Knowledge for Climate

Regional peaks-over-threshold modeling with respect to climate change

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joint work with A. Buishand (KNMI), G. Jongbloed (TU Delft), A. Klein Tank (KNMI), and H. v. Zanten (TU Eindhoven)

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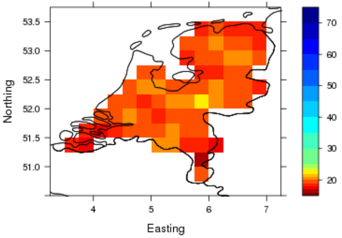
Goals

- Estimate site specific quantiles / return levels
- Assess the temporal trends in these quantiles
- Reduce the estimation uncertainty by spatial pooling

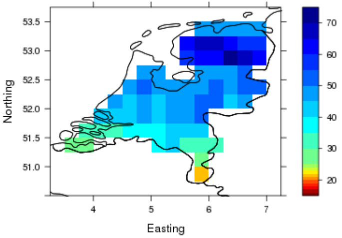
Inspired by the work of M. Hanel, A. Buishand and C. Ferro (2009) for block maxima data.

Data

- Daily, gridded precipitation data (E-OBS v. 5.0)
- Netherlands (high station density)
- Winter (DJF) data from 1950 - 2010

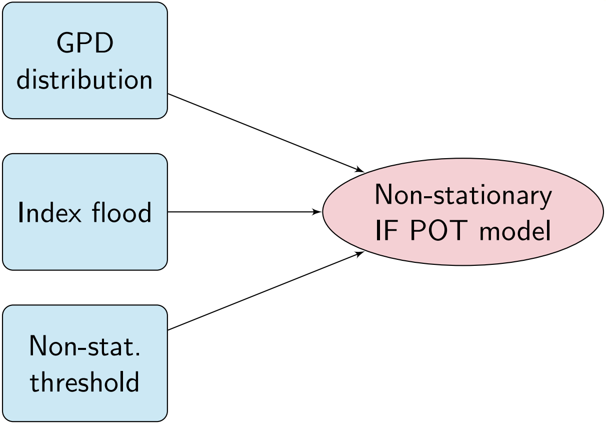


Mean winter maxima



Event on December 3, 1960

Components of the Model



GEV and GPD

Generalized Extreme Value distribution (GEV) for block maxima (BM)

$$
\begin{aligned}
P(M \leq x) &= H_{\xi^*,\sigma^*,\mu^*}(x) \\
&= \begin{cases} \exp\left\{ -\left[1 + \xi^* \left(\frac{x-\mu^*}{\sigma^*} \right) \right]^{-1/\xi^*} \right\}, & \xi^* \neq 0, \\ \exp\left[-\exp(-\frac{x-\mu^*}{\sigma^*}) \right], & \xi^* = 0, \end{cases}
\end{aligned}
$$

Generalized Pareto distribution (GPD) for excesses

$$
\begin{aligned}
P(Y \leq y | Y \geq 0) &= G_{\xi,\sigma}(y) \\
&= \begin{cases} 1 - \left(1 + \frac{\xi y}{\sigma} \right)^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp\left(-\frac{y}{\sigma} \right), & \xi = 0, \end{cases}
\end{aligned}
$$

GEV/GPD relation

If, for a threshold u, the excesses follow a GPD distribution with shape ξ and scale σ (denoted by $G_{\xi,\sigma}$) and the exceedance times follow a Poisson process with intensity λ, then we have that the maxima above u are GEV distributed with the following parameters:

$$\begin{aligned}
\mu^* &= \begin{cases} u - \frac{\sigma}{\xi}(1-\lambda^{\xi}), & \xi \neq 0, \\ u + \sigma \ln(\lambda), & \xi = 0, \end{cases} \\
\sigma^* &= \sigma\lambda^{\xi}, \\
\xi^* &= \xi,
\end{aligned} \tag{1}$$

Index flood for POT I

The index flood method assumes that all site specific distributions are identical apart from a site specific scaling factor, the index variable[1]. For exceedances this means, that

$$P\left(\frac{X_s}{\eta_s} \leq x | X_s \geq u_s\right) = \psi(x) \quad \forall s \in \mathcal{S}, \tag{2}$$

where X_s is a random variable representing the site-specific daily precipitation, u_s is the site specific threshold, η_s is the index variable and ψ does not depend on site s.

[1] Hosking and Wallis (1997)

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Index flood for POT II

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Index variable equals threshold

$$\psi(u_s/\eta_s) = P\left(X_s \leq u_s | X_s \geq u_s\right) = 0, \qquad \forall s \in \mathcal{S}.$$
$$\Rightarrow u_s/\eta_s = c.$$

Without loss of generality we can set $\eta_s = u_s$.

