Robustness in stochastic programs

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Outline

- Robustness in stochastic programs with exogenous randomness using contamination techniques
- Motivation for decision dependent randomness
- Tractable reformulations
- Stability results and contamination bounds
- Numerical examples: mean-risk models
- Main quoted references

The model

We shall deal with robustness properties of risk constrained stochastic programs

$$\min_{x \in \mathcal{X}} F_0(x, P)$$

subject to

$$F_j(x, P) \leq 0,\ j = 1, \ldots, J; \quad (1)$$

- $\mathcal{X} \subset \mathbb{R}^n$ is a fixed nonempty convex set,
- functions $F_j(x, P)$, $j = 0, \ldots, J$ may depend on P
- P is the probability distribution of a random vector ω with range $\Omega \subset \mathbb{R}^m$

Denote $\mathcal{X}(P)$ set of feasible solutions, $\mathcal{X}^*(P)$ set of optimal solutions, $\varphi(P)$ optimal value of the objective function in (1).

Stability results and robustness wrt. P

Complete knowledge of the probability distribution is rare in practice – stability, robustness, output analysis wrt. P is needed .

• Quantitative stability cf. Theorem 5 of Römisch (2003) applied to (1) provides upper semicontinuity of the set of optimal solutions and a local Lipschitz property of the optimal value function for stochastic programs (1) with smooth, convex objective and one expectation type smooth convex constraint $F(x, P) \leq 0$ if at the optimal solution $x^*(P)$ of the unperturbed problem

$$\min_{x \in \mathcal{X}} F_0(x, P) \text{ s.t. } F(x, P) := E_P f(x, \omega) \leq 0$$

the constraint is not active, or if $\nabla F(x^*(P), P) \neq 0$.

To get metric regularity for multiple expectation type smooth convex constraints $F_j(x, P) \leq 0$, $j = 1, \ldots, J$, general constraint qualification should be used or constraints reformulated as $F(x, P) := \max_j F_j(x, P) \leq 0$ – again convex function.

• Contamination bounds for the optimal value function

• Another possibility – incorporate the incomplete knowledge of P into the model – ambiguity, minimax.

Robustness analysis via Contamination

was firstly derived by Dupacova (1990) for (2), i.e. for $\mathcal{X}(P)$ independent of P and for expectation type objective $F_0(x, P)$.
Assume that SP

$$\min_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x}, \omega) \tag{2}$$

was solved for P, denote $\varphi(P)$ optimal value. Changes in probability distribution P are modeled using contaminated distributions P_t,

$$P_t := (1-t)P + tQ, \; t \in [0, 1]$$

with Q another *fixed* probability distribution.
Via contamination, robustness analysis wrt. changes in P gets reduced to much simpler analysis wrt. scalar parameter t (see e.g. resist).
Objective function in (2) is linear in $P \Longrightarrow F_0(x, P_t)$ is linear wrt. $t \Longrightarrow$ optimal value function

$$\varphi(t) := \min_{x\in\mathcal{X}} F_0(x, P_t)$$

is concave on $[0, 1] \Longrightarrow$ continuity and existence of directional derivatives in $(0, 1)$. Continuity at $t = 0$ is property related with stability for SP (2).
In general, one needs set of optimal solutions $\mathcal{X}^*(P) \neq \emptyset$, bounded.

Contamination Bounds

Concave $\varphi(t) \Longrightarrow$ contamination bounds

$$\varphi(0) + t\varphi'(0^+) \geq \varphi(t) \geq (1-t)\varphi(0) + t\varphi(1),\ t \in [0,1]. \qquad (3)$$

Using arbitrary optimal solution $x(P)$ of (2) $\rightarrow$ upper bound

$$\varphi'(0^+) \leq F(x(P), Q) - \varphi(0).$$

Contamination bounds (3) are global, valid for all $t \in [0,1]$. They quantify the change in optimal value due to considered perturbations of (2); cf. application to stress test of CVaR. The approach can be generalized to objective functions $F(x, P)$ convex in x and concave in P.

Stress testing and robustness analysis via contamination with respect to changes in probability distribution P is straightforward for expected disutility models (objective function is linear in P). Also stress testing for convex risk or deviation measures via contamination can be developed: When the risk or deviation measures are concave with respect to probability distribution P they are concave wrt. parameter t of contaminated probability distributions P_t.

Contamination bounds – constraints dependent on P

New problems – $\varphi(t)$ is no more concave in t.

Use $P_t := (1-t)P + tQ$, $t \in (0,1)$ in SP (1) at the place of P. Set of feasible solutions of (1) for contaminated probability distribution P_t

$$\mathcal{X}(P_t) = \mathcal{X} \cap \{x \mid F_j(x, P_t) \le 0, j = 1, \ldots, J\}. \tag{4}$$

Denote $\mathcal{X}(t)$, $\varphi(t)$, $\mathcal{X}^*(t)$ the set of feasible solutions, the optimal value $\varphi(P_t)$ and the set of optimal solutions $\mathcal{X}^*(P_t)$ of contaminated problem

$$\text{minimize } F_0(x, P_t) \text{ on the set } \mathcal{X}(P_t). \tag{5}$$

The task is to construct computable lower and upper bounds for $\varphi(t)$ & exploit them for robustness analysis in risk-shaping with CVaR or for a stochastic dominance test with respect to inclusion of additional scenarios. Thanks to the assumed structure of perturbations

- lower bound can be derived for $F_j(x, P)$, $j = 0, \ldots, J$, linear or concave with respect to P without any smoothness or convexity assumptions with respect to x,
- convexity of SP (1) is essential for directional differentiability of the optimal value function,
- further assumptions are needed for derivation of the upper bound.

Lower bound

1. One constraint dependent on P and objective F_0 *independent* of P:

$$\min_{x \in \mathcal{X}} F_0(x) \text{ subject to } F(x, P) \leq 0. \tag{6}$$

For contaminated probability distribution P_t we get

$$\min_{x \in \mathcal{X}} F_0(x) \text{ subject to } F(x, t) := F(x, P_t) \leq 0 \tag{7}$$

– nonlinear parametric program with scalar parameter $t \in [0, 1]$, set of feasible solutions $\mathcal{X}(t) := \{x \in \mathcal{X} \mid F(x, t) \leq 0\}$ depends on t.
In general, the optimal value function is not concave.

Theorem

Let $F(x, \bullet)$ be concave function of $t \in [0, 1]$. Then the optimal value function of (7)

$$\varphi(t) := \min_{x \in \mathcal{X}} F_0(x) \textit{ subject to } F(x, t) \leq 0$$

is quasiconcave in $t \in [0, 1]$ with the lower bound

$$\varphi(t) \geq \min\{\varphi(1), \varphi(0)\}. \tag{8}$$

Lower bound – cont.

Proof is based on inclusion

$$\mathcal{X}((1-\lambda)t_1+\lambda t_2) \subset \{x \in \mathcal{X} \mid (1-\lambda)F(x,t_1)+\lambda F(x,t_2) \leq 0\} \subset \mathcal{X}(t_1) \cup \mathcal{X}(t_2) \quad (9)$$

valid for arbitrary $t_1,\ t_2 \in [0,1]$ and $0 \leq \lambda \leq 1$.

