5$

All computations were made in **scilab-5.5.0**

If $n = m$, then $\beta_{n,m}(\Delta)$ – symmetric about $\Delta = 0$ and the test ϕ is unbiased.
Otherwise $\beta_{n,m}(\Delta)$ – skew and the test ϕ is biased.

n	m	α	β_{teor}^{min}	$-\Delta_{min}$
3	3	0.1	0.1	0
19	1	0.1	0.0789	0.006
3	2	0.2	0.19818	0.002
8	3	0.012	0.01155	0.0021
10	6	0.00025	0.00024	0.0008
20	10	$6.66e-08$	$6.5e-08$	$5.3e-04$

β_{teor}^{min} – minimum of the numerically computed function $\beta_{n,m}(\Delta)$;

Δ_{min} – the point at which this minimum was obtained.

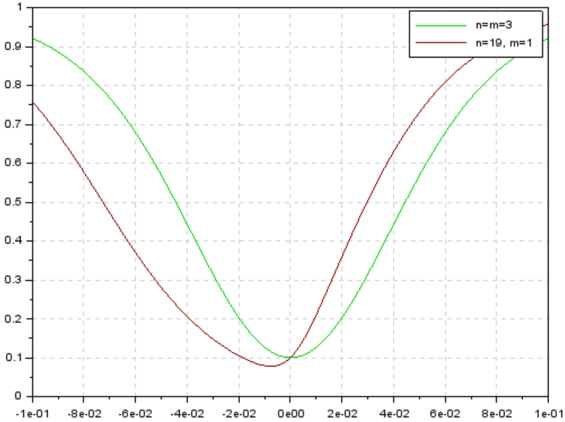
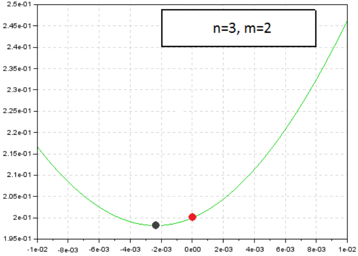
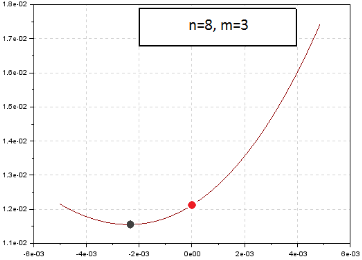
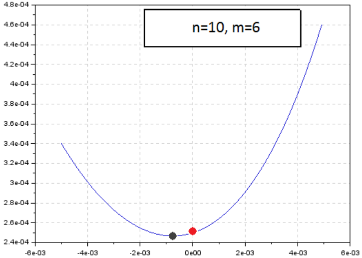


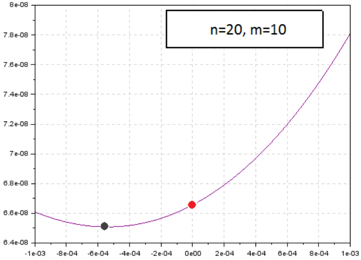
Figure: The power function $\beta_{n,m}(\Delta)$ for n=m=3 and for n=19, m=1. This example is very important, because in both cases the significance level of the test ϕ is equal to 0.1, meanwhile in the first case the test is unbiased but in the second it is biased.

On the following plots the power functions are shown for different combinations of n and m.









n	m	α	N	k_0	β_{prac}	$-\Delta_0$
19	1	0.1	10^5	9302	0.093	0.003
3	2	0.2	10^5	19853	0.1985	0.001
8	3	0.012	10^5	1168	0.01168	0.0013
10	6	0.00025	10^8	24577	0.000246	0.0002
20	10	$6.66e-08$	10^{11}	6487	$6.5e-08$	$2.3e-04$

$\Delta_0 \in [\Delta_{min}; 0]$ – a point for simulation of sample $Y_1, \ldots, Y_m$

N – the number of simulations of samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$;

k_0 – the number of experiments in which $\phi(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = 1$

$\beta_{prac} = \dfrac{k_0}{N}$ – empirical power of the test ϕ.

In the case of the skew logistic distribution with a scale parameter $b > 0$ and a skew parameter $\lambda > 0, \lambda \neq 1$ we have

- **if $m = n$, then the test ϕ is unbiased**
- **if $m \neq n$, then the test ϕ is biased.**

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2) Amrhein, P. (1995). An example of a two-sided Wilcoxon signed rank test which is not unbiased . *Ann. Inst. Statist. Math.* **47** 167–170.
3) Sugiura, N., Murakami, H., Lee, S.K. & Maeda, Y. (2006). Biased and unbiased two-sided Wilcoxon tests for equal sample sizes. *Ann. Inst. Statist. Math.* **58**, 93–100.
4) Jurečková, J., & Kalina, J. (2012). Nonparametric multivariate rank tests and their unbiasedness. *Bernoulli* **18**, 229–251.
5) Jurečková, J. & Milhaud, X. (2003). Derivative in the mean of a density and statistical applications. *IMS Lecture Notes* **42**, 217–232.
6) D. Sastry, D. Bhati (2014) A New Skew Logistic Distribution: Properties and Applications. *Brazilian Journal of Probab. and Statist.* **Dec. 2014**

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