Testing in the Growth Curve Model

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The growth curve model

- introduced by Potthoff and Roy in 1964
- generalized multivariate analysis of variance model or the Potthoff and Roy model
- connection between regression analysis and analysis of variance

$$Y = XBZ + \varepsilon, \qquad E(\varepsilon) = 0, \qquad \text{var}(\text{vec } \varepsilon) = \Sigma \otimes I,$$

where

- $Y_{n\times p}$ is matrix of observations
- $X_{n\times m}$ is ANOVA matrix
- $B_{m\times r}$ is matrix of unknown parameters
- $Z_{r\times p}$ is matrix of regress constants
- $\varepsilon_{n\times p}$ is matrix of random errors with normal distribution
- $I_{n\times n}$ denotes identity matrix
- $\Sigma_{p\times p}$ is variance matrix of rows of matrix Y

Example of Potthoff's and Roy's dental data

Potthoff and Roy tried to answer the question: Is the distance between the center of the pituitary and pterygomaxillary fissure the same for boys and girls? Is its growth rate the same for both groups?
Observations: 11 girls, 16 boys at ages $8, 10, 12, 14$

$$Y = \begin{pmatrix} \mathbf{1}_{16} & \mathbf{0}_{16} \\ \mathbf{0}_{11} & \mathbf{1}_{11} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 10 & 12 & 14 \end{pmatrix} + \varepsilon$$

$$\mathbf{1}_{16} = (1, 1, \ldots, 1)'$$

Uniform correlation structure

$$\Sigma_{p\times p} = \sigma^2[(1-\rho)I + \rho\mathbf{1}\mathbf{1}'] = \sigma^2 \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \dots & 1 \end{pmatrix},$$

where $\sigma^2 > 0$ and $\rho \in (-\frac{1}{p-1}, 1)$ are unknown parameters.

Testing in M.S. Srivastava, C.G. Khatri:

- Likelihood ratio test for B=0
- Test that covariance matrix is an Identity matrix

L-R test for $\psi = 0$

Let $X \sim N_{p,n}(B\psi A, \Sigma, I)$ and we wish to test hypothesis

$$H_0 : \psi = 0 \text{ against } H_1 : \psi \neq 0.$$

Then ratio is given by:

$$\lambda = \frac{|S + (I - TS^{-1})S_1(I - S^{-1}T)|}{|S + S_1|},$$

where $S = X(I - A'(AA')^{-1}A)X'$, $T = B(B'S^{-1}B)^{-1}B'$, and $S_1 = \frac{1}{n}XA'(AA')^{-1}AX'$
Reject H_0 if $\lambda \leq c$, where c is determined from $P(\lambda \leq c|H) = \alpha$.

Test that covariance matrix is an Identity matrix

We want to test hypothesis

$$H_0 : \Sigma = I \text{ againts } H_1 : \Sigma \neq I.$$

Ratio is given by:

$$\lambda = \left(\frac{e}{n}\right)^{np/2} |S + (I - TS^{-1})S_1(I - S^{-1}T)|^{n/2} \times$$

$$\times e^{-\frac{1}{2}Tr(S+(I-B(B'B)^{-1}B')S_1)}$$

h**th** moment of λ :

$$E(\lambda^h) = \frac{\left(\frac{2e}{n}\right)^{nph/2} \Gamma_q(\frac{n(1+h)-p+q}{2})\Gamma_{p-q}(\frac{n(1+h)+m}{2})}{(1+h)^{-np(1+h)/2-m(p-q)/2}\Gamma_q(\frac{n-p+q}{2})\Gamma_{p-q}(\frac{n+m}{2})}$$

$\boldsymbol{H_0 : \Sigma_U = \sigma_1 I + \sigma_2 11'}$ against $\boldsymbol{H_1 : \Sigma > 0}$, where $\sigma_1 = \sigma^2(1-\rho)$ and $\sigma_2 = \sigma^2\rho$ are unknown. Ratio is given by:

$$\boldsymbol{\lambda^\star = \frac{1}{y_1(p-1)^{p-1}} \frac{(TrA_1A_1' + TrVV')^{p-1}}{|A_1 M_{A_2'} A_1'||VV'|}},$$

h**th** moment of λ :

$$E(\lambda^{\star h}) = \frac{1}{(p-1)^{h(p-1)}} \frac{\Gamma(\frac{n-t}{2}+h)}{\Gamma(\frac{n-t}{2})} \frac{\Gamma_{p-r}(\frac{n}{2})}{\Gamma_{p-r}(\frac{n}{2}+h)} \frac{\Gamma_r(\frac{n-t-p+r}{2})}{\Gamma_r(\frac{n-t-p+r}{2}+h)} \times$$

$$\times \frac{\Gamma(\frac{np-n-tr+t}{2}+(p-1)h)}{\Gamma(\frac{np-n-tr+t}{2})}$$

