On near-optimality conditions for controlled Forward-Backward Stochastic Systems

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Outline

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W is a standard Wiener process on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, with $(\mathcal{F}_t)_{t\in[0,T]}$ being his completed canonical filtration.

Cost function

Further, define for each admissible $u(\cdot)$ the functional

$$J(u(\cdot)) = \mathbf{E}\Big[\int_0^T \ell(t, X_t, u_t)dt + h(X_T)\Big], \tag{2}$$

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Usually, the goal is to find such a strategy $u^*(\cdot) \in \mathcal{U}_{ad}$ so that

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We call $u^*(\cdot)$ optimal control to control problem (1)-(3).

Hamiltonian function

Now, define the Hamiltonian of the problem by

$$H(t,x,u,y,z) = b'(t,x,u)y + Tr\big(\sigma'(t,x,u)z\big) - \ell(t,x,u).$$

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This will help to reduce the infinite dimensional optimization problem to finite dimensional one.

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the solution is a couple $(Y_t, Z_t)_{t \in [0,T]}$ and these are the (stochastic) Lagrange multipliers.

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In other words, u_t^* maximizes the function $H(t, X_t^*, \cdot, Y_t^*, Z_t^*)$ over U.

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Conversely, if the variational inequality

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and send $\rho \to 0_+$. The variational inequality is obtained by expanding the difference on the r.h.s.

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- If $\mathcal{O}\left(\varepsilon\right) = C\varepsilon^{\lambda}$ for some $\lambda > 0$ independent of the constant C then $u^{\varepsilon}(\cdot)$ is called near-optimal control of order λ.

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holds for all $u \in U$.

In other words, $u^\varepsilon(\cdot)$ near-maximizes the function $H\left(t, X_t^\varepsilon, \cdot, Y_t^\varepsilon, Z_t^\varepsilon\right)$ over U in an integral sense with order $C\varepsilon^\lambda$.

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Conversely, if the variational inequality

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holds for any $\varepsilon > 0$ *and some admissible family* $(u^\varepsilon(\cdot))_{\varepsilon>0}$, $H\left(t, \cdot, \cdot, Y_t^\varepsilon, Z_t^\varepsilon\right)$ *is concave and* $h(\cdot)$ *is convex*

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What is on the poster?

On the poster, a result by M.Hafayed, P.V. and S.Abbas is presented. We consider the state equation of the form

$$\begin{cases} dx(t) = f\left(t, x(t), u(t)\right) dt + \sigma\left(t, x(t), u(t)\right) dW(t) \\ \qquad\qquad + \int_\Theta c\left(t, x(t_-), u(t), \theta\right) \widetilde{N}\left(d\theta, dt\right), \\ -dy(t) = \int_\Theta g\left(t, x(t), y(t), z(t), r_t\left(\theta\right), u(t)\right) \mu\left(d\theta\right) dt - z(t) dW(t) \\ \qquad\qquad - \int_\Theta r_t\left(\theta\right) \widetilde{N}\left(d\theta, dt\right); \quad x(0) = \zeta, \; y(T) = \phi\left(x(T)\right), \end{cases}$$

with the functional to be minimized

$$J\left(u(\cdot)\right) = \mathbb{E}\Big[\int_0^T \int_\Theta \ell\left(t, x(t), y(t), z(t), r_t\left(\theta\right), u(t)\right) \mu\left(d\theta\right) dt \\ + h\left(x(T)\right) + \gamma\left(y(0)\right)\Big].$$

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holds for all $u \in U$.

Here, H is the Hamiltonian of the problem, $\Lambda_t^\varepsilon(\theta) = (x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t), r_t^\varepsilon(\theta))$ and $\Psi_t^\varepsilon(\theta) = (p_t^\varepsilon, q_t^\varepsilon, k_t^\varepsilon, R_t^\varepsilon(\theta))$ are the solutions to state and adjoint equations respectively, corresponding to $u^\varepsilon(\cdot)$.

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Conversely, if the variational inequality

$$\mathbb{E}\int_0^T (\nabla_u H)'\left(t, \Lambda_t^\varepsilon(\cdot), u^\varepsilon(t), \Psi_t^\varepsilon(\cdot)\right)(u - u^\varepsilon(t))dt \geq -C\varepsilon^\lambda,$$

holds for any $\varepsilon > 0$ *and some admissible family* $(u^\varepsilon(\cdot))_{\varepsilon>0}$, $H(t, \cdot, \cdot, \Psi_t^\varepsilon(\cdot))$ *is concave and* $h(\cdot), \gamma(\cdot)$ *are convex then* $(u^\varepsilon(\cdot))_{\varepsilon>0}$ *is near-optimal control of order* λ*.*

That's the end, my friend...

Thank you for your attention.

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