Testing for a change in covariance operator

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We observe two independent sequences of i.i.d. zero mean Gaussian processes $X_1(t), \dots, X_{N_1}(t)$ and $Y_1(t), \dots, Y_{N_2}(t)$ defined for $t \in [0,1]$ such that $E\,\|X_1(t)\|^2 = E\int_0^1 X_1^2(t)\,dt < \infty$ and $E\,\|Y_1(t)\|^2 = E\int_0^1 Y_1^2(t)\,dt < \infty$.

($L^2[0,1]$ is a Hilbert space of square integrable functions on $[0,1]$ with a scalar product $< f, g >= \int_0^1 f(t)\,g(t)\,dt$ and $\|\ \|$ the corresponding L_2 norm.)

The covariance functions $A(t,s) = E\,X_1(t)X_1(s)$ and $B(t,s) = E\ Y_1(t)Y_1(s)$ are continuous functions on $[0,1]^2$. $\mathcal{A}$ is the corresponding covariance operator of X_1 defined by the kernel $A(t,s)$ and $\mathcal{B}$ is the covariance operator of Y_1 defined by the kernel $B(t,s)$:

$$(\mathcal{A}v)(t) = \int_0^1 A(t,s)\,v(s)\,ds, \quad (\mathcal{B}v)(t) = \int_0^1 B(t,s)\,v(s)\,ds.$$

According to the Mercer lemma there exist expansions:

$$A(t,s) = \sum_{i=1}^{\infty} \lambda_i \, u_i(t) \, u_i(s), \quad B(t,s) = \sum_{i=1}^{\infty} \mu_i \, v_i(t) \, v_i(s),$$

where $\{\lambda_i\}$, $\{u_i(t),\ t \in [0,1]\}$ are eigenelements of $\mathcal{A}$ and $\{\mu_i\}$, $\{v_i(t),\ t \in [0,1]\}$ are eigenelements of $\mathcal{B}$. Clearly

$$\lambda_k = \int_0^1 \int_0^1 u_k(t) A(t,s) u_k(s) \, dt \, ds =< u_k, \mathcal{A} u_k >,$$

$$\mu_k = \int_0^1 \int_0^1 v_k(t) B(t,s) v_k(s) \, dt \, ds =< v_k, \mathcal{B} v_k > .$$

$\lambda_1 > \lambda_2 > \cdots > \lambda_K > \ldots,$

$\mu_1 > \mu_2 > \cdots > \mu_K > \ldots.$

$X_1, \ldots, X_{N_1}$ and $Y_1, \ldots, Y_{N_2}$ are $p-$dimensional zero mean Gaussian vectors. The scalar product $< X, Y > = X^T Y$. The covariance operators $\mathcal{A}$ and $\mathcal{B}$ are given by the matrices $A = E\,XX^T$, resp. by $B = E\,YY^T$ that can be expanded as

$$A = \sum_{i=1}^{p} \lambda_i u_i\, u_i^T, \quad B = \sum_{i=1}^{p} \mu_i v_i\, v_i^T.$$

We estimate the covariance function $A(t,s)$ and $B(t,s)$ by

$$\widehat{A}(t,s) = \frac{1}{N_1}\sum_{i=1}^{N_1} X_i(t)X_i(s), \quad \text{resp.} \quad \widehat{B}(t,s) = \frac{1}{N_2}\sum_{i=1}^{N_2} Y_i(t)Y_i(s).$$

The corresponding operators are $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$:

$$(\widehat{\mathcal{A}}v)(t) = \int_0^1 \widehat{A}(t,s)\,v(s)\,ds, \quad (\widehat{\mathcal{B}}v)(t) = \int_0^1 \widehat{B}(t,s)\,v(s)\,ds.$$