2. When also objective function depends on probability distribution, i.e. on contamination parameter t, the problem is

$$\min_{x \in \mathcal{X}} F_0(x,t) := F_0(x,P_t) \text{ subject to } F(x,t) \leq 0. \quad (10)$$

For $F_0(x,P)$ linear or concave in P, lower bound can be obtained by application of the above quasiconcavity result (8) separately to $F_0(x,P)$ and $F_0(x,Q)$:

$$\varphi(t) = \min_{x \in \mathcal{X}(t)} F_0(x,(1-t)P+tQ) \geq \min_{x \in \mathcal{X}(t)} [(1-t)F_0(x,P)+tF_0(x,Q)] \geq$$
$$(1-t)\min\{\varphi(0), \min_{\mathcal{X}(Q)} F_0(x,P)\} + t\min\{\varphi(1), \min_{\mathcal{X}(P)} F_0(x,Q)\}. \quad (11$$

The bound is more complicated but still computable.

Lower bound -cont.

3. For multiple constraints and contaminated probability distribution it would be necessary to prove first the inclusion $\mathcal{X}(t) \subset \mathcal{X}(0) \cup \mathcal{X}(1)$ and then the lower bound (8) for the optimal value $\varphi(t) = \min_{x \in \mathcal{X}(t)} F_0(x, P_t)$ can be obtained as in the case of one constraint.

Denote $\mathcal{X}_j(t) = \{x \mid F_j(x, P_t) \leq 0\}$. Then according to (9), $\mathcal{X}_j(t) \subset \mathcal{X}_j(0) \cup \mathcal{X}_j(1)$, hence

$$\mathcal{X}(t) \subset \mathcal{X} \cap \bigcap_j [\mathcal{X}_j(0) \cup \mathcal{X}_j(1)] := \mathcal{X}_0.$$

To evaluate the corresponding lower bound $\min_{x \in \mathcal{X}_0} F_0(x, P_t)$ would mean to solve a facial disjunctive program.

Notice that no convexity assumptions with respect to x were needed.

Directional derivative

Assume now that problem (1) is convex with respect to x. Then directional derivative of optimal value function $\varphi(0)$ can be obtained acc. to Gol'shtein (1970), Theorem 17 applied to Lagrange function

$$L(x, u, t) = F_0(x, t) + \sum_j u_j F_j(x, t)$$

when the set of optimal solutions $\mathcal{X}^*(P) = \mathcal{X}^*(0)$ and the set of Lagrange multipliers $\mathcal{U}^*(P) = \mathcal{U}^*(0)$ are nonempty and compact and all functions F_j are linear in P – linearity in the contamination parameter t:

$$\varphi'(0^+) = \min_{x \in \mathcal{X}^*(0)} \max_{u \in \mathcal{U}^*(0)} \frac{\partial}{\partial t} L(x, u, 0) = \min_{x \in \mathcal{X}^*(0)} \max_{u \in \mathcal{U}^*(0)} (L(x, u, Q) - L(x, u, P)). \tag{12}$$

Formula (12) simplifies substantially when $\mathcal{U}^*(0)$ is a singleton. When the constraints do not depend on P we get

$$\varphi'(0^+) = \min_{x \in \mathcal{X}^*(0)} \frac{\partial}{\partial t} F_0(x, 0^+) = \min_{x \in \mathcal{X}^*(0)} F_0(x, Q) - \varphi(0)). \tag{13}$$

These formulas can be exploited to construct an upper bound. More general cases are treated in e.g. Bonnans-Shapiro.

Upper bound

To derive an upper bound for optimal value of the contaminated problem with probability dependent constraints we shall assume that all functions $F_j(x,t)$, $j = 0, \dots, J$, are linear in t on interval $[0,1]$. Denote

$$F(x, P_t) = F(x,t) := \max_j F_j(x,t).$$

For convex $F_j(\bullet, P)\, \forall j$ the max function $F(\bullet, P)$ is convex and

$$\mathcal{X}(t) = \mathcal{X} \cap \{x : F(x,t) \le 0\}$$

with one linearly perturbed convex constraint.
1. Assume first that for optimal solution $x^*(0)$ of (1), $F(x^*(0), P) = 0$ and $F(x^*(0), Q) \le 0$. Then at least one of constraints is active at optimal solution and $x^*(0) \in \mathcal{X}(t)\, \forall t$:

$$F(x^*(0), t) = \max_j [(1-t)F_j(x^*(0), P) + tF_j(x^*(0), Q)]$$

$$\le (1-t)F(x^*(0), P) + tF(x^*(0), Q) \le 0.$$

$\rightsquigarrow$ *trivial global upper bound* $F_0(x^*(0), t) \ge \varphi(t)$; if $F_0(x, P)$ is linear in P

$$\varphi(t) \le F_0(x^*(0), t) = (1-t)\varphi(0) + tF_0(x^*(0), Q)\, \forall t \in [0,1]; \quad (14)$$

Local upper bound via NLP stability results

In convex case – analyze optimal value function by 1st order methods: If $x^*(0)$ is nondegenerate point, $\mathcal{X}$ in (4) convex polyhedral, the contaminated problem reduces locally into problem with parameter independent set of feasible solutions e.g. [Robinson] $\rightarrow$ for t small enough optimal value function $\varphi(t)$ is concave and its upper bound equals

$$\varphi(t) \leq \varphi(0) + t\varphi'(0^+)\,\forall t \in [0, t_0]. \tag{15}$$

Nondegenerate point: for $\mathcal{X} = \mathbb{R}^n$ means independence of gradients of active constraints at $x^*(0)$ or nondegeneracy for LP.

If also strict complementarity holds true, one faces locally an unconstrained minimization problem. More detailed insight can be obtained by a second order analysis; e.g. if $\exists$ continuous trajectory $[x^*(t), u^*(t)]$ of optimal solutions and Lagrange multipliers of (5) emanating from the unique optimal solution $x^*(0)$ and unique Lagrange multipliers $u^*(0)$ of (1) we get (15) with

$$\varphi'(0^+) = (L(x^*(0), u^*(0), Q) - \varphi(P)). \tag{16}$$

Illustrative example – mean-CVaR models

Consider $S = 53$ equiprobable scenarios of weakly returns ϱ of $N = 9$ assets (9 European stock market indexes: AEX, ATX, BCII, BFX, FCHI, GDAXI, PSI20, IBEX, ISEQ) in period 5.10.2007 - 3.10.2008. The scenarios can be collected in the matrix

$$R = \begin{pmatrix} r^1 \\ r^2 \\ \vdots \\ r^S \end{pmatrix}$$

where $r^s = (r_1^s, r_2^s, \ldots, r_N^s)$ is the s-th scenario. We will use x for the vector of portfolio weights and the portfolio possibilities are given by

$$\mathcal{X} = \{x \in \mathbb{R}^N | 1'x = 1, \ \ x_n \geq 0, \ \ n = 1, 2, ..., N\}$$

that is, the short sales are not allowed. The historical data comes from pre-crisis period. The data are contaminated by a scenario $\mathbf{r}^{S+1}$ from 10.10.2008 when all indexes strongly fell down. The additional scenario can be understood as a stress scenario or the worst-case scenario.

