How to choose threshold in a POT model?

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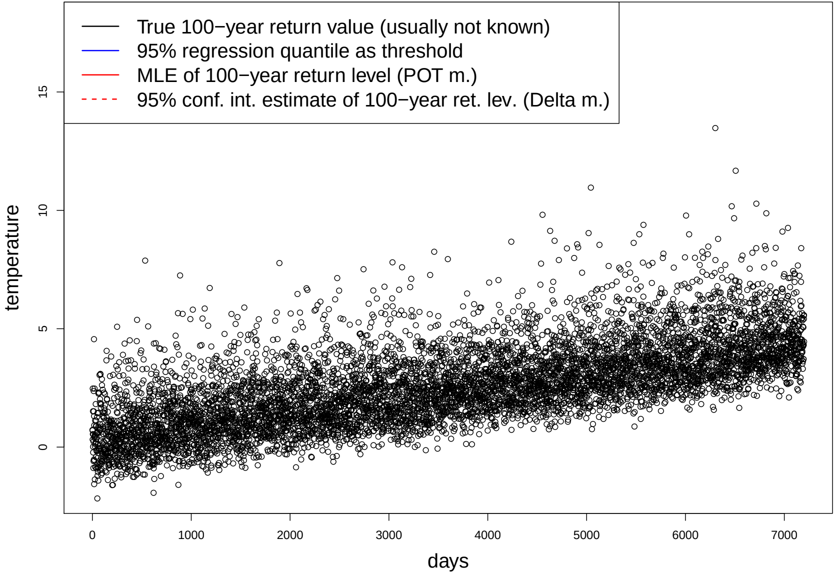
1 Introduction

Outline

Max. daily temperatures with linear trend. Sample size $7200 = 80$ years $\times$ 90



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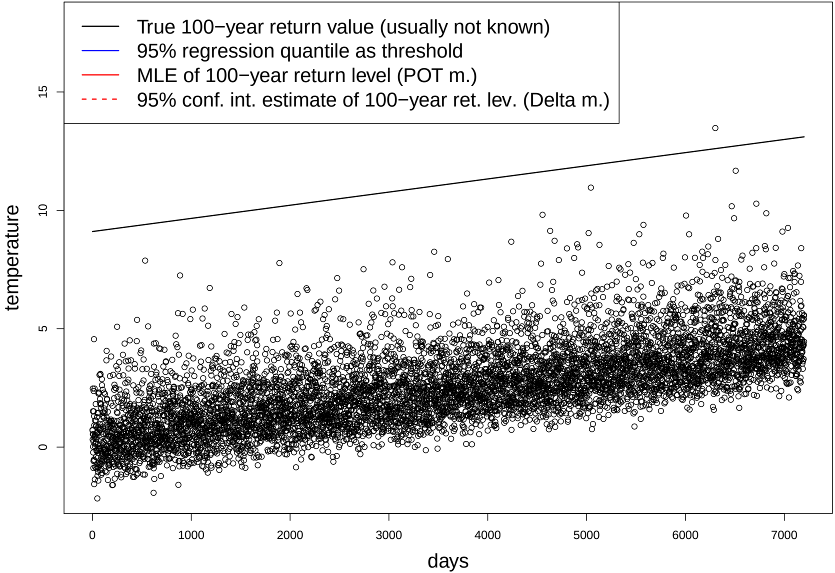

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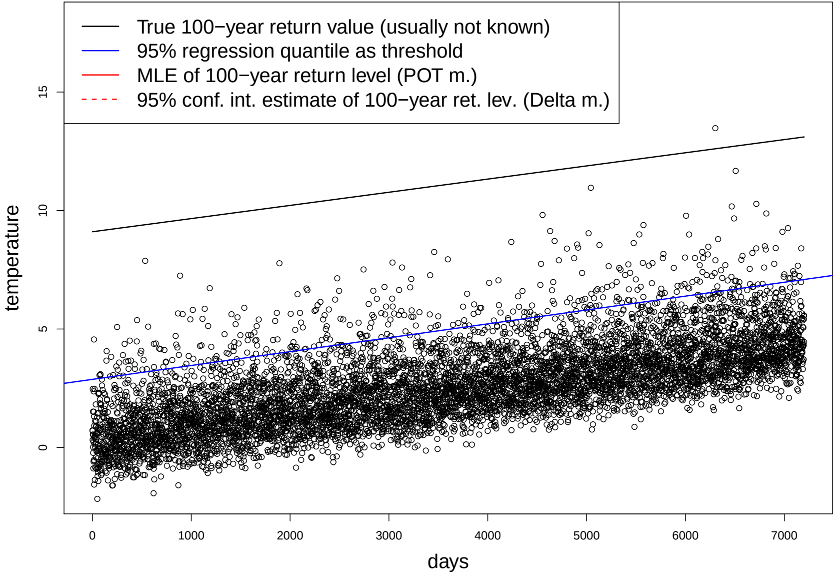