We denote $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots$ eigenvalues and $\widehat{u}_1, \widehat{u}_2, \dots$ eigenfunctions of $\widehat{\mathcal{A}}$ while $\widehat{\mu}_1 \geq \widehat{\mu}_2 \geq \dots$ eigenvalues and $\widehat{v}_1, \widehat{v}_2, \dots$ eigenfunctions of $\widehat{\mathcal{B}}$. $(\int \widehat{u}_k u_k \geq 0, \int \widehat{v}_k v_k \geq 0, k = 1, \dots)$
We introduce an operator $\widehat{\mathcal{C}}$ with the kernel

$$\widehat{C}(t,s) = \frac{1}{N_1 + N_2}\Big(\sum_{i=1}^{N_1} X_i(t)X_i(s) + \sum_{i=1}^{N_2} Y_i(t)Y_i(s)\Big).$$

Two-sample problem

$$H_0 : \mathcal{A} = \mathcal{B} \qquad A : \mathcal{A} \neq \mathcal{B}$$

We define for $i = 1, \ldots, N_1$ and $k = 1, 2, \ldots$:

$$\beta^u_{X_i}(k) = < u_k, X_i > .$$

They are uncorrelated and because of normality also independent.

We define for $i = 1, \ldots, N_1$ and $k = 1, 2, \ldots,\; k' = k, k+1, \ldots$:

$$\eta^u_{X_i}(k, k') = < u_k, X_i >< u_{k'}, X_i > .$$

It holds

$$\begin{aligned} E\,\eta^u_{X_i}(k,k) = \lambda_k \quad & var\,\eta^u_{X_i}(k,k) = 2\lambda_k^2 \\ E\,\eta^u_{X_i}(k,k') = 0 \quad & var\,\eta^u_{X_i}(k,k') = \lambda_k \lambda_{k'} \end{aligned}$$

The variables $\{\eta_{X_i}(k, k'), k = 1, \ldots,\; k' = k, k+1, \ldots,\; i = 1, \ldots\}$ are uncorrelated.

For $k = 1, 2, \ldots$ it holds

$$|\widehat{\lambda}_k - \lambda_k| \leq |||\widehat{\mathcal{A}} - \mathcal{A}|||,$$
$$||\widehat{u}_k - u_k|| \leq const(k)|||\widehat{\mathcal{A}} - \mathcal{A}|||.$$

For $N_1 \to \infty$:

$$|||\widehat{\mathcal{A}} - \mathcal{A}||| = O_P(1/\sqrt{N_1}),$$
$$|\widehat{\lambda}_k - \lambda_k| = O_P(1/\sqrt{N_1}),$$
$$||\widehat{u}_k - u_K|| = O_P(1/\sqrt{N_1}).$$

Moreover

$$\sqrt{N_1}\,\frac{\widehat{\lambda}_k - \lambda_k}{\sqrt{2}\lambda_k} \sim N(0, 1).$$

$$|||\mathcal{K}||| = \left(\int_0^1 \int_0^1 K(t, s)^2\, dt\, ds\right)^{1/2}$$

Panaratos et al. (2010):

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}\left(\widehat{\mathcal{J}}_N(k,k')\right)^2=\sum_{k=1}^{K}\left(\widehat{\mathcal{J}}_N(k,k)\right)^2+2\sum_{k=1}^{K}\sum_{k'=1}^{k-1}\left(\widehat{\mathcal{J}}_N(k,k')\right)^2,$$

where

$$\widehat{\mathcal{J}}_N(k,k')=\sqrt{\frac{N_1N_2}{2N}}\frac{<\widehat{\phi}_k,(\widehat{\mathcal{A}}-\widehat{\mathcal{B}})\widehat{\phi}_{k'}>}{\sqrt{<\widehat{\phi}_k,\widehat{\mathcal{C}}\widehat{\phi}_k><\widehat{\phi}_{k'},\widehat{\mathcal{C}}\widehat{\phi}_{k'}>}},$$

where the functions $\{\widehat{\phi}_k(t),\ k=1,\ldots,K\}$ are eigenfunctions (corresponding to K largest eigenvalues) of the operator $\widehat{\mathcal{C}}$.