Illustrative example – mean-CVaR models

Index	Country	Mean	Max	Min	A.S.
AEX	Netherlands	-0.0098	0.10508	-0.12649	-0.24551
ATX	Austria	-0.01032	0.067022	-0.06982	-0.28503
BCII	Italy	-0.01051	0.047976	-0.06044	-0.19581
BFX	Belgium	-0.00997	0.051099	-0.07386	-0.2253
FCHI	France	-0.00795	0.050254	-0.06292	-0.21704
GDAXI	Germany	-0.00742	0.040619	-0.07568	-0.21151
PSI20	Portugal	-0.00998	0.049866	-0.07404	-0.18116
IBEX	Spain	-0.00625	0.053098	-0.06992	-0.2074
ISEQ	Ireland	-0.01378	0.113174	-0.14689	-0.26767

Table: Descriptive statistics of 9 European stock indexes and the additional scenario

We will apply the contamination bounds to mean-risk models with CVaR as a measure of risk. Two formulations are considered: In the first one, we are searching for a portfolio with minimal CVaR and at least the prescribed expected return. Secondly, we minimize the expected loss of the portfolio under the condition that CVaR is below a given level.

Illustrative example – CVaR minimizing

Mean-CVaR model with CVaR minimization is a special case of the general formulation (1) when $F_0(x,P) = \mathrm{CVaR}(-\varrho' x)$ and $F_1(x,P) = E_P(-\varrho' x) - \mu(P)$; $\mu(P)$ is the maximal allowable expected loss. We choose

$$\mu(P) = -E_P \varrho' (\frac{1}{9}, \frac{1}{9}, ..., \frac{1}{9})' = \frac{1}{53} \sum_{s=1}^{53} -\mathbf{r}^s (\frac{1}{9}, \frac{1}{9}, ..., \frac{1}{9})'.$$

It means that the minimal required expected return is equal to the average return of the equally diversified portfolio. The significance level $\alpha = 0.95$ and $\mathcal{X}$ is a fixed convex polyhedral set representing constraints that do not depend on P.

We construct:

- Lower bound (globally for $t \in [0,1]$):

$$(1-t)\min\{\varphi(0), \min_{\mathcal{X}(Q)} F_0(x,P)\} + t\min\{\varphi(1), \min_{\mathcal{X}(P)} F_0(x,Q)\}$$

Illustrative example – CVaR minimizing

- Trivial upper bound (globally for $t \in [0,1]$): Since $x^*(0)$ is a feasible solution of fully contaminated problem, we may use the trivial global bound:

$$F_0(x^*(0), P_t) = \mathrm{CVaR}_\alpha(x^*(0), (1-t)P + tQ))$$

The disadvantage of this trivial bound is the fact, that it would require evaluation of the CVaR for each t. Linearity with respect to t does not hold true, but using concavity of CVaR with respect to t, we may derive an upper estimate for $F_0(x^*(0), t)$:

- Upper estimate of upper bound (globally for $t = [0,1]$):

$$\mathrm{CVaR}_\alpha(x^*(0), (1-t)P + tQ)$$

$$\leq (1-t)\mathrm{CVaR}_\alpha(x^*(0), P) + t\Phi_\alpha(x^*(0), v^*(x, P), Q),$$

see [D-P].

Illustrative example – CVaR minimizing

The lower bound is linear, the upper bound is piecewise linear in t and for small values of t it coincides with the estimated upper bound.

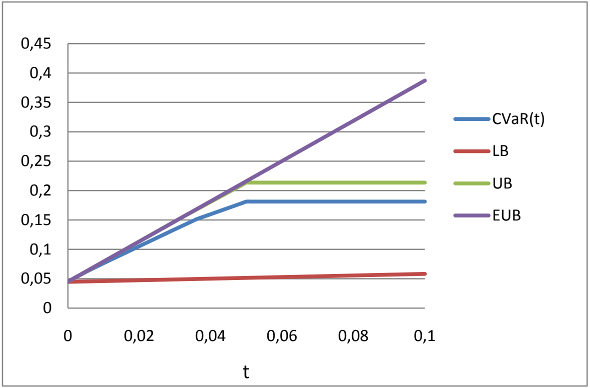

Figure: Comparison of optimal values (CVaR(t)) of mean-CVaR models with lower bound (LB), upper bound (UB) and the estimated upper bound (EUB) for the contaminated data.

Illustrative example – Expected loss minimizing

As the second example, consider the mean-CVaR model minimizing the expected loss subject to a constraint on CVaR. This corresponds to (1) with $F_0(x,P) = E_P(-\varrho' x)$ and $F_1(x,P) = \mathrm{CVaR}(-\varrho' x) - c$ where $c = 0.19$ is the maximal accepted level of CVaR. For simplicity, this level does not depend on the probability distribution. Similarly to the previous example, we compute the optimal value $\varphi(t)$ and its lower and upper bound.

- Lower bound (globally for $t \in [0,1]$):

$$(1-t)\min\{\varphi(0), \min_{\mathcal{X}(Q)} F_0(x,P)\} + t\min\{\varphi(1), \min_{\mathcal{X}(P)} F_0(x,Q)\}$$

- Upper bound (locally for $t \in [0,t_0]$): In this case $x^*(0) \notin \mathcal{X}(Q)$, hence the trivial upper bound can not be used. Therefore we apply the more general upper bound:

$$\varphi(t) \le \varphi(0) + t\varphi'(0^+)\,\forall t \in [0,t_0].$$

that leads to:

$$\varphi(t) \le (1-t)\varphi(0) + tF_0(x^*(0),Q)\,\forall t \in [0,t_0].$$

Illustrative example – Expected loss minimizing

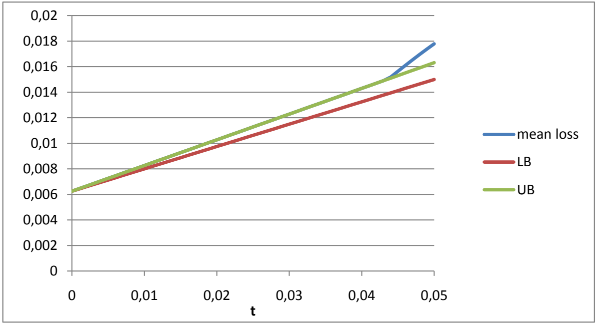

Figure: Comparison of minimal mean loss values with its lower bound (LB) and upper bound (UB) for the contaminated data.

The upper bound coincides with $\varphi(t)$ for $t \leq 0.043$. It illustrates the fact that the local upper bound is meaningful if the probability of the additional scenario is not too large, i.e. no more than the double of probabilities of the original scenarios for our example.

Markowitz model example – Expected loss minimizing

Consider the mean-var (Markowitz) model minimizing the expected loss subject to a constraint on var. This corresponds to (1) with $F_0(x,P) = E_P(-\varrho' x)$ and $F_1(x,P) = x^\top \Sigma x - v$ where $v = 0.001$ is the maximal accepted level of var. We compute the optimal value $\varphi(t)$ and its lower and upper bound.