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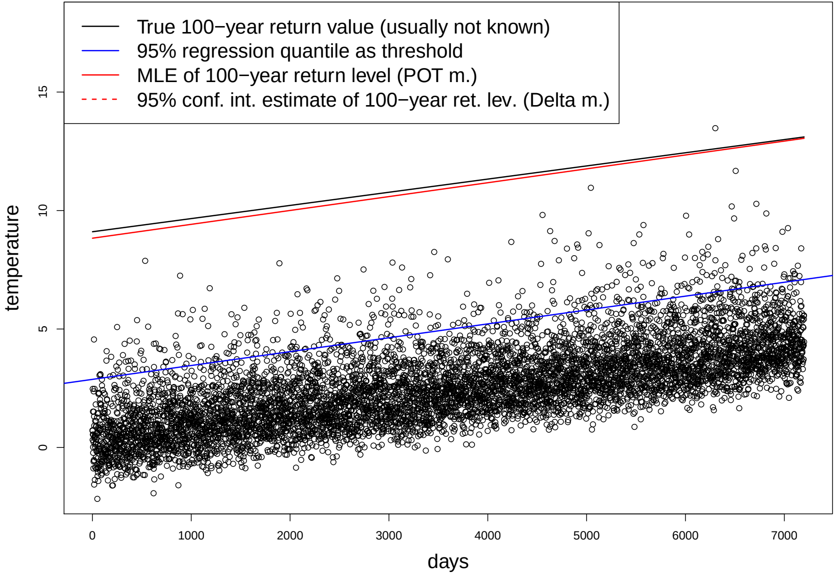

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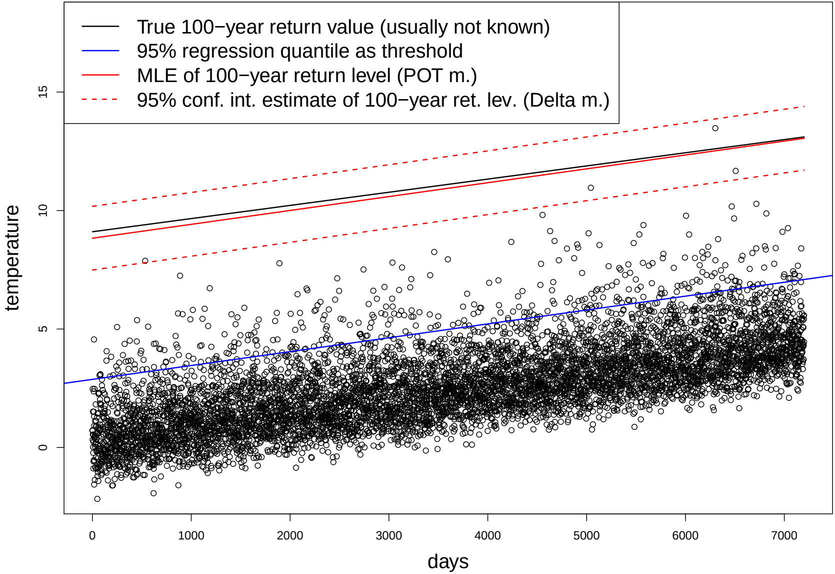

Peak over threshold (POT) method

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Situation (assumptions):

- data $y_1, \ldots, y_n$ follow a model $y_i = e_i + \beta_0 + \beta_1 \cdot i, \qquad i = 1, \ldots, n,$ where e_i is a random sample with distribution function $F(x)$.
- m-year return ”level” is

$$F^{-1}\left(1 - \frac{1}{m}\right) + \beta_1 \cdot i, \qquad 1 \leq i \leq n$$

which is the population $(1 - \frac{1}{m})$-th regression quantile line.

