Parameters Estimates of AR for Change-Point Problem

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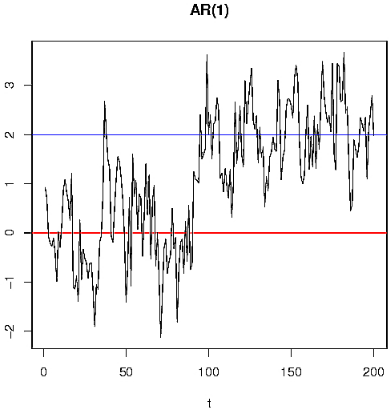
Obsah

$$\text{AR(p)}: \; Y_i = \alpha_0 + \sum_{j=1}^{p} \alpha_j Y_{i-j} + \varepsilon_i$$

- Gaussian likelihood ratio test statistic (see Davis et. al.)
- Score test statistic (see Gombay)



AR(p) with change at time τ:

$$Y_i - \mu = \sum_{j=1}^{p} \phi_j (Y_{i-j} - \mu) + \varepsilon_i \qquad i \leq \tau$$

$$Y_i - \mu^* = \sum_{j=1}^{p} \phi_j^* (Y_{i-j} - \mu^*) + \varepsilon_i \qquad i > \tau$$

- $\{\varepsilon_i, i \leq \tau\}$ martingale difference sequence with $\mathbb{E}[\varepsilon_i^2] = \sigma^2$
- $\xi = (\mu, \sigma^2, \phi_1, \ldots, \phi_p)$ - before change
- $\xi^* = (\mu^*, \sigma^{*2}, \phi_1^*, \ldots, \phi_p^*)$ - after change

- Efficient score test statistic:

$$\hat{\mathbf{B}}(u) = n^{-1/2} I^{-1/2}(\hat{\xi}_n) \begin{pmatrix} \frac{\partial}{\partial \mu} \ell_{[nu]}(\hat{\xi}_n) \\ \frac{\partial}{\partial \sigma^2} \ell_{[nu]}(\hat{\xi}_n) \\ \nabla_\phi \ell_{[nu]}(\hat{\xi}_n) \end{pmatrix}$$

- I is an information matrix (in this case block-diagonal)
- The main advantage of this test statistic is, that we may test the change in every parameter separately.
- For testing change in any of the parameters of AR(p) we use $\max_{j=1,\dots,p+2} \hat{B}^{(j)}(u)$, where $\hat{B}^{(j)}(u)$ are the elements of $\hat{\mathbf{B}}(u)$.
- Which estimates should we use as $\hat{\xi}_n$?

Theorem

Let us assume that there is no change in any of the parameters. Let $\{\varepsilon_t\}$ be an i.i.d. sequence with $\mathbb{E}[\varepsilon_i^2] = \sigma^2$ and $\mathbb{E}|\varepsilon_i|^\kappa < \infty$ for some $\kappa > 4$. Assume that characteristic polynomial $\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j$ satisfies $\phi(z) \neq 0$ for all $|z| \leq 1$. $\hat{\xi}_k = (\hat{\mu}_k, \hat{\sigma}_k^2, \hat{\phi}_{1k}, \ldots, \hat{\phi}_{pk})'$

(i) *Under the hypothesis $\sigma^2 = \sigma_0^2$ and $\phi = \phi_0$*

$$|\hat{\mu}_k - \mu| = O\left(\sqrt{\frac{\log\log k}{k}}\right) \quad a.s.$$

(ii) *Under the hypothesis $\mu = \mu_0$ and $\sigma^2 = \sigma_0^2$*

$$\|\hat{\phi}_k - \phi\| = O\left(\sqrt{\frac{\log\log k}{k}}\right) \quad a.s.$$

(iii) *Under the hypothesis $\mu = \mu_0$ and $\phi = \phi_0$*

$$|\hat{\mu}_k - \mu| = O\left(\sqrt{\frac{\log\log k}{k}}\right) \quad a.s.$$

- The parameters of mean

$$\tilde{\mu}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i$$

- The autoregressive parameters

$$\tilde{\phi}_k = (\mathbf{X}_k' \mathbf{X}_k)^{-1} \mathbf{X}_k' \mathbf{Z}_k$$

$$\mathbf{X}_k = \begin{pmatrix} Y_0 - \tilde{\mu}_k & \dots & Y_{-p+1} - \tilde{\mu}_k \\ \vdots & \vdots & \vdots \\ Y_{k-1} - \tilde{\mu}_k & \dots & Y_{k-p} - \tilde{\mu}_k \end{pmatrix}, \mathbf{Z}_k = \begin{pmatrix} Y_1 - \tilde{\mu}_k \\ \vdots \\ Y_k - \hat{\mu}_k \end{pmatrix}$$

- The variance of noise sequence

$$\tilde{\sigma}_k^2 = \frac{1}{k} \sum_{i=1}^{k} \left(Y_i - \tilde{\mu}_k - \sum_{j=1}^{p} \tilde{\phi}_{kj} (Y_{i-j} - \tilde{\mu}_k) \right)^2$$

Theorem

Let us assume that $\{Y_i\}$ satisfy AR(p). Let $\{\varepsilon_t\}$ be a stationary and ergodic martingale difference sequence with $\mathbb{E}[\varepsilon_i^2] = \sigma^2$, $\mathbb{E}[\varepsilon_i^2 | \mathcal{F}_{i-1}] = \sigma^2$ and $\mathbb{E}|\varepsilon_i|^\kappa < \infty$ for some $\kappa > 4$. Assume that the characteristic polynomial $\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j$ satisfies $\phi(z) \neq 0$ for all $|z| \leq 1$. Then the following results hold:

(i) $|\tilde{\mu}_k - \mu| = O\left(\sqrt{k^{-1} \log \log k}\right)$ *a.s.;*

(ii) $\|\tilde{\phi}_k - \phi\| = O\left(\sqrt{k^{-1} \log \log k}\right)$ *a.s.;*

(iii) $|\tilde{\sigma}_k^2 - \sigma^2| = O\left(\sqrt{k^{-1} \log \log k}\right)$ *a.s..*

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