Multidimensional stochastic dominance for discrete distribution

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Contents

Contents

Motivation

The concept of stochastic dominance is widely used in seeking for the optimal investment strategy:

$$\begin{aligned} &\text{maximize} && \mathbb{E}h(x) \\ &\text{subject to} && G(x) \succeq_{(k)} Y, \\ & && x \in X_0. \end{aligned}$$

How to extend the concept of stochastic dominance to multiple dimension?

One possible solution (Dentcheva and Ruszczýnski (2009)) is to use a mapping which converts random vectors to random variables. Nevertheless reduction of dimensions leads to loss of information.

In our research we build up the theory of stochastic dominance of random vectors which does not utilize reduction of dimensions. Definition of multidimensional stochastic dominance formulated in Sriboonchitta, Wong, Dhompongsa, and Nguyen (2009) serves as a springboard for the whole theory.

Contents

One dimensional (first order) SD I

Definition (Levy (2006))

Let X and Y be two random variables with distribution functions F and G. Then X stochastically dominates Y in the first order, denotes as $X \succeq Y$, if $F(x) \leq G(x)$ for every $x \in \mathbb{R}$.

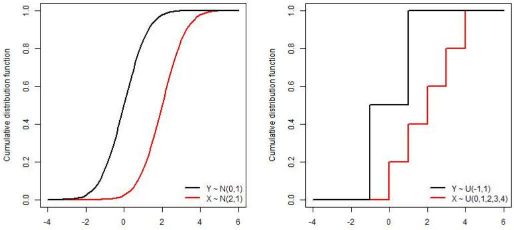

Figure: The random variable X stochastically dominates the random variable Y.

One dimensional (first order) SD II

Theorem (Levy (2006))

Let X and Y be two random variables with distribution functions F,G and quantile functions F^{-1}, G^{-1}. Then the following statements are equivalent:

1. *X stochastically dominates Y in the first order,*
2. $\mathbb{E}u(X) \geq \mathbb{E}u(Y)$ *for any $u \in \mathcal{U}_1$,*

where $\mathcal{U}_1$ is the space of all nondecreasing functions $u : \mathbb{R} \to \mathbb{R}$.

Is it possible to define the multidimensional stochastic dominance via cumulative distribution functions ($\boldsymbol{X} \succeq \boldsymbol{Y}$ if $F(\boldsymbol{x}) \leq G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$)?

What is then the generator of the dominance in the sense of von Neumann Morgenstern utility functions?

Contents

Definition of strong SD

Definition (Sriboonchitta, Wong, Dhompongsa, and Nguyen (2009))

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors. Then $\boldsymbol{X}$ (strongly) stochastically dominates $\boldsymbol{Y}$, denoted as $\boldsymbol{X} \succeq \boldsymbol{Y}$, if for every upper set $M \in \mathcal{M}$ one has $\mathbb{P}(\boldsymbol{X} \in M) \geq \mathbb{P}(\boldsymbol{Y} \in M)$.

A subset $M \subset \mathbb{R}^d$ is called an upper set if for each $\boldsymbol{y} \in \mathbb{R}^d$ such that $\boldsymbol{y} \geq \boldsymbol{x}$ one also has $\boldsymbol{y} \in M$ whenever $\boldsymbol{x} \in M$

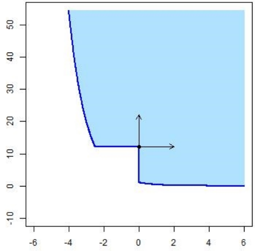

Figure: Example of an upper set in $\mathbb{R}^2$.

Definition of weak SD I

Definition

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors. Then $\boldsymbol{X}$ (weakly) stochastically dominates $\boldsymbol{Y}$, denoted as $\boldsymbol{X} \succeq \boldsymbol{Y}$, if for every upper set $M^* \in \mathcal{M}$ one has $\mathbb{P}(\boldsymbol{X} \in M^*) \geq \mathbb{P}(\boldsymbol{Y} \in M^*)$.

The simplest upper sets in $\mathbb{R}^d$ are set with a single generator:

$$\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} = \boldsymbol{m}^* + \boldsymbol{t}, \boldsymbol{t} \geq \boldsymbol{0}\} = (\boldsymbol{m}^*, \infty) = (m_1^*, \infty) \times \cdots \times (m_d^*, \infty).$$

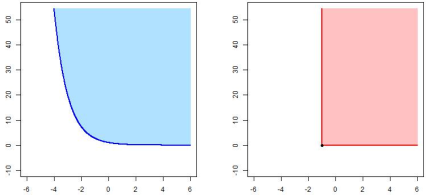

Figure: Example of an upper set M in $\mathbb{R}^2$ (left picture) and an upper set M^* in $\mathbb{R}^2$ (right picture).