Index flood also for the excesses

$$P\left(\frac{Y_s}{\eta_s} \leq y | Y_s \geq 0\right) = \widetilde{\psi}(y) \quad \forall s \in \mathcal{S}, \tag{3}$$

where $\widetilde{\psi}(y) := \psi(y+1)$ is independent of site s.

Index flood for POT III

Site specific threshold

The τ-th quantile ($\tau >> 0.9$) of the daily precipitation amounts is a natural choice for a site specific threshold.
$\Rightarrow$ λ_s will be approximately constant over the region.

Restriction on the GPD parameters

The distribution of the scaled excesses has the following form:

$$P\left(\frac{Y_s}{\eta_s} \leq y \middle| Y_s \geq 0\right) = G_{\xi_s, \frac{\sigma_s}{u_s}}(y) \equiv \widetilde{\psi}(y). \tag{4}$$

Therefore we have:

$$\frac{\sigma_s}{u_s} \equiv \gamma, \qquad \xi_s \equiv \xi \quad \forall s \in \mathcal{S}. \tag{5}$$

We refer to γ as the dispersion coefficient.

Index flood for block maxima

Assuming constant λ, γ and ξ gives for the GEV parameters:

$$\xi_s^* \equiv \xi$$

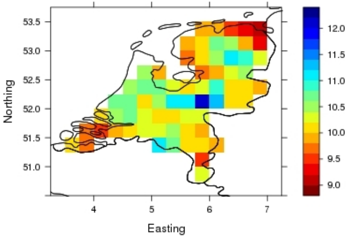
$$\gamma_s^* := \frac{\sigma_s^*}{\mu_s^*} = \left\{ \begin{array}{ll} \frac{\lambda^\xi}{\gamma^{-1} - \frac{1}{\xi}(1-\lambda^\xi)}, & \xi \neq 0 \\ \frac{1}{\gamma^{-1} + \ln(\lambda)}, & \xi = 0. \end{array} \right\} \equiv \gamma^*$$

Therefore the parameters ξ^* and γ^* fulfill the IF assumption for BM data[2]. This does not apply for the IF model for POT data proposed by Madsen and Rosbjerg (1997).

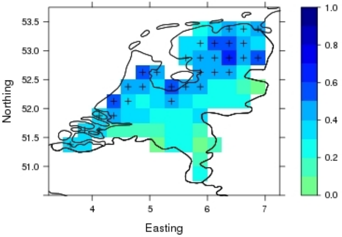
[2] Hanel, Buishand and Ferro (2009)

Nonstationary Threshold

The threshold is determined as the 0.96 linear regression quantile[3]:



Mean of the threshold for the 1950–2010 period in mm.



Trend in the threshold for the 1950–2010 period in mm per decade.

[3]Koenker (2005)

Nonstationary Version of the IF Model

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IF restrictions on the GPD parameters

$$\xi_s(t) \equiv \xi(t), \quad \frac{\sigma_s(t)}{u_s(t)} \equiv \gamma(t).$$

Quantile estimates

$$\begin{aligned} q_\alpha(s,t) &= u_s(t) + G^{-1}_{\xi(t),\sigma_s(t)}\big(1 - \frac{\alpha}{\lambda}\big) \\ &= \begin{cases} u_s(t) \cdot \Big(1 - \frac{\gamma(t)}{\xi(t)}\big[1 - \big(\frac{\alpha}{\lambda}\big)^{-\xi(t)}\big]\Big), & \xi(t) \neq 0, \\ u_s(t) \cdot \big(1 + \gamma(t)\ln(\lambda/\alpha)\big), & \xi(t) = 0. \end{cases} \end{aligned}$$

Note the factorization into a time and site dependent index variable and a quantile function, which depends on time only.

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Independence Likelihood

- Maximum likelihood estimation (MLE) popular method
- Since late 1980s used for regional estimation approaches
- Difficult dependence structure was neglected using an artificial independence assumption (independence likelihood)
- Dependence influences mainly the uncertainty
- Smith (1990) studies the uncertainty in an extended manner
- Special case of composite likelihood[4], which is a class of simplified (not true) likelihoods (e.g. also pairwise likelihood)

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Composite Likelihood

- Allows to assess for spatial dependence
- Specify a certain structure for the parameters, e.g.

$$\gamma(t) = \gamma_1 + \gamma_2 \cdot (t - \bar{t}), \qquad \xi(t) = \xi_1.$$

- Maximize:

$$\ell_I(\theta) = \sum_{s=1}^{S} \sum_{\substack{t=1 \\ y_s(t) \geq 0}}^{T} \ln\Big(f_{\underbrace{\gamma(t) u_s(t)}_{\sigma_s(t)}, \xi(t)}(y_s(t)) \Big),$$

where $f_{\sigma,\xi}(y)$ is the density of the GPD distribution.