Solution: natural choice for the threshold is an α-th regression quantile

$$(\hat{\beta}_0(\alpha), \hat{\beta}_1(\alpha))^T = \arg\min_{\mathbf{t}\in\mathbf{R}^2} \sum_{i=1}^{n} \rho_\alpha(y_i - t_0 - t_1 \cdot i), \text{ where}$$

$$\rho_\alpha(u) = |u|\{(1-\alpha)I[u<0] + \alpha I[u>0]\}, \quad u \in \mathbf{R},\ \alpha \in (0,1)$$

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Goal: Find the α such that the α-th regression quantile line $\hat{\beta}_0(\alpha) + \hat{\beta}_1(\alpha) \cdot i$ is the "optimal" threshold for the POT method.
Criterion: Maximization of the probability that a confidence belt covers the real return "level".
Methods: MLE is used to estimate the GPD parameters, and Delta method to construct the confidence belts for return "levels".

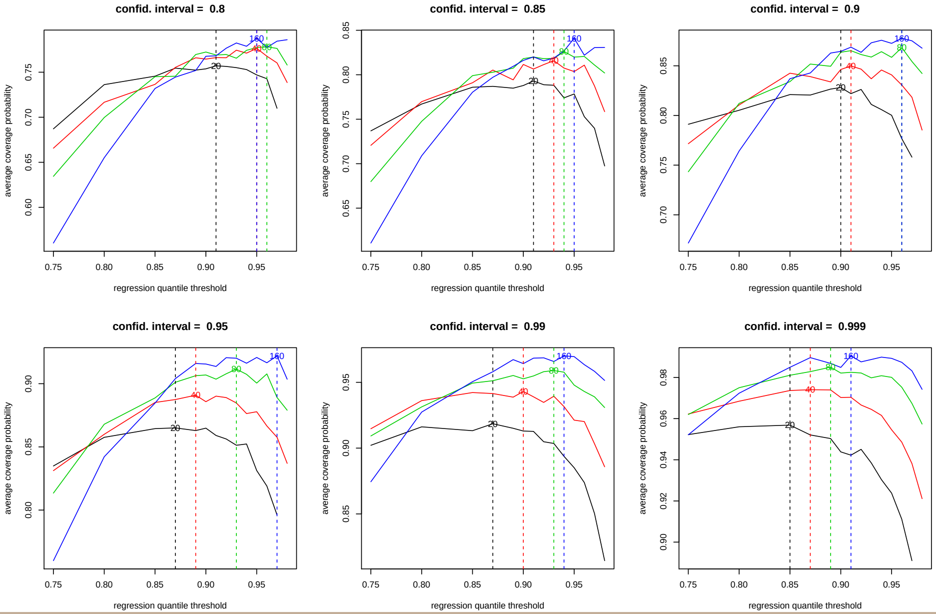
Outline

Monte Carlo simulations

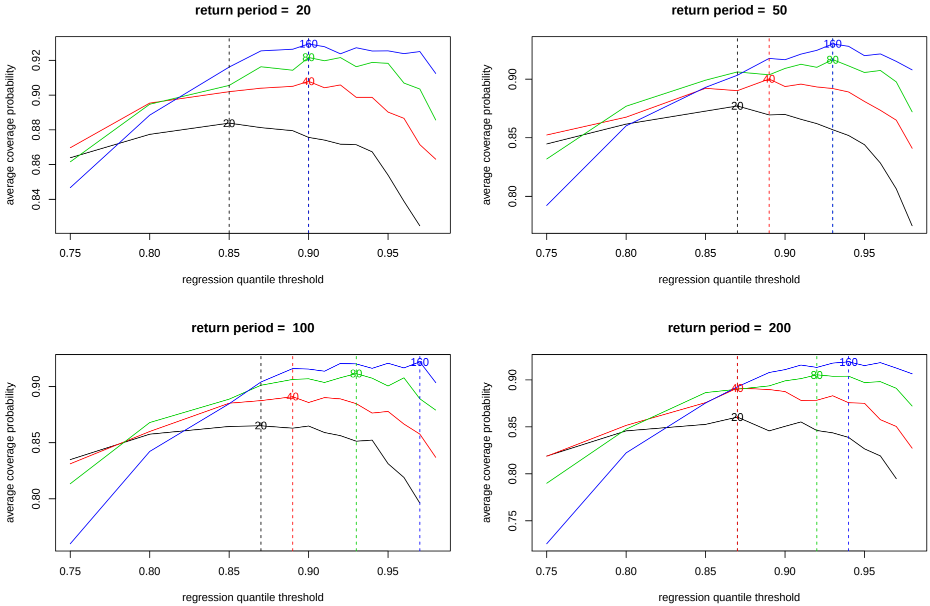
Parameters of simulated data:

- Choice of $F(x)$: standard Gumbel distribution
- Sample size $n = (20, 40, 80, 160)$-years $\cdot\ 90$
- We set trend $\beta_1 = 0.05/90$
- We wish to estimate $(20, 50, 100, 200)$-year return ”levels”
- We use $(75, 80, 85, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98)\%$ regression quantile as threshold
- We compute $(80, 85, 90, 95, 99, 99.9)\%$ confidence belts to estimate the return ”levels”
- For every setting of the parameters we generate 6600 sets of data.

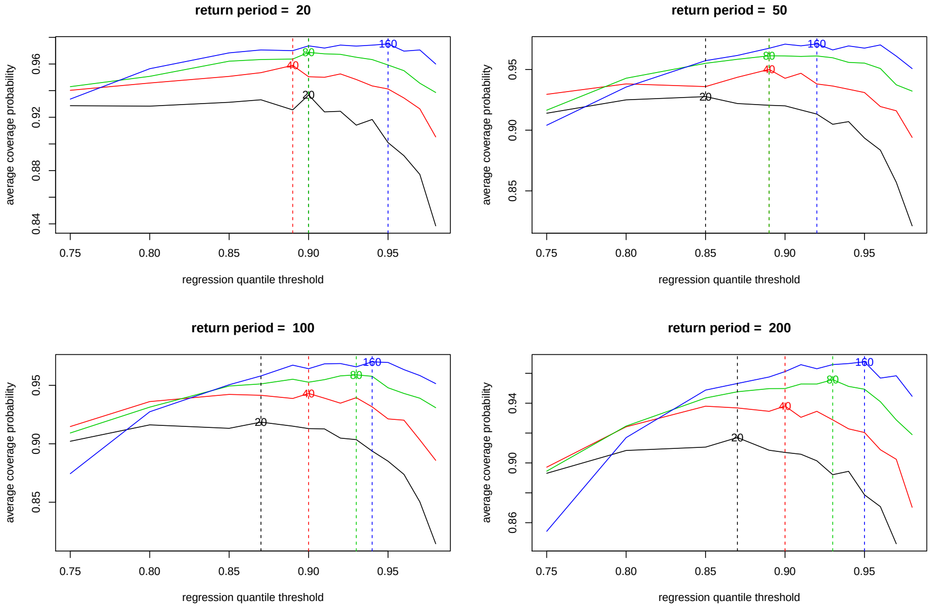
Average coverage probability for different sample size. Return "level": 100 years