Let $\mathcal{A} = \mathcal{B}$. Replace $\widehat{\phi}_k$ by u_k.

$$< u_k, \widehat{\mathcal{A}} u_{k'} > =$$

$$= \int_0^1 \int_0^1 u_k(t) \Big(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i(t) X_i(s) \Big) u_{k'}(s)\, ds\, dt =$$

$$= \frac{1}{N_1} \sum_{i=1}^{N_1} \int_0^1 u_k(t) X_i(t)\, dt \int_0^1 u_{k'}(s) X_i(s)\, ds =$$

$$= \frac{1}{N_1} \sum_{i=1}^{N_1} \beta_{X_i}(k) \beta_{X_i}(k') = \frac{1}{N_1} \sum_{i=1}^{N_1} \eta_{X_i}(k, k')$$

Under H_0 supposing that $N_1/N \to \alpha \in (0,1)$ the variables $(k \neq k')$

$$J_N(k,k) = \sqrt{\frac{N_1 N_2}{N}} \frac{< u_k, (\widehat{\mathcal{A}} - \widehat{\mathcal{B}}) u_k >}{\sqrt{2\,\lambda_k^2}} = \frac{\overline{\eta^u_{X_i}(k,k)} - \overline{\eta^u_{Y_i}(k,k)}}{\sqrt{2}\lambda_k \sqrt{1/N_1 + 1/N_2}}$$

and

$$J_N(k,k') = \sqrt{\frac{N_1 N_2}{N}} \frac{< u_k, (\widehat{\mathcal{A}} - \widehat{\mathcal{B}}) u_{k'} >}{\sqrt{\lambda_k\,\lambda_{k'}}} = \frac{\overline{\eta^u_{X_i}(k,k')} - \overline{\eta^u_{Y_i}(k,k')}}{\sqrt{\lambda_k \lambda_{k'}} \sqrt{1/N_1 + 1/N_2}}$$

are asymptotically $N(0,1)$ distributed. It follows that the suggested test statistic has asymptotically χ^2 distribution with $K(K+1)/2$ degrees of freedom.

What is the alternative?

What is K?

Two-sample problem

$$H_0 : \mathcal{A} = \mathcal{B} \qquad A : \mathcal{A}_K \neq \mathcal{B}_K,$$

where $\mathcal{A}_K$ corresponds to $A_K(t,s)$ and $\mathcal{B}_K$ corresponds to $B_K(t,s)$:

$$A_K(t,s) = \sum_{k=1}^{K} \lambda_k \, u_k(t) \, u_k(s), \quad B_K(t,s) = \sum_{k=1}^{K} \mu_k \, v_k(t) \, v_k(s).$$

$$\sum_{k=1}^{K} \frac{\left(\widehat{T}_u(k)\right)^2 + \left(\widehat{T}_v(k)\right)^2}{2},$$

where

$$\widehat{T}_u(k) = \sqrt{\frac{N_1 N_2}{2N}} \frac{< \widehat{u}_k, (\widehat{\mathcal{A}} - \widehat{\mathcal{B}})\widehat{u}_k >}{< \widehat{u}_k, \widehat{\mathcal{C}}\, \widehat{u}_k >} = \sqrt{\frac{N_1 N_2}{2N}} \frac{\widehat{\lambda}_k - \widetilde{\lambda}_k}{< \widehat{u}_k, \widehat{\mathcal{C}}\, \widehat{u}_k >}$$

and

$$\widehat{T}_v(k) = \sqrt{\frac{N_1 N_2}{2N}} \frac{< \widehat{v}_k, (\widehat{\mathcal{A}} - \widehat{\mathcal{B}})\widehat{v}_k >}{< \widehat{v}_k, \widehat{\mathcal{C}}\, \widehat{v}_k >} = \sqrt{\frac{N_1 N_2}{2N}} \frac{\widetilde{\mu}_k - \widehat{\mu}_k}{< \widehat{v}_k, \widehat{\mathcal{C}}\, \widehat{v}_k >}.$$

Let $\mathcal{A}_K \neq \mathcal{B}_K$ then there exists $k \leq K$ such that $< u_k, (\mathcal{A} - \mathcal{B}) u_k > \neq 0$ or $< v_k, (\mathcal{A} - \mathcal{B}) v_k > \neq 0$.