Asymptotic Normality

$\hat{\theta}_I$ is asymptotically normal with mean θ and covariance matrix $G^{-1}(\theta)$

Godambe (sandwich) information

$$G(\theta) = H(\theta) J^{-1}(\theta) H(\theta)$$

- $H(\theta)$ is the expected negative Hessian of $\ell_I(\theta, \mathbf{Y})$
  Fisher information or sensitivity matrix
- $J(\theta)$ is the covariance matrix of the score $\nabla_\theta \ell_I(\theta, \mathbf{Y})$
  referred to as variability matrix
- In the independent case we have

$$J(\theta) = H(\theta) \Rightarrow G(\theta) = H(\theta)$$

Simulation

- Marginal parameters:
  $u \sim \mathcal{N}(10, 1)$, $\xi = 0.1$ and $\gamma = 0.5$
- Dependence model: Normal copula with auto-regressive correlation structure governed by one parameter rho
- Dimension: 10 sites and 100 (common) excesses and 2500 samples

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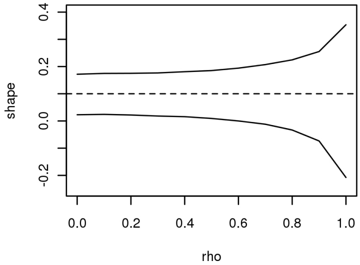

Mean confidence interval (shape parameter)

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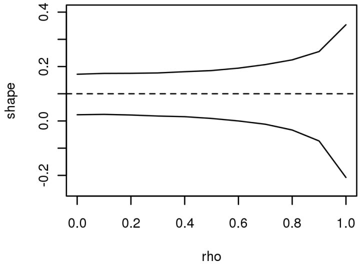

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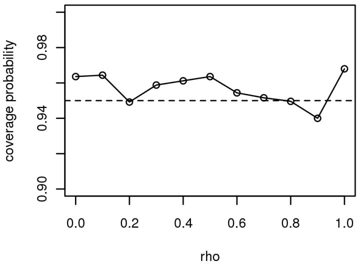

Coverage probability

Composite Information Criteria

Composite likelihood adaptations of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC)

$$
\begin{aligned}
AIC &= -2\ell_I(\hat{\theta}_I, Y) + 2\dim(\theta), \\
BIC &= -2\ell_I(\hat{\theta}_I, Y) + \ln(n)\dim(\theta),
\end{aligned}
$$

where $\dim(\theta)$ is an effective number of parameters:

$$
\dim(\theta) = \mathrm{tr}\left(\hat{H}(\theta)\hat{G}(\theta)^{-1}\right),
$$

Composite Likelihood Ratio Test

Adaptation of the likelihood ratio test

$$W = 2\left[\ell_I\left(\hat{\theta}_{M_1}; y\right) - \ell_I\left(\hat{\theta}_{M_0}; y\right)\right].$$

The asymptotic distribution of W is given by a linear combination of independent χ^2 variables, and can be determined using the Godambe information.

Bootstrap

- Transform excesses to standard exponentials using the full model M_1
- Sample monthly blocks of the whole region
- Transform the sampled data back using the nested model M_0

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Models and Information Criteria

IF models used

Model	**dispersion** γ	**shape** ξ
no trend	γ_1	ξ_1
trend in dispersion	$\gamma_1 + \gamma_2 * (t - \bar{t})$	ξ_1
trend in shape	γ_1	$\xi_1 + \xi_2 * (t - \bar{t})$

Information criteria for the IF models

Model	AIC	BIC
no trend	78387.28	**78715.59**
trend in dispersion	78435.60	78880.41
trend in shape	**78333.28**	78748.95

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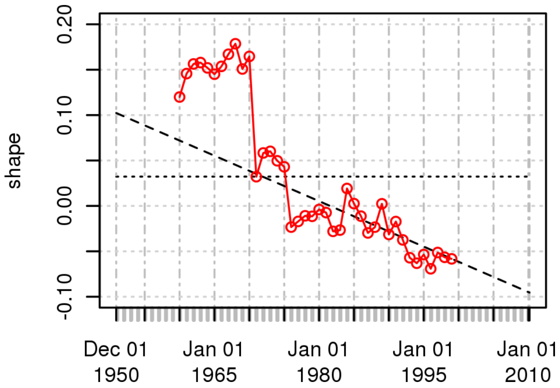
Shape