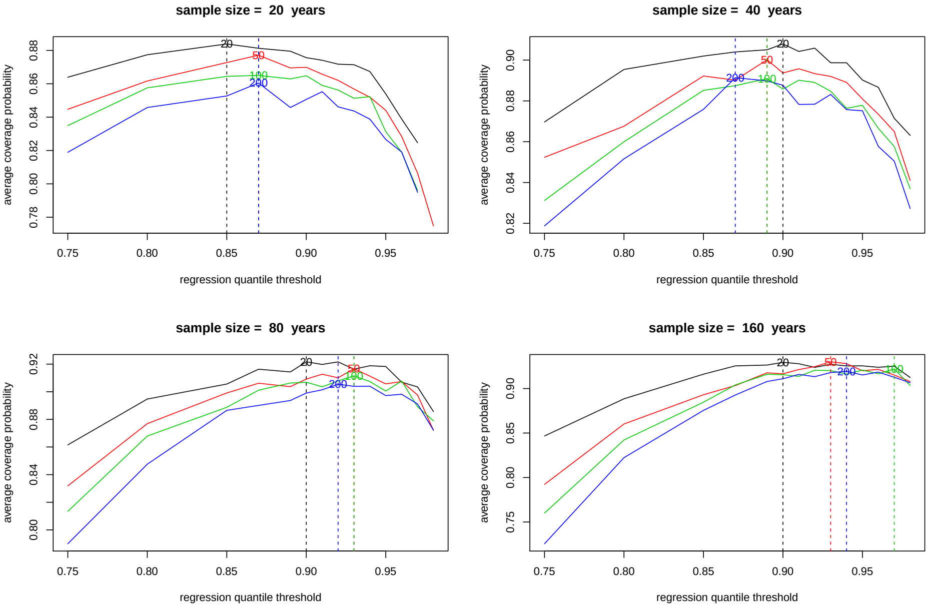
Average coverage probability for different sample size. Confidence interval: 95%



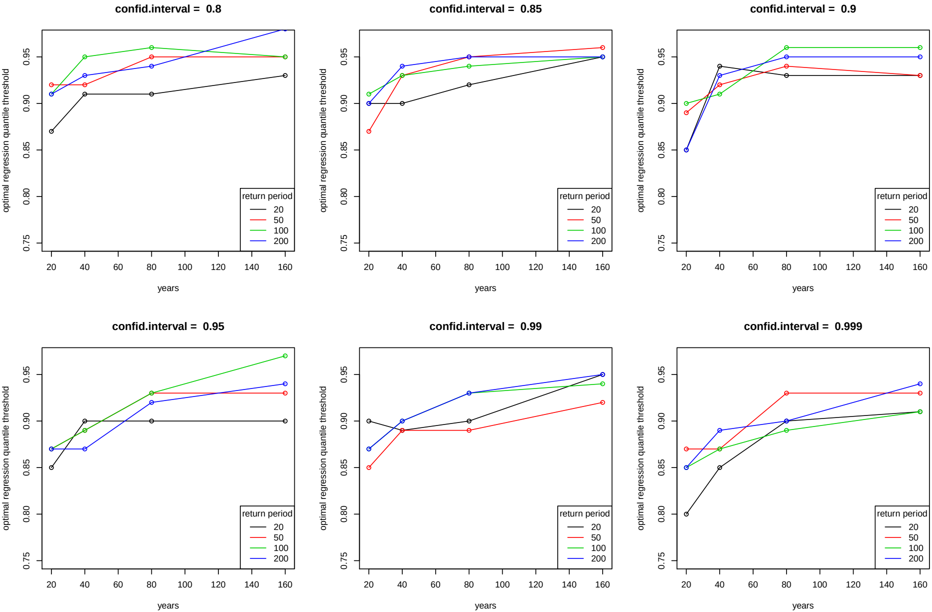
Average coverage probability for different sample size. Confidence interval: 99%



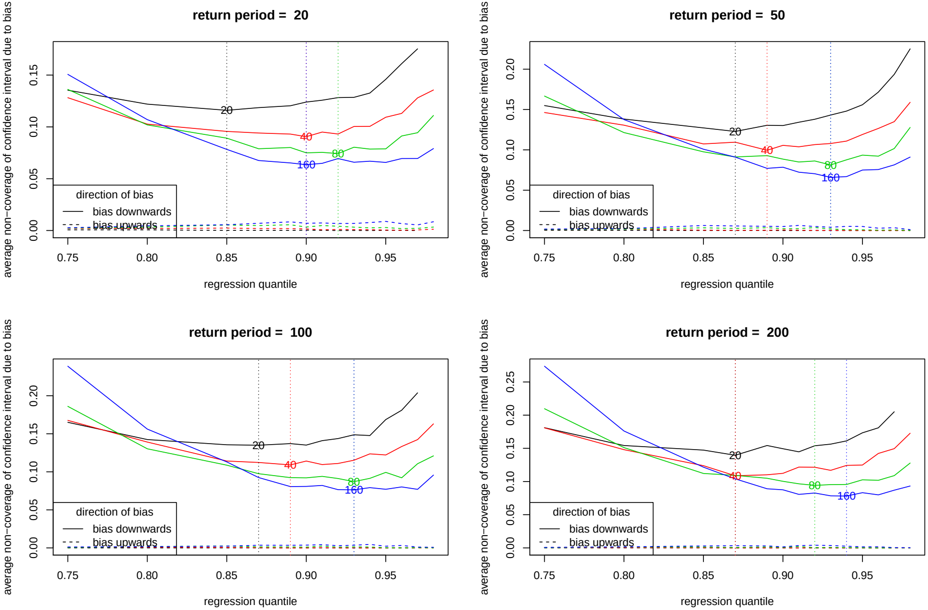
Average coverage probability for different return periods. Confidence interval: 95%