- original distribution - 40 monthly return scenarios before the crises
- alternative distribution - 40 monthly return scenarios during the crises
- Lower bound (globally for $t \in [0,1]$):

$$(1-t)\min\{\varphi(0), \min_{\mathcal{X}(Q)} F_0(x,P)\} + t\min\{\varphi(1), \min_{\mathcal{X}(P)} F_0(x,Q)\}$$

- Upper bound (locally for $t \in [0, t_0]$): We apply the local upper bound:

$$\varphi(t) \le \varphi(0) + t\varphi'(0^+)\, \forall t \in [0, t_0].$$

that leads to:

$$\varphi(t) \le (1-t)\varphi(0) + tF_0(x^*(0), Q)\, \forall t \in [0, t_0].$$

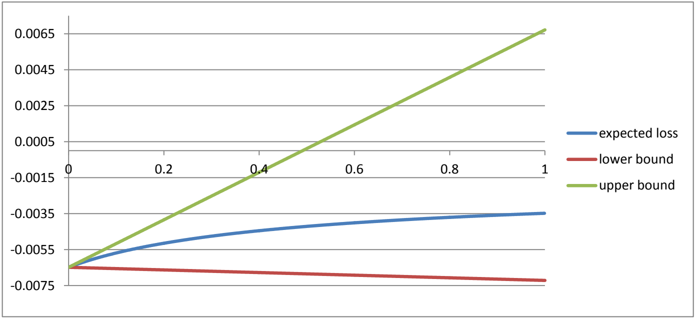

Figure: Comparison of minimal mean loss values with its lower bound (LB) and upper bound (UB) for the contaminated data.

The upper bound holds true all $t \in [0, 1]$.

Illustrative example – Mean-VaR$_{0.97}$ *model*

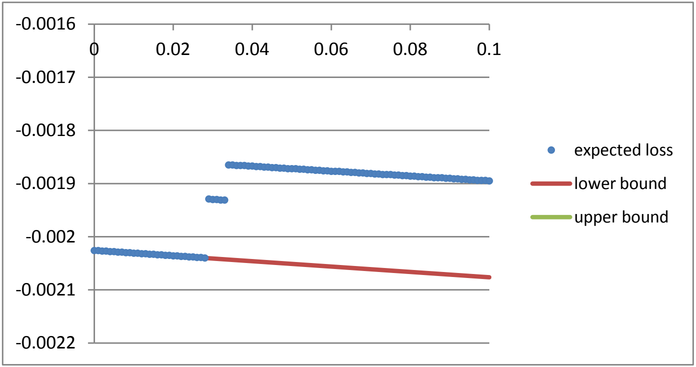

Figure: Comparison of minimal mean loss values with its lower bound (LB)

The upper bound holds true for $t \leq 0.028$.

Illustrative example – Mean-VaR$_{0.95}$ *model*

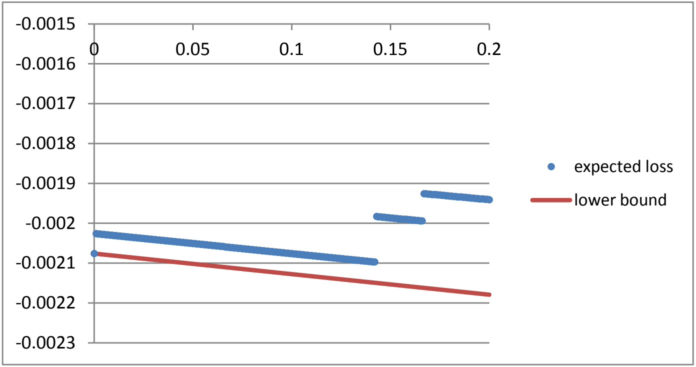

Figure: Comparison of minimal mean loss values with its lower bound (LB)

Motivation for decision dependent randomness

Common assumption – decisions do not infuence probability distribution of random parameters – need not be adequate.

Important applications
Jonsbraten, Wets, Woodruff – production line (1998)
Plambeck, Robinson, Suri – PERT (1996)
Goel, Gupta, Grossmann – oil and gass field development, (2004, 2007, 2012, ...)
Many new chalenging problems – network interdiction

Finance:

- Decision dependent randomness is typically observed on the illiquid markets or/and in the case of intraday trading, when the distribution of rate of returns may change if a relatively high percentage of stocks are bought or sold by the decision maker.
- Another example of rate of returns depending on the investment volume can be found in retail savings, some banks offer higher interest rates for small volumes of savings as a marketing action. Several ranges for investment volumes are considered.

Extension of newsboy problem

Additional decision variable – price p of product. Random demand depends on p; cf. BRNO group 2012.

Newsboy sells newspapers at p and has to buy them at c before he starts selling. Newspapers left over at the end of the day cannot be stored. The demand ω is random, with distribution function $F(t)$, the newsboy wants to maximize his expected profit:

$$\max_x (p-c)x - p\int_{x\geq t}(x-t)dF(t).$$

Known formulas for case of ω uniformly distributed $U(a,b)$.

Demand depends on price p – additional decision variable. Form of dependence – $\omega(p)$ uniformly distributed on $[a(p), b(p)]$ and expected profit

$$(p-c)x - \frac{p(x-a(p))^2}{2(b(p)-a(p))}$$

is maximized wrt. $b(p) \geq x \geq 0$, $p \in [p_0, p_1]$.

Similar approach appears in PERT problem by Plambeck et al.

Model with fixed feasibility set

The easiest case of decision dependent randomness occurs if the set of feasible solutions does not depend on the random vector. In this case the problem with decision dependent randomness takes the form:

$$\min_{x \in \mathcal{X}} F_0(x, P(x)) \tag{17}$$

where

- $\mathcal{X} \subseteq \mathbb{R}^n$ is a nonempty convex set,
- $P(x)$ is the probability distribution of a random vector $\omega(x)$ with range $\Omega \subset \mathbb{R}^m$ for all $x \in \mathcal{X}$. We asume that $\omega(x)$ and $P(x)$ are uniquely assigned to each $x \in \mathcal{X}$. Moreover, we shall denote $\mathcal{P}$ the set of all such $P(x)$, i.e. $\mathcal{P} = \{P(x) : x \in \mathcal{X}\}$
- $F_0 : \mathcal{X} \times \mathcal{P} \to \mathbb{R}$ is a function which may depend on $P(x)$,
- we assume that a solution of (17) exists.

Increased complexity; search for tractable cases, e.g.

- decision dependent parameters of the distribution or scenario probabilities
- suitable transformation of decision dependent probability distribution
- finite number of considered probability distributions

Decision dependent probabilities - model with fixed feasibility set A

Now, $\mathcal{P}$ denotes set of probability distributions on a fixed finite set of atoms/scenarios $\Omega = \{\omega_1, \dots, \omega_I\}$. Hence, $P(x) \in \mathcal{P}$ are given by probabilities $p(x, \omega_i)) \geq 0, i = 1, \dots, I, \sum_i p(x, \omega_i) = 1$ for $x \in \mathcal{X}$. Zero probabilities are not excluded and degenerated distributions can be considered. With objective functions

$$F_0(x, P(x)) = \sum\nolimits_{i=1}^{I} p(x, \omega_i) f(x, \omega_i),$$

two plausible assumptions are $p(x, \omega_i)$ and $f(x, \omega_i)$ continuous functions of $x \, \forall i$ or the set of feasible decisions $\mathcal{X}$ is discrete.