Simultaneous testing

$\boldsymbol{H_0 : \Sigma_U = \sigma_1 I + \sigma_2 11', B = 0}$ against $\boldsymbol{H_1 : \Sigma > 0, B \neq 0.}$

$$\lambda^\star = \frac{1}{(p-1)^{p-1}e}\frac{(TrA_1A_1' + TrVV')^{p-1}}{y_1|A_1M_{A_2'}A_1'||VV'|}e^{-\frac{z'z+z_2'z_2}{z'z}}e^{(p-1)\frac{TrA_4A_4'+z'z+z_2'z_2}{TrA_1A_1'+TrVV'}},$$

Let us denote:

$$E(\lambda^{\star h}) = \frac{1}{(p-1)^{h(p-1)}e^h}E\left(\left(\frac{(TrA_1A_1' + TrVV')^{p-1}}{y_1|A_1M_{A_2'}A_1'||VV'|}\right)^h\right)\times$$

$$\times E\left(e^{-\frac{hz_2'z_2}{z'z}}\right)E\left(e^{(p-1)h\frac{TrA_4A_4'}{TrA_1A_1'+TrVV'}}\right)E\left(e^{(p-1)h\frac{z'z+z_2'z_2}{TrA_1A_1'+TrVV'}}\right)=$$

$$= \frac{1}{(p-1)^{h(p-1)}e^h}E(A)E(B)E(C)E(D).$$

Simultaneous testing

$$\frac{z_2' z_2}{z' z} \sim B'\left(\frac{t}{2}, \frac{n-t}{2}\right)$$

$$\frac{(p-1)TrA_4A_4'}{TrA_1A_1' + TrVV'} \sim B'\left(\frac{(r-1)t}{2}, \frac{n(p-1)-t(r-1)}{2}, 1, p-1\right)$$

$$\frac{(p-1)(z_2'z_2 + z'z)}{TrA_1A_1' + TrVV'} \sim B'\left(\frac{n}{2}, \frac{n(p-1-t(r-1))}{2}, 1, \frac{p-1}{\sigma_1}\right)$$

Transformation theorem is used to find out h-th moment of e^x using Beta prime distribution of x.
If $x \sim B'(\alpha, \beta, p, q)$ then $E(e^{xh}) = \int_0^\infty c^{-1} e^{xh} x^{\alpha-1}(1+x)^{-\alpha-\beta} dx$.

Simultaneous testing

$$E(B) = E\left(e^{-\frac{hz_2'z_2}{z'z}}\right) = 1 - \frac{ht}{n-t-2} + \frac{h^2t(t+2)}{2(n-t-2)(n-t-4)}$$

$$E(C) = E\left(e^{\frac{h(p-1)TrA_4A_4'}{TrVV'+TrA_1A_1'}}\right) = 1 + \frac{h(p-1)(r-1)t}{n(p-1)-t(r-1)-2} +$$

$$+\frac{h^2(p-1)^2(r-1)t}{2(n(p-1)-t(r-1)-2)}\frac{(r-1)t-2}{n(p-1)-t(r-1)-4}$$

$$E(D) = E\left(e^{\frac{h(p-1)(z'z+z_2'z_2)}{TrVV'+TrA_1A_1'}}\right) = 1 + \frac{h(p-1)n}{n(p-1)-t(r-1)-2} +$$

$$+\frac{h^2(p-1)^2(n+2)n}{2\sigma_1(n(p-1)-t(r-1)-2)(n(p-1)-t(r-1)-4)}$$

Simultaneous testing

$$E(\lambda^{\star h}) = \frac{1}{(p-1)^{h(p-1)}e^h} \frac{\Gamma(\frac{n-t}{2}+h)}{\Gamma(\frac{n-t}{2})} \frac{\Gamma_{p-r}(\frac{n}{2})}{\Gamma_{p-r}(\frac{n}{2}+h)} \frac{\Gamma_r(\frac{n-t-p+r}{2})}{\Gamma_r(\frac{n-t-p+r}{2}+h)} \frac{\Gamma(\frac{np-n-tr+t}{2}+(p-1)h)}{\Gamma(\frac{np-n-tr+t}{2})} \times$$

$$\times \left(1 - \frac{ht}{n-t-2} + \frac{h^2 t(t+2)}{2(n-t-2)(n-t-4)}\right) \times$$

$$\times \left(1 + \frac{h(p-1)(r-1)t}{n(p-1)-t(r-1)-2} + \frac{h^2(p-1)^2(r-1)t}{2(n(p-1)-t(r-1)-2)} \frac{(r-1)t-2}{n(p-1)-t(r-1)-4}\right) \times$$

$$\times \left(1 + \frac{h(p-1)n}{n(p-1)-t(r-1)-2} + \frac{h^2(p-1)^2(n+2)n}{2\sigma_1(n(p-1)-t(r-1)-2)(n(p-1)-t(r-1)-4)}\right)$$

Thank you for your attention.