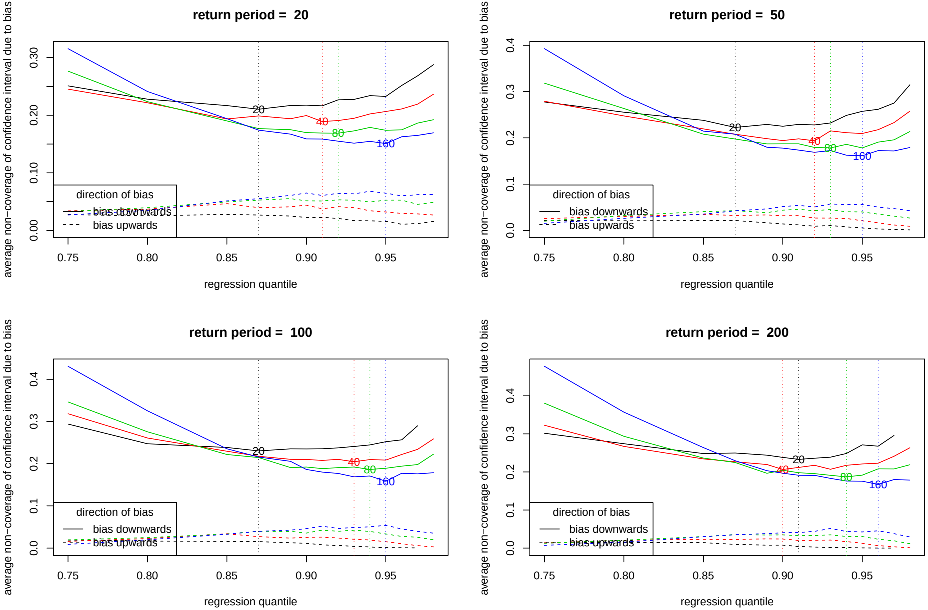
Optimal regression quantile wrt. sample size



Average non-coverage prob. for different sample sizes. Confidence interval: 95%



Average non-coverage prob. for different sample sizes. Confidence interval: 80%



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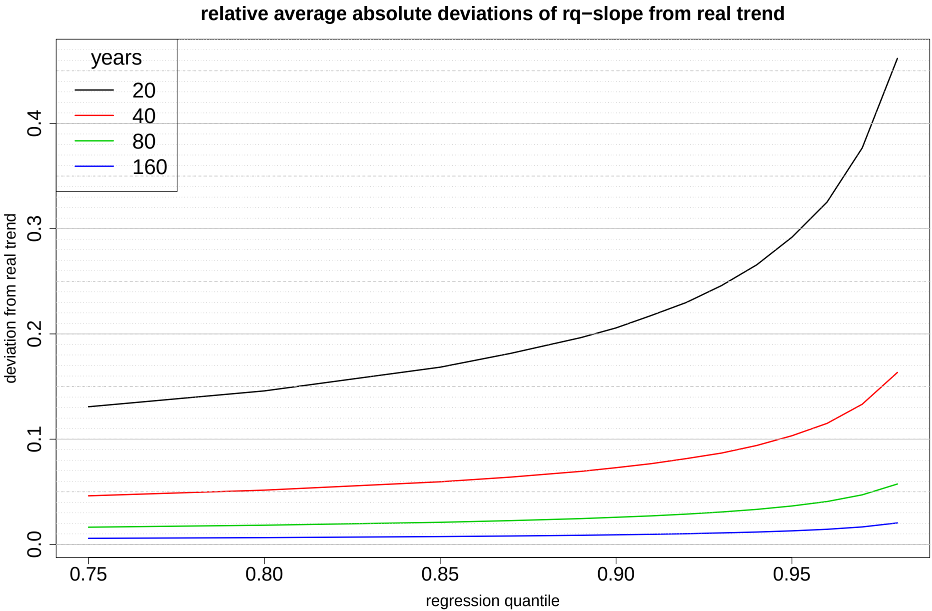
  - Confidence int. $\nearrow$ $\Longrightarrow$ Optimal regression quantile $\searrow$
- Real coverage probability of confid. int. is much lower than expected.
  - For real 95% confidence we need to construct approx. 99% confid. int.
  - Sample size $\searrow$ $\Longrightarrow$ Real confidence $\searrow$
  - Return period $\nearrow$ $\Longrightarrow$ Real confidence $\searrow$
  - Confidence interval is biased downwards, i.e. return level is too often higher than the confidence interval indicate.