Shape parameter for different models (dotted – constant, dashed – linear trend, solid red – 20 year moving window estimates)

Significance Tests

Trend in the GPD parameters

p-values of the test for trend in the GPD parameters

Model	asymptotic	bootstrap
trend in dispersion	82.9%	81.3%
trend in shape	26.7%	12.2%

Index flood assumption

We compare the composite likelihoods of an IF model without trend in the parameters with that of a model with site specific dispersion coefficient and common shape parameter using the bootstrap method. We obtain a *p*-value of 0.103, i.e. the IF assumption is not rejected.

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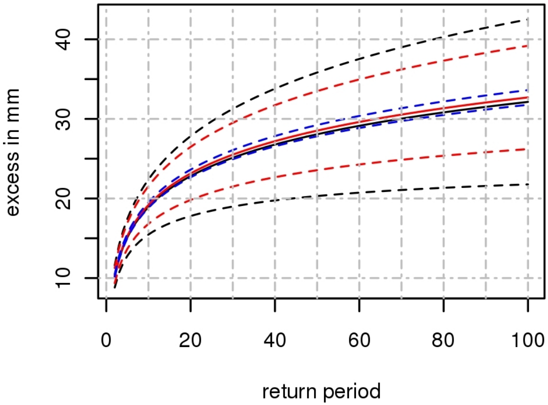
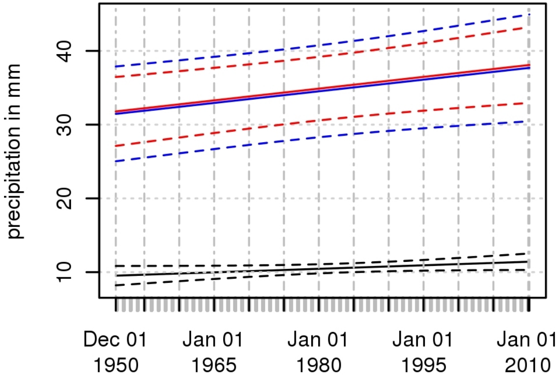
Uncertainty I - Excess distribution



Estimated return levels of the excesses (solid lines) with 95% pointwise confidence bands (dashed lines) for the year 1980 at the grid box around De Bilt (black – site-specific, red – IF, blue – no correction of the standard error for spatial dependence)

Uncertainty II - Threshold and Return Level



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Estimated threshold with 95% pointwise confidence band (black) and 25-year return level based on the at-site estimation (blue) and the IF approach (red), together with pointwise confidence bands, for the grid box around De Bilt.

Spatial Dependence I

- Transform observed peaks to standard exponentials
- Consider the maximum $M_{s,j}$ for each site s and winter season j. This maximum is approximately Gumbel distributed with location parameter $\ln(\lambda)$ and scale parameter 1
- Determine the spatial mean Gumbel plot of these maxima
- Determine Gumbel plot of $\max_s M_{s,j}$ over the grid. For independent observations this should be Gumbel distributed with location parameter $\ln(\lambda * S)$ and scale parameter 1[5]

[5] Reed and Stewart (1994)

Spatial Dependence II

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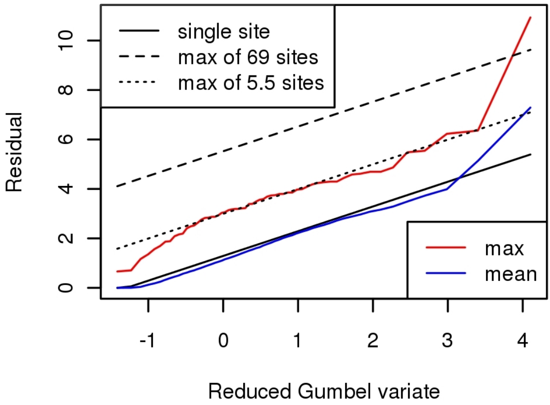

Spatial mean Gumbel plot (blue), Gumbel plot of the maxima over the grid (red) and theoretical distributions for the maximum of a different number of Gumbel variables (black)

Conclusions and Further Research

- Positive trends in the threshold are observed, which are significant in the coastal region
- No trend in the dispersion coefficient, i.e. proportional increase of the GPD scale parameter
- Negative trend in the shape parameter not significant
- Uncertainty is substantially reduced by regional modeling

- Application to climate model data
- Validity of the bootstrap needs to be explored

Literature

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