Statistical Inference for Stochastic Partial Differential Equations

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Definition of the scalar fractional Brownian motion

Definition

$\mathbb{R}$-valued Gaussian process $(B_t^H \geq 0)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called (scalar) fractional Brownian motion with Hurst parameter $H \in (0,1)$, if this process satisfies

- *(1) $\mathbb{E}B_t^H = 0$ for any $t \in \mathbb{R}_+$,*
- *(2) $R(t,s) = \mathbb{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \quad s,t \in \mathbb{R}_+$,*
- *(3) $(B_t^H, t \geq 0)$ has continuous paths $\mathbb{P} - a.s.$*

- From (1) and (2) follows that $B^H(0) = 0 \quad \mathbb{P} - a.s.$
- (FBm) with $H = \frac{1}{2}$ is the standard Brownian motion.

Riemann–Liouville operators

- Let $(V, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space.
- If $\varphi \in L^1([0,T],V)$, then for $\alpha > 0$ the left-side and the right-side fractional (Riemann–Liouville) integrals of φ are defined by

$$(I_{0+}^{\alpha}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds \tag{1}$$

and

$$(I_{T-}^{\alpha}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1}\varphi(s)\,ds, \tag{2}$$

respectively, where $\Gamma(\cdot)$ is the gamma function.

Fractional derivatives

- For $\alpha \in (0,1)$ the inverse operators of these fractional integrals are called fractional derivatives and can be given by the following representations

$$(D^{\alpha}_{0+}\psi)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,ds\right) \quad (3)$$

and

$$(D^{\alpha}_{T-}\psi)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,ds\right), \quad (4)$$

where $\psi \in I^{\alpha}_{0+}(L^1([0,T],V))$ and $\psi \in I^{\alpha}_{T-}(L^1([0,T],V))$, respectively.

Kernel $K_H(t,s)$

- Let $K_H(t,s)$ for $0 \leq s \leq t \leq T$ and $H \in (0,1/2)$ be the real-valued kernel function

$$K_H(t,s) = \frac{\tilde{c}_H (t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} + \\ + \frac{\tilde{c}_H(\frac{1}{2}-H)}{\Gamma(H+\frac{1}{2})} \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du. \quad (5)$$

- If $H \in (1/2,1)$, then K_H has a simpler form as

$$K_H(t,s) = \frac{\hat{c}_H}{\Gamma(H-\frac{1}{2})} s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, du. \quad (6)$$

- The terms $\tilde{c}_H$ and $\hat{c}_H$ are constants that depend only on H.

Integral operator $\mathbb{K}_H$

- Define the integral operator $\mathbb{K}_H$ induced from the kernel K_H by

$$\mathbb{K}_H h(t) = \int_0^t K_H(t,s)h(s)\,ds \tag{7}$$

for $h \in L^2([0,T],V)$.

- It is known that

$$\mathbb{K}_H : L^2([0,T],V) \to I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T],V)\right) \tag{8}$$

is a bijection.

Integral operator $\mathbb{K}_H$

- For $H \in (0, 1/2]$ one may describe $\mathbb{K}_H$ as

$$\mathbb{K}_H h(s) = \bar{c}_H I_{0+}^{2H}\left(u_{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}} h\right)\right)(s). \qquad (9)$$

- For $H \in [1/2, 1)$ one may describe $\mathbb{K}_H$ as

$$\mathbb{K}_H h(s) = c_H I_{0+}^{1}\left(u_{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}}\left(u_{\frac{1}{2}-H} h\right)\right)(s), \qquad (10)$$

where

$$c_H = \left[\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right]^{\frac{1}{2}},$$

$$\bar{c}_H = c_H \Gamma(2H)$$

and for $s \geq 0$ and $a \in \mathbb{R}$

$$u_a(s) = s^a I.$$

Inverse operator $\mathbb{K}_H^{-1}$

- The inverse operator

$$\mathbb{K}_H^{-1} : I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T],V)\right) \to L^2([0,T],V)$$

is given by

$$\mathbb{K}_H^{-1}\varphi(s) = \bar{c}_H^{-1} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}} D_{0+}^{2H}\varphi\right)(s), \quad H \in (0,1/2]$$

$$\mathbb{K}_H^{-1}\varphi(s) = c_H^{-1} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}}\left(u_{\frac{1}{2}-H} D\varphi\right)(s), \quad H \in [1/2,1).$$

for $\varphi \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T],V)\right)$.