Outline

How deviations of rq slope from real trend effect the optimal choice? IN NO WAY!



Noncoverage induced by variance of reg. quant. is constant

- For our model we can show that the α-th regression quantile $\hat{\beta}_1(\alpha)$ is approximately normal $\mathcal{N}\left(\mu = \beta_1, \sigma^2 = \frac{\alpha(1-\alpha)}{f^2(F^{-1}(\alpha))}\frac{12}{n(n^2-1)}\right)$
- Width of a confidence int. (belt) is proportional to $1/\sqrt{(1-\alpha)n}$
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$\Rightarrow$ e.g., for fixed α and F, if we quadruple sample size to $4n$, the conf. int. gets $2\times$ narrower and $4\times$ longer which compensates with the fact that mean absolute deviation of $\hat{\beta}_1(\alpha)$ from real trend β_1 gets smaller $\sqrt{4^3}\times = 8\times$.

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- Expanding the reciprocal around $\alpha = 1$ we get

$$\frac{1-\alpha}{\alpha(\ln\alpha)^2} \doteq \frac{1}{(1-\alpha)-\frac{(1-\alpha)^3}{12}-\frac{(1-\alpha)^4}{12}-\frac{13(1-\alpha)^5}{180}-\frac{11(1-\alpha)^6}{180}} \overset{\alpha\in(0.7,1)}{\doteq} \frac{1}{1-\alpha}$$

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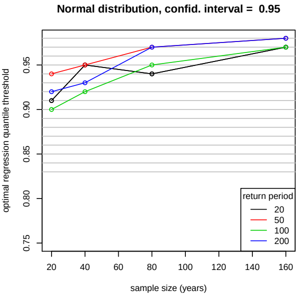
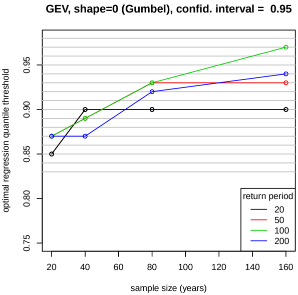
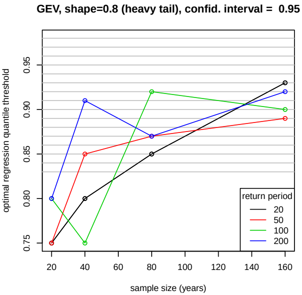
Conclusion: The optimal quantile as threshold is the same as if there was no trend in data and we used ordinary quantiles as threshold.

Other distributions F







Form of $\frac{\alpha(1-\alpha)}{f^2(F^{-1}(\alpha))}$ for different distributions F

Normal distribution	GEV,shape = 0 (Gumbel)	GEV,shape = 0.8 (heavy)
$\approx \frac{1}{\sqrt{1-\alpha}}$	$= \frac{1-\alpha}{\alpha(\ln \alpha)^2} \approx \frac{1}{1-\alpha}$	$\approx \frac{1}{(1-\alpha)^{2.6}}$

Outline

Future work

- Take data with different structure
  - Investigate autocorrelated data.
  - Different forms of trend.
- Use other methods to estimate the return ”levels”. Bootstrap?

Bibliography

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