Model with fixed feasibility set A - contamination

Consider alternative probability distributions $Q(x) \in \mathcal{P}$ determined by probabilities $q(x, \omega_i)$, $i = 1, \dots, I$. For a fixed $x \in \mathcal{X}$, contaminated distribution $P(x, \lambda) = (1 - \lambda)P(x) + \lambda Q(x)$ is defined by probabilities $(1 - \lambda)p(x, \omega_i) + \lambda q(x, \omega_i)$. Linearity of $F_0(x, P(x, \lambda))$ in λ implies that contaminated optimal value

$$\varphi(\lambda) = \min_x F_0(x, P(x, \lambda))$$

is concave wrt. λ provided that optimal solutions exist. Hence, the lower contamination bounds are valid, the upper bounds based on directional derivatives are more involved:

$$\varphi(0) + \lambda\varphi'(0^+) \geq \varphi(\lambda) \geq (1 - \lambda)\varphi(0) + \lambda\varphi(1),\ \lambda \in [0, 1]. \tag{18}$$

Identical supports $\Omega_P, \Omega_Q \subset \Omega$ correspond to changing probabilities $p_i(x)$, different supports $\Omega_P, \Omega_Q \subset \Omega$ correspond e.g. to a pooled sample of two scenario beds.
This scheme fits well the static and two-stage stochastic programs. To apply it to linear multistage problems, it will be necessary to use the forms with explicit nonanticipativity constraints for x and to keep identical constraints both for $P(x)$ and $Q(x)$.

Simple example

Consider a simple maximizing expected utility problem:

$$\max_{x \in \mathcal{X}} Eu(\omega(x)^{\mathrm{T}} x)$$

where $\omega(x)$ is a random vector of returns depending on x as follows:

- $\omega(x)$ takes only two values (scenarios): ω_1 and ω_2.
- probability of taking ω_1 depends on x: $p(x)$

The problem can be reformulated as follows:

$$\max_{x \in \mathcal{X}} p(x)u(\omega_1^{\mathrm{T}} x) + (1 - p(x))u(\omega_2^{\mathrm{T}} x)$$

Alternative distribution $Q(x)$ differs only in the probability of taking scenario ω_1: $q(x)$ and contaminated problem is of the form:

$$\max_{x \in \mathcal{X}} ((1-\lambda)p(x)+\lambda q(x))u(\omega_1^{\mathrm{T}} x)+((1-\lambda)(1-p(x))+\lambda(1-q(x)))u(\omega_2^{\mathrm{T}} x)$$

Example of $p(x)$: $p(x) = p_1$ iff $x \in \mathcal{X}_1 \subset \mathcal{X}$ and $p(x) = p_2$ otherwise.

Transformation function - model with fixed feasibility set B

Under specific assumptions, dependence of P on x can be removed by a suitable transformation of the decision-dependent probability distribution $P(x)$ cf. Varayia-Wets, Pflug 1990, 1999.
ASSUME expectation type of objective function and existence of densities $p(x,\omega)$ of $P(x) \in \mathcal{P}$ with respect to a common probability measure μ.

Objective function can be rewritten as

$$F_0(x, P(x)) = \int f(x,\omega)p(x,\omega)\mu(d\omega) = \int_\Omega \tilde{f}(x,\omega)\mu(d\omega)$$

with $\tilde{f}(x,\omega) := f(x,\omega)p(x,\omega)$ and decision-independent distribution μ.

Problems

When using transformation, convenient properties of random objective function $f(x,\omega)$ can get lost $\longrightarrow$ difficulties in evaluation of subgradients of $F(x)$, etc. Properties of the resulting objective function depend on structure of the problem & on type of dependence of P on x.

Model with fixed feasibility set B – Contamination

Meaning of contamination?
Instead of $P(x)$ consider other distributions, say $Q(x) \in \mathcal{P}$ with densities $q(x,\omega)$ with respect to $\mu \to$ two objectives

$$F_0(x, P(x)) = \int_\Omega f(x,\omega)p(x,\omega)\mu(d\omega)$$

$$F_0(x, Q(x)) = \int_\Omega f(x,\omega)q(x,\omega)\mu(d\omega)$$

Contamination means here contamination of density $p(x,\omega)$ by $q(x,\omega)$:

$$\varphi(\lambda) = \min_x F_0(x, P(x,\lambda)) = \min_x \int f(x,\omega)((1-\lambda)p(x,\omega)+\lambda q(x,\omega))\mu(d\omega)$$

Hence $F_0(x, P(x,\lambda))$ is linear in λ.

Model with fixed feasibility set B – Contamination

Assuming continuous integrands and fixed set $\mathcal{X}$ some of contamination results could be obtained (optimal value is again concave in λ). Concavity of the contaminated objective function will be OK also for other choices of $Q(x)$. Expectation form of the objective can be relaxed to objective functions concave in distribution P to include risk criteria. For a fixed x (in fact for a fixed tripple $x, P(x), Q(x)$)

$$F_0(x,(1-\lambda)P(x)+\lambda Q(x)) \geq (1-\lambda)F_0(x,P(x))+\lambda F_0(x,Q(x))$$

However, the optimal values of the two objectives $F_0(x, P(x))$, $F_0(x, Q(x))$ may be attained in different points, say x^P, x^Q and for corresponding distributions $P(x^P)$, $Q(x^Q)$. Anyway, for $x(\lambda) \in \mathrm{argmin}_x F_0(x,(1-\lambda)P(x)+\lambda Q(x))$:

$$\begin{aligned}
\varphi(\lambda) &= F_0(x(\lambda), P(x(\lambda),\lambda)) \\
&\geq (1-\lambda)F_0(x(\lambda),P(x(\lambda)))+\lambda F_0(x(\lambda),Q(x(\lambda))) \\
&\geq (1-\lambda)\min_x F_0(x,P(x))+\lambda \min_x F_0(x,Q(x)) \\
&\geq (1-\lambda)\varphi(0)+\lambda\varphi(1).
\end{aligned}$$

Model with fixed feasibility set B – upper bound

Although $\varphi(\lambda)$ is concave in λ it may be DIFFICULT to compute the directional derivative for the upper bound. In this case (since $F_0(x, P(x, \lambda))$ is linear in λ) at least the following triavial upper bound can be used:

$$\begin{aligned}
\varphi(\lambda) &= F_0(x(\lambda), P(x(\lambda), \lambda)) \\
&\leq F_0(x(0), P(x(0), \lambda)) \\
&\leq \int_\Omega f(x(0), \omega)\, ((1-\lambda)p(x(0), \omega) + \lambda q(x(0), \omega))\, \mu(d\omega) \\
&\leq (1-\lambda)\varphi(0) + \lambda F_0(x(0), Q(x(0)))
\end{aligned}$$

Summarizing:

$$(1-\lambda)\varphi(0)+\lambda\varphi(1) \leq \varphi(\lambda) \leq (1-\lambda)\varphi(0)+\lambda F_0(x(0), Q(x(0))) \;\; \forall \lambda \in [0, 1]$$

Neither the assumption on convexity in x nor on differentiability in x is needed. However, the trivial global upper bound will not hold if $F_0(x, P(x, \lambda))$ is not linear in λ.