Inverse operator $\mathbb{K}_H^{-1}$

- If $\varphi \in H^1([0, T], V)$, the Sobolev space, then

$$\mathbb{K}_H^{-1}\varphi(s) = \bar{c}_H^{-1} s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H}\left(u_{\frac{1}{2}-H}\varphi'\right)(s) \tag{11}$$

for $H \in (0, 1/2]$.

Operator $\mathcal{K}_H^*$

- Family of linear operators $(\mathcal{K}_H^*, H \in (0,1))$ is provides an isometry between Wiener-type integrals of a fractional Brownian motion and $L^2([0, T], V)$.
- Let $\mathcal{K}_H^* : \mathcal{E} \to L^2([0, T], V)$ be the linear map given by

$$\mathcal{K}_H^* \varphi(t) = \varphi(t) K_H(T, t) + \int_t^T (\varphi(s) - \varphi(t)) \frac{\partial K_H}{\partial s}(s, t)\, ds \tag{12}$$

for $\varphi \in \mathcal{E}$, where $\mathcal{E}$ is the linear space of V-valued step functions on $[0, T]$.

Definition of the stochastic integral

- $\varphi \in \mathcal{E}$ if
$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathbf{1}_{[t_i, t_{i+1})}(t), \tag{13}$$
where $x_i \in V$ for $i \in 1, \ldots, n-1$ and
$0 = t_1 < t_2 < \ldots < t_n = T$.
- Define the stochastic integral as
$$\int_0^T \varphi \, d\beta := \sum_{i=1}^{n-1} x_i \left(\beta(t_{i+1}) - \beta(t_i) \right). \tag{14}$$

Definition of the stochastic integral

- It follows that

$$\mathbb{E}\left\|\int_0^T \varphi\, d\beta\right\|^2 = |\mathcal{K}_H^* \varphi|^2_{L^2([0,T],V)}, \tag{15}$$

  where $|\cdot|_{L^2([0,T],V)}$ is the norm in $L^2([0,T],V)$ induced by the inner product.
- Let $(\mathcal{H}, |\cdot|_{\mathcal{H}}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\mathcal{E}$ with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \langle \mathcal{K}_H^* \varphi, \mathcal{K}_H^* \psi \rangle_{L^2([0,T],V)} \tag{16}$$

  for $\varphi, \psi \in \mathcal{E}$. The stochastic integral is extended to an arbitrary $\varphi \in \mathcal{H}$ by the isometry (15).

Definition of the stochastic integral

- $\mathcal{H}$ is a linear space of integrable functions.
- If $H \in (1/2, 1)$, then $\mathcal{H} \supset L^{\frac{1}{H}}([0, T], V) \supset L^2([0, T], V)$.
- If $H \in (0, 1/2)$, then $\mathcal{H} \supset C^{\beta}([0, T], V)$ for each $\beta > \frac{1}{2} - H$.

Cylindrical fractional Brownian motion

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A cylindrical process $\langle B, \cdot \rangle : \Omega \times \mathbb{R}_+ \times V \to \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if

- *(1) for each $x \in V \backslash \{0\}$, $\frac{1}{\|x\|_V} \langle B(\cdot), x \rangle$ is a standard scalar fractional Brownian motion with the Hurst parameter H*
- *(2) for $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$*

$$\langle B(t), \alpha x + \beta y \rangle = \alpha \langle B(t), x \rangle + \beta \langle B(t), y \rangle \quad \mathbb{P} - a.s. \tag{17}$$

Cylindrical fractional Brownian motion

- For a complete orthonormal basis $(e_n, n \in \mathbb{N})$ of V, letting $\beta_n(t) = \langle B(t), e_n \rangle$ for $n \in \mathbb{N}$, the sequence of scalar processes $(\beta_n, n \in \mathbb{N})$ is independent and B can be represented by the formal series
$$B(t) = \sum_{n=1}^{\infty} \beta_n(t) e_n \tag{18}$$
that does not converge a.s. in V.