Definition of weak SD II

Definition (Equivalent definition)

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors with survival distribution functions $\bar{F}$ and $\bar{G}$. Then $\boldsymbol{X}$ weakly stochastically dominates $\boldsymbol{Y}$, denoted as $\boldsymbol{X} \succeq_w \boldsymbol{Y}$, if for each $\boldsymbol{m} \in \mathbb{R}^d$ one has $\bar{F}(\boldsymbol{m}) \geq \bar{G}(\boldsymbol{m})$.

Why do strong and weak stochastic dominance merge in one dimension?

When we consider random variables instead of random vectors, both definition are equivalent and we only talk about stochastic dominance, since in $\mathbb{R}$ all upper sets are intervals of the form (m, ∞).

Generator of strong and weak SD

A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be nondecreasing if $\boldsymbol{x} \leq \boldsymbol{y}$ implies $u(\boldsymbol{x}) \leq u(\boldsymbol{y})$. We define by $\mathcal{U}_1$ the space of all nondecreasing functions $u : \mathbb{R}^d \to \mathbb{R}$.

Theorem (Sriboonchitta, Wong, Dhompongsa, and Nguyen (2009))

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors. Then $\boldsymbol{X}$ stochastically dominates $\boldsymbol{Y}$ if and only if $\mathbb{E}u(\boldsymbol{X}) \geq \mathbb{E}u(\boldsymbol{Y})$ for all $u \in \mathcal{U}_1$.

Theorem

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors. Then $\boldsymbol{X}$ weakly stochastically dominates $\boldsymbol{Y}$ if and only if $\mathbb{E}u(\boldsymbol{X}) \geq \mathbb{E}u(\boldsymbol{Y})$ for all $u \in \mathcal{U}_1^*$.

Since $\mathcal{M}^* \subset \mathcal{M}$, the strong dominance implies the weak dominance and therefore we must have $\mathcal{U}_1^* \subset \mathcal{U}_1$.

How can be functions from $\mathcal{U}_1^*$ described analytically?

Comparison of SD in one dimension and multiple dimension

- One dimensional first order SD

$$\begin{aligned} X \succeq Y \quad &\Longleftrightarrow \quad \mathbb{P}(X \in (m,\infty)) \geq \mathbb{P}(Y \in (m,\infty)) \\ &\Longleftrightarrow \quad \bar{F}(x) \geq \bar{G}(x) \text{ for all } x \in \mathbb{R} \\ &\Longleftrightarrow \quad \mathbb{E}u(X) \geq \mathbb{E}u(Y) \text{ for all } u \in \mathcal{U}_1 . \end{aligned}$$

- Multidimensional SD

$$\begin{aligned} \boldsymbol{X} \succeq \boldsymbol{Y} \quad &\Longleftrightarrow \quad \boxed{\mathbb{P}(\boldsymbol{X} \in M) \geq \mathbb{P}(\boldsymbol{Y} \in M) \text{ for all } M \in \mathcal{M}} \ \text{hard to verify} \\ &\Longleftrightarrow \quad \boxed{\mathbb{E}u(X) \geq \mathbb{E}u(Y) \text{ for all } u \in \mathcal{U}_1} \ \text{general class of utility functions} \\ \boldsymbol{X} \succeq_w \boldsymbol{Y} \quad &\Longleftrightarrow \quad \boxed{\bar{F}(\boldsymbol{x}) \geq \bar{G}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^d} \ \text{possible to verify} \\ &\Longleftrightarrow \quad \boxed{\mathbb{E}u(X) \geq \mathbb{E}u(Y) \text{ for all } u \in \mathcal{U}_1^*} \ \text{subset of } \mathcal{U}_1 \end{aligned}$$

Contents

Conclusion and future work

In the research concerning the multidimensional SD we have already:

- stated definitions of the strong and weak SD and described the their generators,
- formulated propositions clarifying relations between two types of SD, as well as clarifying relations between multidimensional SD suggested by Dentcheva and Ruszczýnski (2009),
- formulated plenty of statements concerning both types of SD,
- proposed algorithm (formulated as an optimization problem) identifying the stochastic dominance between two random vectors with uniform as well as general discrete distribution,
- seeking for an exact conditions of SD for another types of distributions.

In the forthcoming research we intend to:

- incorporate the multidimensional SD constraints into the problem of finding the optimal investment strategy,
- examine whether it is possible to define the SD of higher orders also in multiple dimensions.

References

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Thank you for your attention