Assume that $N_1/N \to \alpha > 0$ as $N \to \infty$. Then under A the test based on my test statistic is consistent. More specifically, it holds

$$\left(\widehat{T}_u(k) - \sqrt{N}\sqrt{\alpha(1-\alpha)} \frac{< u_k, (\mathcal{A} - \mathcal{B}) u_k >}{\sqrt{2}(\alpha\lambda_k + (1-\alpha)\kappa_k)} \right) = O_P(1).$$

$$\left(\widehat{T}_v(k) - \sqrt{N}\sqrt{\alpha(1-\alpha)} \frac{< v_k, (\mathcal{A} - \mathcal{B}) v_k >}{\sqrt{2}(\alpha\nu_k + (1-\alpha)\mu_k)} \right) = O_P(1).$$

For N_1, N_2 finite the variables $\widehat{T}_u(k)$ and $\widehat{T}_v(k)$ are not unbiased. How large are the biases?

Hall and Hosseini-Nasab (2009)

$$\widehat{\lambda}_k - \lambda_k = \frac{1}{\sqrt{n}} < u_k, \mathcal{Z}_A u_k > + \frac{1}{n} \sum_{j:j \neq k} \frac{1}{(\lambda_k - \lambda_j)} < u_k, \mathcal{Z}_A u_j >^2 +$$

$$+ \frac{1}{n^{3/2}} U_{n,k},$$

where $\sup_{1 \leq k \leq K} E \left| U_{n,k} \right|^r = O(1)$ as $n \to \infty$ for K an arbitrary fixed number and for any $r = 1, 2, \ldots$.

$$\widehat{u}_k - u_k = \frac{1}{\sqrt{n}} \sum_{j:j \neq k} \frac{1}{(\lambda_k - \lambda_j)} u_j < u_k, \mathcal{Z}_A u_j >$$

$$- \frac{1}{2n} u_k \sum_{j:j \neq k} \frac{1}{(\lambda_k - \lambda_j)^2} < u_k, \mathcal{Z}_A u_j >^2$$

$$+ \frac{1}{n} \sum_{j:j \neq k} u_j \Bigg(\frac{1}{(\lambda_k - \lambda_j)} \sum_{j':j' \neq k} \frac{1}{(\lambda_k - \lambda_{j'})} < u_k, \mathcal{Z}_A u_{j'} >< u_j, \mathcal{Z}_A u_{j'} >$$

$$- \frac{1}{(\lambda_k - \lambda_j)^2} < u_k, \mathcal{Z}_A u_k >< u_k, \mathcal{Z}_A u_j > \Bigg) + \frac{1}{n^{3/2}} V_{n,k},$$

where $sup_{1 \leq k \leq K} E \, ||V_{n,k}||^r = O(1)$ as $n \to \infty$ for K an arbitrary fixed number and for any $r = 1, 2, \ldots$.

Let $N_1/N \to \alpha$ as $N \to \infty$ then

$$N_1\, E\left(\widehat{\lambda_k} - \lambda_k\right) = \sum_{j:j\neq k} \frac{\lambda_k \lambda_j}{(\lambda_k - \lambda_j)} + o(1),$$

$$N_1\, E\left(\widehat{\lambda_k} - \lambda_k\right)^2 = 2\lambda_k^2 + o(1).$$

$$N_1\, E\left(\widetilde{\lambda_k} - \lambda_k\right) = -\sum_{j:j\neq k} \frac{\lambda_k \lambda_j}{(\lambda_k - \lambda_j)} + o(1),$$

$$N_2\, E\left(\widetilde{\lambda_k} - \lambda_k\right)^2 = 2\lambda_k^2 + o(1),$$

$$N\, E\left(\widetilde{\lambda_k} - \lambda_k\right) = o(1),$$

$$N\, E\left(\widehat{\lambda_k} - \lambda_k\right)\left(\widetilde{\lambda_k} - \lambda_k\right) = o(1),$$

where $\widehat{\lambda_k} =< \widehat{u}_k, \widehat{\mathcal{A}}\widehat{u}_k >$ and $\widetilde{\lambda_k} =< \widehat{u}_k, \widehat{\mathcal{B}}\widehat{u}_k >$.