Definition of the stochastic integral

- For $x \in V \setminus \{0\}$, let $\beta_x(t) = \langle B(t), x \rangle$. There is a scalar Wiener process $(w_x(t), t \geq 0)$ such that

$$\beta_x(t) = \int_0^t K_H(t,s)\, dw_x(s) \tag{19}$$

  for $t \in \mathbb{R}_+$.
- Dually $w_x(t) = \beta_x\left((\mathcal{K}_H^*)^{-1}\mathbb{I}_{[0,t)}\right)$ if $V = \mathbb{R}$.
- Note that for any $\varphi \in \mathcal{H}$ and $x \in V$, there is the equality

$$\int_0^T \varphi\, d\beta_x = \int_0^T \mathcal{K}_H^* \varphi\, dw_x. \tag{20}$$

Definition of the stochastic integral

Definition

Let $G : [0, T] \to \mathcal{L}(V)$ *be Borel measurable, let* $(e_n, n \in \mathbb{N})$ *be a complete orthonormal basis in* V*, let* $G(\cdot)e_n \in \mathcal{H}$ *for each* $n \in \mathbb{N}$ *and let* B *be a standard cylindrical fractional Brownian motion for some fixed* $H \in (0, 1)$*. The stochastic integral* $\int_0^T G\,dB$ *is defined as*

$$\int_0^T G\,dB := \sum_{n=1}^{\infty} \int_0^T Ge_n\,d\beta_n, \tag{21}$$

provided the infinite series converges in $L^2(\Omega, V)$*.*

Girsanov theorem

Theorem

Let $H \in (0,1)$ and $T > 0$ be fixed and let $(u(t), t \in [0,T])$ be a V-valued, $(\mathcal{F}_t)$-adapted process such that

- *(1)*

$$\int_0^T \|u(t)\|\, dt < \infty \quad \mathbb{P} - a.s. \tag{22}$$

- *(2)*

$$U(\cdot) := \int_0^{\cdot} u(s)\, ds \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T],V)\right) \quad \mathbb{P} - a.s. \tag{23}$$

Girsanov theorem

Theorem

Furthermore, it is assumed that

$$\mathbb{E}\xi_T = 1,$$

where

$$\xi_T = \exp\left[\int_0^T \left\langle \mathbb{K}_H^{-1}(U)(t), dW(t) \right\rangle - \frac{1}{2}\int_0^T \left\|\mathbb{K}_H^{-1}(U)(t)\right\|^2 dt\right], \tag{24}$$

where $(W_t, t \in [0, T])$ *is a standard cylindrical Wiener process in* V *associated with* $(B_t^H, t \in [0, T])$.

Girsanov theorem

Theorem

Then the process $(\tilde{B}_t, t \in [0, T])$ *given by*

$$\tilde{B}_t := B_t^H - U_t$$

is a standard cylindrical fractional Brownian motion in V *with the Hurst parameter* H *on the probability space* $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$, *where*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi_T \quad a.s. \tag{25}$$

Many cases...

- We can be interested in the following cases

$$dX_t = (AX_t + \theta F(X_t))\ dt + \Phi\, dB_t^H \tag{26}$$

$$dX_t = \theta F(X_t)\, dt + \Phi\, dB_t^H \tag{27}$$

$$dX_t = \theta A X_t\, dt + \Phi\, dB_t^H \tag{28}$$

$$dX_t = \theta\,(AX_t + F(X_t))\ dt + \Phi\, dB_t^H \tag{29}$$

$$dX_t = (\theta A X_t + F(X_t))\ dt + \Phi\, dB_t^H \tag{30}$$

$$dX_t = (\theta A X_t + \mu F(X_t))\ dt + \Phi\, dB_t^H, \tag{31}$$

with $X(0) = x_0 \in V$ in all equations.

Statistical inference in general case

- Consider the following stochastic differential equation

$$dX_t = (AX_t + \theta F(X_t))\, dt + \Phi\, dB_t^H, \quad X_0 = x_0, \quad t \geq 0, \quad (32)$$

  where
  - $\Phi \in \mathcal{L}(V)$
  - $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$ and A is the infinitesimal generator of a strongly continuous semigroup $(S(t),\, t \geq 0)$ on V,
  - $F : V \to V$ is nonlinear function.
- Denote $G := \Phi^{-1} F$.

Ornstein–Uhlenbeck process

- Let us begin with this helpful equation

$$\begin{aligned} dZ_t &= AZ_t\,dt + \Phi\,dB_t^H \\ Z_0 &= x_0. \end{aligned} \tag{33}$$

- The solution of this equation is the fractional Ornstein–Uhlenbeck process

$$\begin{aligned} Z_t &= S(t)x_0 + \int_0^t S(t-r)\Phi\,dB_r^H \\ &= S(t)x_0 + \tilde{Z}_t. \end{aligned} \tag{34}$$