Finite cardinality of $\mathcal{P}$ - model with fixed feasibility set C

Assume now that there exists a partition of $\mathcal{X}$: $(\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_I)$ such that:

(i) $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset \quad \forall i \neq j$,
(ii) $\bigcup_{i=1}^{I} \mathcal{X}_i = \mathcal{X}$,
(iii) $P(x) = P_i \quad \forall x \in \mathcal{X}_i, \quad i = 1, \ldots, I$,
(iv) $F_0(x, P_i)$ is continuous in x on $\mathrm{clo}(\mathcal{X}_i), i = 1, \ldots, I$.

Then (17) can be solved using the following I auxiliary problems:

$$\varphi_i(P_i) = \min_{x \in \mathrm{clo}(\mathcal{X}_i)} F_0(x, P_i), i = 1, \ldots, I \tag{19}$$

After solving these programs, using standard stochastic programming techniques and algorithms we identify a set of indexes I^* for which the auxiliary problem has a solution. Then problem (17) is equivalent to:

$$\varphi(P) = \min_{i \in I^*} \min_{x \in \mathrm{clo}(\mathcal{X}_i)} F_0(x, P_i).$$

If the i-th auxiliary problem does not have a solution, i.e. $i \notin I^*$, it can not contribute to the solution of (17). Therefore only $i \in I^*$ are considered. x_i^* $(\mathcal{X}_i^*(P_i))$ denotes optimal solutions (sets).

Model with fixed feasibility set C – Contamination considerations

Consider alternative probability distribution $Q(x)$ such that optimal value $\varphi(Q)$ and optimal solution exist.

ASSUME that Q has the same structure as P, i.e. there is an identical partition with $Q(x) = Q_i$ on $\mathcal{X}_i$.

ASSUME that $F_0(x, P_i)$ are concave in P_i, $\forall i$.

For contaminated distributions $P_\lambda(x) = (1-\lambda)P(x) + \lambda Q(x)$ contamination bounds for

$$\varphi_i(\lambda) = \min F_0(x, (1-\lambda)P_i + \lambda Q_i) \text{ s.t. } x \in \text{clo}\mathcal{X}_i$$

based on concavity can be constructed under modest assumptions:

$$L_i(\lambda) = (1-\lambda)\varphi_i(P_i) + \lambda\varphi_i(Q_i)$$

$$U_i(\lambda) = \varphi_i(P_i) + \lambda\varphi_i'(O^+)$$

$\rightarrow$ Contamination bounds for
$\varphi(\lambda) = \min_i \min_x \{F_0(x, (1-\lambda)P_i + \lambda Q_i) \text{ s.t. } x \in \mathcal{X}\}$
$\min_i L_i(\lambda) \leq \varphi(\lambda) \leq \max_i U_i(\lambda)$

Model with fixed feasibility set C – Contamination considerations

Under modest additional assumptions,

$$\varphi_i'(0^+) = \frac{d}{d\lambda}\varphi_i(0^+) = \min_{x \in \mathcal{X}^*(P_i)} \frac{d}{d\lambda} F_i(x, 0^+).$$

Cf. Chapter 4.3.1 of B-S for results concerning the directional derivative. Hence, the upper bound $U_i(\lambda)$ follows.

ASSUME $\exists i^*$ such that $\varphi(\lambda) = \varphi_{i^*}(\lambda)$ for λ small enough. Then we have contamination bounds for $\varphi_{i^*}(\lambda)$ with directional derivative computed at the optimal solution $x^*_{i^*}$.

ASSUME FURTHER that $\varphi_i(P_i) > \varphi_{i^*}(P_{i^*}) i \neq i^*$. Consider two possibilities.

- Only P_{i^*} contaminated; $\Longrightarrow$ $\varphi_i(\lambda) = \varphi_i(P_i) \geq \varphi_{i^*}(\lambda)$ for λ small enough and $i \neq i^*$.

- Change of another P_i

In both cases $x^*_{i^*}$ remains optimal for λ small enough.

Model with RANDOM feasibility set

If the set of feasible solutions depends on the probability distribution $P(x)$, the problem takes the form:

$$\min_{x \in \mathcal{X}} F_0(x, P(x)) \tag{20}$$

subject to

$$(x, P(x)) \quad \in \quad \mathcal{Y} \tag{21}$$

where

- $\mathcal{X} \subseteq \mathbb{R}^n$ is a fixed nonempty convex set,
- $P(x)$ is the probability distribution of a random vector $\omega(x)$ with range $\Omega \subset \mathbb{R}^m$ for all $x \in \mathcal{X}$. We asume that $\omega(x)$ and $P(x)$ are uniquely assigned to each $x \in \mathcal{X}$. Moreover, we shall denote $\mathcal{P}$ the set of all such $P(x)$, i.e. $\mathcal{P} = \{P(x) : x \in \mathcal{X}\}$,
- set $\mathcal{Y}$ expresses the constraints depending on probability distributions,
- $F_0 : \mathcal{X} \times \mathcal{P} \to \mathbb{R}$ is a function which may depend on $P(x)$,
- we assume that a solution of (20) - (21) exists.

Model with RANDOM feasibility set A - decision dependent probabilities

ASSUME again that $P(x) \in \mathcal{P}$ are given by probabilities $p(x, \omega_i)) \geq 0, i = 1, \ldots, I, \sum_i p(x, \omega_i) = 1$ of fixed scenarios $\omega_1, \ldots, \omega_I$ and only the probabilities are affected by the contamination. The alternative probabilities are $q(x, \omega_i)) \geq 0, i = 1, \ldots, I, \sum_i q(x, \omega_i) = 1$. Let objective function (20) be expressed as follows:

$$F_0(x, P(x)) = \sum_{i=1}^{I} p(x, \omega_i) f_0(x, \omega_i),$$

and condition (21) is in the form:

$$G_j(x, P(x)) \leq 0, \quad j = 1, ..., J.$$

Then, taken $F_1(x, P(x)) = \max_j G_j(x, P(x))$ and assuming that there exists $f_1(x, \omega)$ such that:

$$F_1(x, P(x)) = \sum_{i=1}^{I} p(x, \omega_i) f_1(x, \omega_i),$$

Model with RANDOM feasibility set A - lower contamination bound

we have:

$$\min_{x\in\mathcal{X}} \quad \sum_{i=1}^{I} p(x,\omega_i)f_0(x,\omega_i) \tag{22}$$

$$s.t. \quad \sum_{i=1}^{I} p(x,\omega_i)f_1(x,\omega_i) \le 0. \tag{23}$$

Let $\mathcal{X}(\lambda) = \{x \in \mathcal{X} : \sum_{i=1}^{I}((1-\lambda)p(x,\omega_i) + \lambda q(x,\omega_i))f_1(x,\omega_i) \le 0\}$.
Since $F_0(x, P(x,\lambda)) = \sum_{i=1}^{I}((1-\lambda)p(x,\omega_i) + \lambda q(x,\omega_i))f_0(x,\omega_i)$ and $F_1(x, P(x,\lambda)) = \sum_{i=1}^{I}((1-\lambda)p(x,\omega_i) + \lambda q(x,\omega_i))f_1(x,\omega_i)$ are linear in λ, lower bound can be obtained by application of (18) separately to $F_0(x, P(x,0))$ and $F_0(x, P(x,1))$:

$$\begin{aligned}
\varphi(\lambda) &= \min_{x\in\mathcal{X}(\lambda)} F_0(x, P(x,\lambda)) = \min_{x\in\mathcal{X}(\lambda)} F_0(x, (1-\lambda)P(x) + \lambda Q(x)) \\
&= \min_{x\in\mathcal{X}(\lambda)} [(1-\lambda)F_0(x, P(x)) + \lambda F_0(x, Q(x))] \\
&\ge (1-\lambda)\min\{\varphi(0), \min_{\mathcal{X}(1)} F_0(x, P(x))\} \\
&\quad +\lambda \min\{\varphi(1), \min_{\mathcal{X}(0)} F_0(x, Q(x))\}.
\end{aligned} \tag{24}$$