Let $N_1/N \to \alpha$ as $N \to \infty$ then

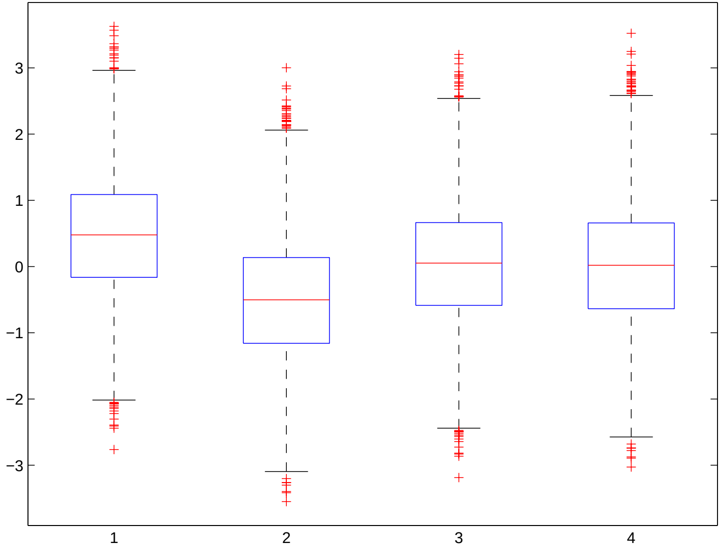
$$E\,\widehat{T}_u(k) = \frac{1}{\sqrt{N}}\sqrt{\frac{(1-\alpha)}{2\,\alpha}}\,2\sum_{j:j\neq k}\frac{\lambda_j}{\lambda_k-\lambda_j} + o(1/\sqrt{N}).$$

$$A = \begin{pmatrix} 10 & -1 & 0 \\ -1 & 9.5 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

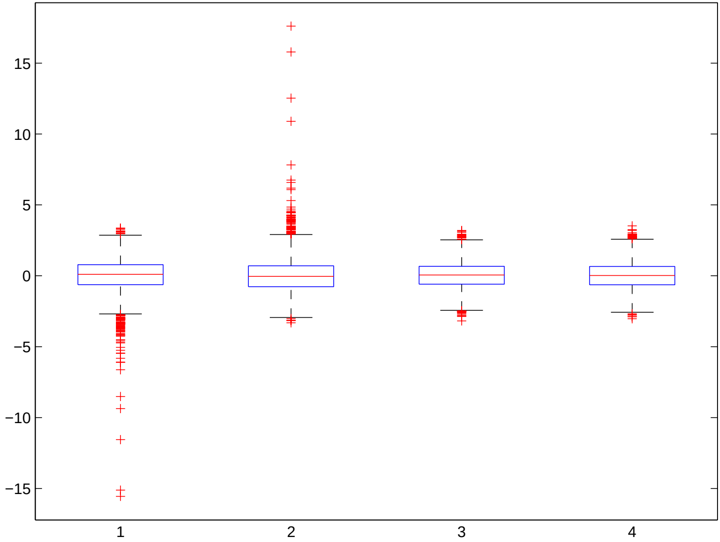
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 10.7808 \\ 8.7192 \\ 0.1 \end{pmatrix}$$

Approximate asymptotic biases for $\widehat{T}_u(1)$ and $\widehat{T}_u(2)$ ($N_1 = N_2 = 100, K = 2$: 0.424,-0.5218.)

Boxplots for Thatu and Thatu corrected for right as. bias



Boxplots for Thatu corrected for estimated as. bias and Thatu corrected for as. bias



Apply permutation principle

$$A = \begin{pmatrix} 9.9665 & -0.4076 & 0 & 0 & 0 & \dots & 0 \\ -0.4076 & 5.0335 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0.01 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & 0.01 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.01 \end{pmatrix}$$

$$B = \begin{pmatrix} 8.9884 & -1.8700 & 0.7334 & 0 & 0 & \dots & 0 \\ -1.8700 & 5.3508 & -1.6956 & 0 & 0 & \dots & 0 \\ 0.7334 & -1.6956 & 1.6608 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0.01 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & 0.01 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.01 \end{pmatrix}$$

size of A and B is 13 x 13.