Construction of weak solution

- Define the process

$$\tilde{B}_t^H = B_t^H - \int_0^t G(Z_s)\,ds \tag{35}$$

- Using Girsanov theorem, we get that $\tilde{B}_t$ is a fractional Brownian motion on some probability space.
- We can make the following calculation

$$\begin{aligned}
B_t^H &= \tilde{B}_t^H + \int_0^t \Phi^{-1} F(Z_s)\,ds \\
dB_r^H &= d\tilde{B}_r^H + \Phi^{-1} F(Z_r)\,dr \\
S(t-r)\Phi\,dB_r^H &= S(t-r)\Phi d\tilde{B}_r^H + S(t-r)\Phi\Phi^{-1} F(Z_r)\,dr \\
\int_0^t S(t-r)\Phi\,dB_r^H &= \int_0^t S(t-r)\Phi\,d\tilde{B}_r^H + \int_0^t S(t-r) F(Z_r)\,dr.
\end{aligned}$$

Construction of weak solution

- On the other hand

$$\int_0^t S(t-r)\Phi\, dB_r^H = Z_t - S(t)x_0.$$

- Together we have

$$Z_t - S(t)x_0 = \int_0^t S(t-r)\Phi\, d\tilde{B}_r^H + \int_0^t S(t-r)F(Z_r)\, dr$$

$$Z_t = S(t)x_0 + \int_0^t S(t-r)\Phi\, d\tilde{B}_r^H +$$

$$+ \int_0^t S(t-r)F(Z_r)\, dr.$$

- It implies that Z is a weak (also mild) solution of semilinear equation (32) on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$.

MLE estimator

- The obtained MLE estimator of the parameter θ has a form

$$\theta_T = \frac{\int_0^T \langle Q_t, dW_t \rangle}{\int_0^T \|Q_t\|^2 \, dt}. \tag{36}$$

- We also have

$$\theta_T - \theta = \frac{\int_0^T \left\langle Q_t, d\tilde{W}_t \right\rangle}{\int_0^T \|Q_t\|^2 \, dt}, \tag{37}$$

  where $\tilde{W}_t = W_t - \theta \int_0^t Q_s \, ds$.

- To prove consistency of the estimator θ_T we need only to show that

$$\lim_{t \to \infty} \int_0^t \|Q_s\|^2 ds = \infty \quad \mathbb{P} - a.s. \tag{38}$$

$\lim_{t\to\infty} \int_0^t \|Q_s\|^2 ds \overset{?}{=} \infty$

- In the case $H \in (0, 1/2)$ we have

$$\int_0^t \|Q_s\|^2\, ds = \int_0^{t_0} \|Q_s\|^2\, ds + \int_{t_0}^t \|Q_s\|^2\, ds$$
$$\geq \int_{t_0}^t \|Q_s\|^2\, ds.$$

- Now we can compute

$$\|Q_s\| = \left\| \mathbb{K}_H^{-1} \left(\int_0^{\cdot} G(Z_r)\, dr \right)(s) \right\|$$
$$= \left\| c_H s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} \left(r^{\frac{1}{2}-H} G(Z_r) \right)(s) \right\|.$$

$\lim_{t\to\infty} \int_0^t \|Q_s\|^2 ds \stackrel{?}{=} \infty$

$$
\begin{aligned}
\|Q_s\| &= c_H s^{H-\frac{1}{2}} \left\| \int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} G(Z_r)\, dr \right\| \\
&\stackrel{?}{=} c_H s^{H-\frac{1}{2}} \int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} \|G(Z_r)\|\, dr \\
&= c_H s^{H-\frac{1}{2}} \int_0^{s_0} (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} \|G(Z_r)\|\, dr + \\
&+ c_H s^{H-\frac{1}{2}} \int_{s_0}^{s} (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} \|G(Z_r)\|\, dr \\
&\geq c_H s^{H-\frac{1}{2}} (s-s_0)^{-\frac{1}{2}-H} s_0^{\frac{1}{2}-H} \int_{s_0}^{s} \|G(Z_r)\|\, dr \\
&\geq c_1 s^{H-\frac{1}{2}} (s-s_0)^{-\frac{1}{2}-H} \frac{C}{2} s.
\end{aligned}
$$

$\lim_{t\to\infty} \int_0^t \|Q_s\|^2 ds \stackrel{?}{=} \infty$

- We get

$$\|Q_s\| \geq c_2 s^{H+\frac{1}{2}} (s - s_0)^{-\frac{1}{2}-H} = c_2 \left(\frac{s}{s - s_0}\right)^{\frac{1}{2}+H}.$$

- Hence we obtain

$$\int_{t_0}^t \|Q_s\|^2 \, ds \geq \int_{t_0}^t c_1^2 \left(\frac{s}{s - s_0}\right)^{1+2H} ds \underset{t\to\infty}{\to} \infty.$$

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