Model with RANDOM feasibility set A - upper contamination bound

Basically, two upper bounds could be considered:

- local bound (for λ sufficiently small) based on the directional derivative $\varphi(0^+)$
- local or global trivial bound (valid for all $\lambda \in [0, 1]$) using the optimal solution of non-contaminated problem

1. If for optimal solution of non-contaminated problem $x(0)$ the constraint is not active, then (from linearity of $F_1(x, P(x, \lambda))$ in λ), there exists $\lambda_0 > 0$ such that $x(0) \in \mathcal{X}(\lambda)$ and, hence, trivial local upper bound is:

$$\varphi(\lambda) \leq F_0(x(0), P(x(0), \lambda)) \quad \forall \lambda \in [0, \lambda_0]$$

and using linearity of $F_0(x, P(x, \lambda))$ in λ:

$$\varphi(\lambda) \leq [(1-\lambda)F_0(x(0), P(x(0))) + \lambda F_0(x(0), Q(x(0)))] \quad \forall \lambda \in [0, \lambda_0].$$

Model with RANDOM feasibility set A - upper contamination bound

2 Assume that $x(0)$ is a feasible solution for fully contaminated problem, too. Then

$$x(0) \in \mathcal{X}(0) \Rightarrow \sum_{i=1}^{I} p(x(0), \omega_i) f_1(x(0), \omega_i) \le 0$$

$$x(0) \in \mathcal{X}(1) \Rightarrow \sum_{i=1}^{I} q(x(0), \omega_i) f_1(x(0), \omega_i) \le 0$$

and, consequently:

$$\sum_{i=1}^{I} ((1-\lambda) p(x(0), \omega_i) + \lambda q(x(0), \omega_i)) f_1(x(0), \omega_i) \le 0 \quad \forall \lambda \in [0,1]$$

hence $x(0) \in \mathcal{X}(\lambda) \;\; \forall \lambda \in [0,1]$ and $x(0)$ is used for the trivial upper bound construction:

$$\begin{aligned} \varphi(\lambda) &\le F_0(x(0), P(x(0), \lambda)) \\ &\le (1-\lambda) F_0(x(0), P(x(0))) + \lambda F_0(x(0), Q(x(0))) \quad \forall \lambda \in [0,1] \end{aligned} \tag{25}$$

i.e. this is again the same upper bound, but now valid for all $\lambda \in [0,1]$.
3. If there exists a feasible solution $x \in \mathcal{X}(\lambda)$ it can be used in (25) instead of $x(0)$.

Model with RANDOM feasibility set A - upper contamination bound

Notice that the trivial upper bound (25) holds true without any convexity or smoothness assumptions and for an arbitrary distributions $Q(x)$ forthe given scenarios.

4. Upper bound based on directional derivative: $\varphi(0) + \lambda\varphi'(0^+)$ would be possible to construct only under quite strong assumptions (for example: differentiability, strong complementarity, uniqueness of optimal solution $x(0)$, ...) - difficult to fulfill

Model with RANDOM feasibility set B - transformation functions

ASSUME again expectation type of objective function and existence of densities $p(x, \omega)$ of $P(x) \in \mathcal{P}$ with respect to a common probability measure μ. Moreover, let condition (21) is in the form:

$$Eg_j(x, \omega(x)) \leq 0, \quad j = 1, ..., J.$$

Then, taken $f_1(x, \omega(x))$ such that $Ef_1(x, (\omega(x))) = \max_j Eg_j(x, \omega(x))$ we reformulate the problem as follows:

$$\min_{x \in \mathcal{X}} \quad \int p(x, \omega) f_0(x, \omega) \mu(d\omega) \tag{26}$$

$$s.t. \quad \int p(x, \omega) f_1(x, \omega) \mu(d\omega) \leq 0. \tag{27}$$

Model with RANDOM feasibility set B - contamination

Contamination means here contamination of density $p(x, \omega)$ by $q(x, \omega)$:

$$\begin{aligned} \varphi(\lambda) &= \min_{x \in \mathcal{X}(\lambda)} F_0(x, P(x, \lambda)) \\ &= \min_{x \in \mathcal{X}(\lambda)} \int f_0(x, \omega)((1-\lambda)p(x, \omega) + \lambda q(x, \omega))\mu(d\omega) \end{aligned}$$

where

$$\mathcal{X}(\lambda) = \{x \in \mathcal{X} : \int f_1(x, \omega)((1-\lambda)p(x, \omega) + \lambda q(x, \omega))\mu(d\omega) \leq 0\}.$$

Again: $F_0(x, P(x, \lambda)) = \int f_0(x, \omega)((1-\lambda)p(x, \omega) + \lambda q(x, \omega))\mu(d\omega)$ and $F_1(x, P(x, \lambda)) = \int f_1(x, \omega)((1-\lambda)p(x, \omega) + \lambda q(x, \omega))\mu(d\omega)$ are linear in λ and $x \in \mathcal{X}(0) \cap \mathcal{X}(1) \Rightarrow x \in \mathcal{X}(\lambda) \quad \forall \quad \lambda \in [0, 1]$.
Hence, the same lower and upper bounds appear as in model A.

Model with RANDOM feasibility set C - finite cardinality of $\mathcal{P}$

For all distributions, assume again that there exists a partition of $\mathcal{X} : (\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_I)$ such that:

(i) $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset \quad \forall i \neq j$,
(ii) $\bigcup_{i=1}^{I} \mathcal{X}_i = \mathcal{X}$,
(iii) $P(x) = P_i \quad \forall x \in \mathcal{X}_i, \quad i = 1, \ldots, I$,
(iv) $F_0(x, P_i)$ is continuous in x on $\mathrm{clo}(\mathcal{X}_i), i = 1, \ldots, I$.

Then (20)-(21) can be solved using the following auxiliary problems:

$$\min F_0(x, P_i) \tag{28}$$

$$(x, P_i) \in \mathcal{Y}_i \tag{29}$$

where $\mathcal{Y}_i = \{(x, P(x)) \in \mathcal{Y} : x \in \mathrm{clo}(\mathcal{X}_i), \quad P(x) = P_i \quad \forall x \in \mathrm{clo}(\mathcal{X}_i)\}$ for each $i = 1, \ldots, I$.

