Parameters Estimates of AR for Change-Point Problem

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SUMMARY

The poster shortly presents score test statistic for detection changes in the parameter values of an autoregressive (AR) time series. For evaluating the statistic we need to estimate the parameters of AR. As we already have the likelihood function (resp. conditional likelihood function), we use the maximum likelihood estimates (MLE). To show the asymptotic properties of score test statistic we need to know the rate of convergence to the true parameters. MLE converge with proper rate, but under a restrictive assumption. Therefore, we use slightly different estimates with the same rate of convergence as MLE without the need of this assumption.

CHANGE-POINT DETECTION

A good model should be able to predict the future values of observed variable with a small error. If there is a change in any parameter and we estimate a model without a chnage, the prediction ability will be lower. Therefore, it is important to keep our model up to date and when there is a change in the model (shifted mean, change in the variance of the white noise or change in the autoregressive parameters), estimate the new values of parameters and use this model to get better prediction than with the model without any change.

AR(p) model

We work with autoregressive model of order p AR(p):

$$Y_i - \mu = \sum_{j=1}^{p} \phi_j (Y_{i-j} - \mu) + \varepsilon_i,$$

where $\{\varepsilon_i\}$ are martingale differences with $\mathrm{E}[\varepsilon_i^2] = \sigma^2$, $\phi_1, \dots, \phi_p$ and μ are real parameters. We denote the vector of parameters $\xi = (\mu, \sigma^2, \phi_1, \dots, \phi_p)'$.

Score Test Statistic

To derive the efficient score statistic we assume that the errors $\{\varepsilon_i\}$ are normally distributed. We denote the logarithmic (conditional) likelihood function of $Y_{-p+1}, \dots, Y_0, Y_1, \dots, Y_k$ as $\ell_k(\xi)$.

We also need the information matrix $I(\xi)$, which in this case is a block-diagonal. This design allows testing the change separately for mean, variance of the noise sequence and autoregressive parameters.

Denote $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}^2, \hat{\phi}_{1n}, \dots, \hat{\phi}_{pn})'$ the maximum likelihood estimators of ξ and consider statistic

$$\hat{\mathbf{B}}(u) = n^{-1/2} I^{-1/2}(\hat{\xi}_n) \begin{pmatrix} \frac{\partial}{\partial \mu} \ell_{[nu]}(\hat{\xi}_n) \\ \frac{\partial}{\partial \sigma^2} \ell_{[nu]}(\hat{\xi}_n) \\ \nabla_\phi \ell_{[nu]}(\hat{\xi}_n) \end{pmatrix}, \quad u \in [0,1], \tag{1}$$

where $[x]$ is the integer part of x and $\nabla_\phi \ell$ is a vector of partial derivations of the function ℓ according to the elements of $(\phi_1, \dots, \phi_p)'$. For independent identically distributed errors is in [2] proved, that (under some assumptions)

$$\max_{1 \le j \le p+2} \sup_{0 \le u \le 1} |\hat{B}^{(j)}(u) - B^{(j)}(u)| = o_p(1),$$

where $B^{(j)}$ are independent Brownian bridges. To prove this we need to know the rate of convergence of the estimates to the true parameters.

MLE AND OTHER ESTIMATORS

Theorem 1: Let us assume that $\{Y_i\}$ satisfy AR(p). Let $\{\varepsilon_t\}$ be an i.i.d. white noise sequence with the variance σ^2 and $\mathrm{E}\,|\varepsilon_i|^\kappa < \infty$ for some $\kappa > 4$. Assume that the characteristic polynomial $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$ satisfies $\phi(z) \neq 0$ for all $|z| \le 1$. Then the following results hold:

(i) Under the hypothesis $\phi = \phi_0$: $|\hat{\mu}_k - \mu| = O\left(\sqrt{k^{-1} \log\log k}\right)$ a.s.

(ii) Under the hypothesis $\mu = \mu_0$: $\|\hat{\phi}_k - \phi\| = O\left(\sqrt{k^{-1} \log\log k}\right)$ a.s.

(iii) Under the hypothesis $\mu = \mu_0$ and $\phi = \phi_0$: $|\hat{\sigma}_k^2 - \sigma^2| = O\left(\sqrt{k^{-1} \log\log k}\right)$ a.s.

This theorem is proved in [3]. We would like to change the assumption of i.i.d. white noise to martingale differences. Moreover, assuming that we know all the other parameters is quite restrictive (we cannot use (i)-(iii) all at once) and it is not possible to drop this assumption for MLE. Therefore, we use this estimates

$$\tilde{\mu}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i; \quad \tilde{\phi}_k = (\mathbf{X}_k' \mathbf{X}_k)^{-1} \mathbf{X}_k' \mathbf{Z}_k;$$

$$\tilde{\sigma}_k^2 = \frac{1}{k} \sum_{i=1}^{k} \left(Y_i - \tilde{\mu}_k - \sum_{j=1}^{p} \tilde{\phi}_{kj} (Y_{i-j} - \tilde{\mu}_k) \right)^2;$$

where $\mathbf{X}_k = \begin{pmatrix} Y_0 - \tilde{\mu}_k & \dots & Y_{-p+1} - \tilde{\mu}_k \\ \vdots & \vdots & \vdots \\ Y_{k-1} - \tilde{\mu}_k & \dots & Y_{k-p} - \tilde{\mu}_k \end{pmatrix}$ and $\mathbf{Z}_k = \begin{pmatrix} Y_1 - \tilde{\mu}_k \\ \vdots \\ Y_k - \tilde{\mu}_k \end{pmatrix}$.

Theorem 2: Let us assume that $\{Y_i\}$ satisfy AR(p). Let $\{\varepsilon_t\}$ be a stationary and ergodic martingale difference sequence with $\mathrm{E}[\varepsilon_i^2] = \sigma^2$, $\mathrm{E}[\varepsilon_i^2 | \mathcal{F}_{i-1}] = \sigma^2$ and $\mathrm{E}\,|\varepsilon_i|^\kappa < \infty$ for some $\kappa > 4$. Assume that the characteristic polynomial $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$ satisfies $\phi(z) \neq 0$ for all $|z| \le 1$. Then the following results hold:

$$\begin{aligned} &\text{(i)} & |\tilde{\mu}_k - \mu| &= O\left(\sqrt{k^{-1} \log\log k}\right) \text{ a.s.;} \\ &\text{(ii)} & \|\tilde{\phi}_k - \phi\| &= O\left(\sqrt{k^{-1} \log\log k}\right) \text{ a.s.;} \\ &\text{(iii)} & |\tilde{\sigma}_k^2 - \sigma^2| &= O\left(\sqrt{k^{-1} \log\log k}\right) \text{ a.s..} \end{aligned}$$

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References.

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