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \dots \\ \lambda_{13} \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 1 \\ 0.01 \\ \dots \\ 0.01 \end{pmatrix}$$

$N_1 = N_2 = 100$ (2500 simulations)

	asympt $K = 3$	permut $K = 3$	Panaretos $K = 2$	Panaretos $K = 3$
H_0 :	0.336	0.053	0.047	0.052
A :	0.613	0.644	0.330	0.983

	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$	$K = 6$
power	0.072	0.084	0.644	0.517	0.461	0.432
5 %*cr.v.*	3.56	5.63	7.54	13.56	16.89	18.53
$\chi^2(0.95)$	3.84	5.99	7.81	9.49	11.07	12.59

Behaviour of the Panaretos et al. statistic under A when $N_1 = N_2$:

$$\sqrt{N}\frac{< \widehat{\phi}_j, (\widehat{\mathcal{A}} - \widehat{\mathcal{B}})\widehat{\phi}_k >}{\sqrt{< \widehat{\phi}_j, \widehat{\mathcal{C}}\widehat{\phi}_j >< \widehat{\phi}_k, \widehat{\mathcal{C}}\widehat{\phi}_k >}},$$

$$\widehat{\phi}_j = \phi_j + \frac{1}{\sqrt{n}} \sum_{l \neq j} \frac{1}{(\phi_j - \phi_l)} \phi_l < \phi_l, \mathcal{Z}_C \phi_j > + O_P(1/n),$$

$$\widehat{\phi}_k = \phi_k + \frac{1}{\sqrt{n}} \sum_{l' \neq k} \frac{1}{(\phi_k - \phi_{l'})} \phi_{l'} < \phi_{l'}, \mathcal{Z}_C \phi_k > + O_P(1/n),$$

$$\widehat{\mathcal{A}} - \widehat{\mathcal{B}} = (\mathcal{A} - \mathcal{B}) + \frac{1}{n} \sum (\widehat{\mathcal{A}}_i - \mathcal{A}) - \frac{1}{n} \sum (\widehat{\mathcal{B}}_i - \mathcal{B})$$

$$\sqrt{N}\Big\langle \big(\phi_j + \frac{1}{\sqrt{N}}\widetilde{\phi}_j + O_P(1/N)\big),\ \Big((\mathcal{A}-\mathcal{B})+$$

$$\frac{1}{N}\sum(\widehat{\mathcal{A}}_i - \mathcal{A}) - \frac{1}{N}\sum(\widehat{\mathcal{B}}_i - \mathcal{B})\Big)\left(\phi_k + \frac{1}{\sqrt{N}}\widetilde{\phi}_k + O_P(1/N)\right)\Big\rangle =$$

$$\sqrt{N}\Big\langle \phi_j, (\mathcal{A}-\mathcal{B})\phi_k \Big\rangle + \Big\langle \widetilde{\phi}_j, (\mathcal{A}-\mathcal{B})\phi_k \Big\rangle + \Big\langle \phi_j, (\mathcal{A}-\mathcal{B})\widetilde{\phi}_k \Big\rangle +$$

$$\Big\langle \phi_j, \mathcal{Z}_A\phi_k \Big\rangle - \Big\langle \phi_j, \mathcal{Z}_B\phi_k \Big\rangle + O_P(1/N)$$

$$\langle \widetilde{\phi}_j, (\mathcal{A}-\mathcal{B})\phi_k \rangle = \sum_{l\neq j} \frac{<\phi_l, (\mathcal{A}-\mathcal{B})\phi_k>}{(\lambda_j - \lambda_l)} \langle \phi_j, \mathcal{Z}_c\phi_l \rangle$$

$$\langle \phi_j, (\mathcal{A}-\mathcal{B})\widetilde{\phi}_k \rangle = \sum_{l'\neq j} \frac{<\phi_{l'}, (\mathcal{A}-\mathcal{B})\phi_j>}{(\lambda_k - \lambda_{l'})} \langle \phi_k, \mathcal{Z}_c\phi_{l'} \rangle$$

$$A = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix} \quad B = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

For $K = 2$

	Panaretos	Jaruskova
$N_1 = N_2$	power	power
100	0.653	0.99
1000	0.307	1.00
10000	0.082	1.00