The solution of (20)-(21) can be identified by minimization over $i \in I^*$, i.e. problem (20)-(21) is equivalent to:

$$\min_{i \in I^*} \min\{F_0(x, P_i) : (x, P_i) \in \mathcal{Y}_i\}.$$

If moreover

$$\mathcal{Y} = \{(x, P(x)) \in \mathcal{X} \times \mathcal{P} : F_j(x, P(x)) \leq 0, j = 1, \ldots, J\} \tag{30}$$

then problem (20)-(21) can alternatively be rewritten using binary variables b_i as follows:

$$\min_{x \in \mathcal{X}} \sum_{i=1}^{I} b_i F_0(x, P_i) \tag{31}$$

subject to

$$\sum_{i=1}^{I} b_i F_j(x, P_i) \leq 0, j = 1, \ldots, J, \tag{32}$$

$$b_i = 1 \text{ iff } \ x \in \mathrm{clo}(\mathcal{X}_i), \quad \text{otherwise } \ b_i = 0, \quad i = 1, \ldots, I. \tag{33}$$

The randomness in this problem again does not depend on the decision vector, however the tractable reformulation of (33) may not be available in some cases.

Model with RANDOM feasibility set C - Contamination

Consider alternative probability distribution $Q(x)$ such that optimal value $\varphi(Q)$ and optimal solution exist.

ASSUME that $Q(x)$ has the same structure as $P(x)$, i.e. there is an identical partition with $Q(x) = Q_i \forall x \in \mathcal{X}_i$.

ASSUME that $F_0(x, P_i)$ are linear (concave) in P_i, $\forall i$.

If for all relevant auxiliary problems (28) - (29) lower bounds L_i and upper bounds U_i can be somehow derived then:
$\min_{i \in I^*} L_i$ is a lower bound and $\max_{i \in I^*} U_i$ is an upper bound for the model C.
Let $x(0) \in \mathcal{X}_{i(0)}$. More tight bounds can be derived under assuption that the optimal solution of contaminated problem $x(\lambda) \in \mathcal{X}_{i(0)}$, too.
However, these bounds would typically be valid only locally, i.e. for sufficiently small λ.

Financial example - mean-variance model

In many applications $\mathcal{Y}$ can be formulated via J inequality constraints as follows:

$$\mathcal{Y} = \{(x, P(x)) \in \mathcal{X} \times \mathcal{P} : F_j(x, P(x)) \leq 0,\ j = 1, \ldots, J\} \tag{34}$$

For example, if:

$$\begin{aligned} F_0(x, P(x)) &= \operatorname{var}_{P(x)}(x'\omega) = \operatorname{var}(x'\omega(x)), && (35) \\ \mathcal{Y} &= \{(x, P(x)) \in \mathcal{X} \times \mathcal{P} : \\ &\quad \mu - \mathbb{E}_{P(x)}(x'\omega) = \mu - \mathbb{E}(x'\omega(x)) \leq 0\}, && (36) \\ \mathcal{X} &= \{x \in \mathbb{R}^n : \mathbf{1}'x = 1\}, && (37) \end{aligned}$$

then one gets the well know mean-variance problem with decision dependent randomness of returns, where μ is a minimal required mean return parameter.
Similarly for other measures of risk...

Toy example

Some banks offer their deposit certificates with the nonrandom rate of return which depends on the investment volume:
Assume that a decision maker wants to invest 1 Million USD. She can invest in deposit certificate and a stock index. The rate of return of the deposit certificate is 2% if at most 0.5 Million USD is invested and it decreases to 1% if the investment volume x_1 exceeds 0.5 Million USD. The rate of return of the stock index is random with expected value 4% and variance 0.01. The distribution of the rate of return of the stock index does not depend on the amount x_2 invested in it. We will formulate the mean-variance model using (35) - (37) with $x = (x_1, x_2)'$ and $\mu = 3\%$:

$$\min_{x_1, x_2} F_0(x, P(x)) = \mathrm{var}_{P(x)}(x'\omega) = 0.01 x_2^2$$

subject to

$$\begin{aligned} \mu - \mathbb{E}(x'\omega(x)) &= 0.03 - 0.02x_1 - 0.04x_2 \leq 0 \ \text{ if } \ x_1 \leq 0.5 \\ &= 0.03 - 0.01x_1 - 0.04x_2 \leq 0 \ \text{ if } \ x_1 > 0.5 \\ x_1 + x_2 &= 1. \end{aligned}$$

The optimal solution is $(x_1^*, x_2^*) = (0.5, 0.5)$ because the objective function is increasing and $x_2 < 0.5$ is not feasible.

Toy example con't

Now let's apply the partitioning. It is easy to check that sets:
$\mathcal{X}_1 = \{x \in \mathbb{R}^2 : x_1 \leq 0.5, x_2 = 1 - x_1\}$ and
$\mathcal{X}_2 = \{x \in \mathbb{R}^2 : x_1 > 0.5, x_2 = 1 - x_1\}$ satisfy all assumptions (i) - (iv).
The first auxiliary problem:

$$\min_{x_1, x_2} 0.01 x_2^2$$

subject to

$$\begin{aligned} 0.03 - 0.02x_1 - 0.04x_2 &\leq 0, \quad x_1 + x_2 = 1 \\ x_1 &\leq 0.5 \end{aligned}$$

has the optimal solution $(x_1^*, x_2^*) = (0.5, 0.5)$ The second auxiliary problem:

$$\min_{x_1, x_2} 0.01 x_2^2$$

subject to

$$\begin{aligned} 0.03 - 0.01x_1 - 0.04x_2 &\leq 0, \quad x_1 + x_2 = 1 \\ x_1 &\geq 0.5 \end{aligned}$$

has no solution, so $I^* = \{1\}$ and hence the optimal solution of (20)-(21) is $(x_1^*, x_2^*) = (0.5, 0.5)$.

Seem to be the most important and most difficult class of problems. Problems of Jonsbraten et al. and also Goel and Grossmann belong there. Similarly as in these papers, we shall assume that there is a finite number of possible probability distributions and each of them has been approximated by a discrete distribution carried by a finite number of atoms / scenarios. For multistage stochastic programs, these discrete probability distributions are used to create scenario trees. We assume that the horizon and position of stages is given and identical path probabilities are attached to scenarios related to the exogenous uncertainty (demand, capacity, etc). However, there is no universal scenario tree when the distribution of some random parameters is decision dependent.

Decision dependent scenario trees

Resulting scenario trees and path probabilities are indexed by a finite number of vector valued indices d. In principle, one may apply full enumeration with respect to d, a version of the branch-and-bound method, cf. Jonsbraten et al., or disjunctive programming techniques, cf. Goel and Grossmann.

For each scenario tree separately, one can solve the multistage stochastic program and also to carry over the stability analysis. However, endogenous uncertainty may lead to different nonanticipativity conditions, i.e. to various scenario trees with different branching schemes attached to scenarios.(Notice that the explicit form of nonanticipativity conditions corresponding to scenario tree d is kept fixed.) Contamination of the original probability distribution P^d leads to contamination bounds

$$l(P^d) \leq \varphi(P^d) \leq u(P^d).$$

Again, it is straightforward to get the lower bound for $\min_d \varphi(P^d)$ which is not the case of the upper bound. The variational analysis of Jonsbraten et al. may give the first hint.

The first stage decisions d can be interpreted as decisions on timing of an operation which helps to get a more precise information about scenarios.

Future work

- More tractable reformulations
- More financial & energy applications
- Stochastic dominance formulations for decision dependent randomness
- Robustness in the sense of worst-case solutions

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Thank you for your attention.

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