COVARIATE EFFECTS IN EXTREMES – REMARKS AND THEORY[1]

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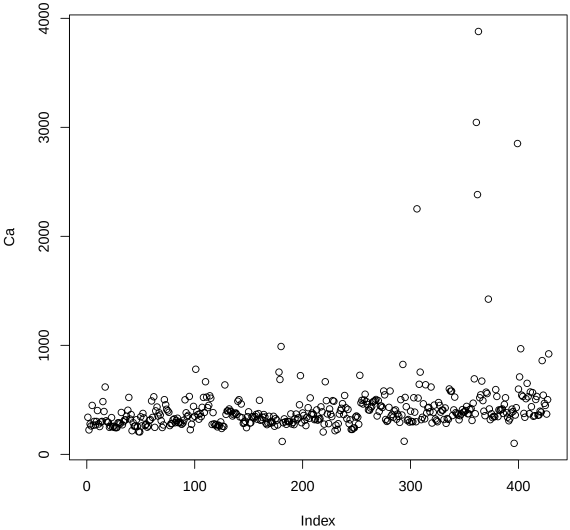
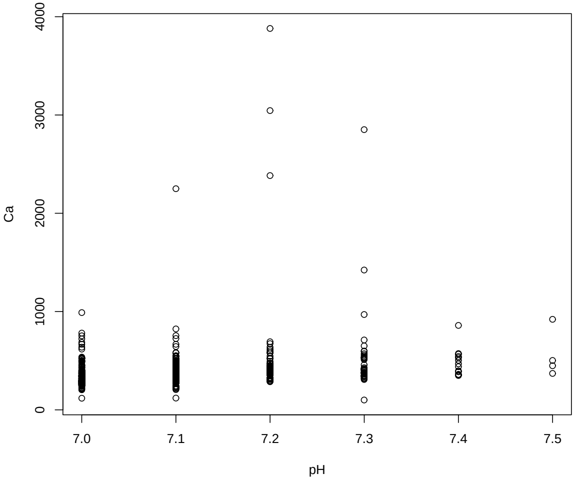
Motivation – sample data



Condroz data

Dataset of Calcium content vs. pH in soil in Condroz region in Belgium.



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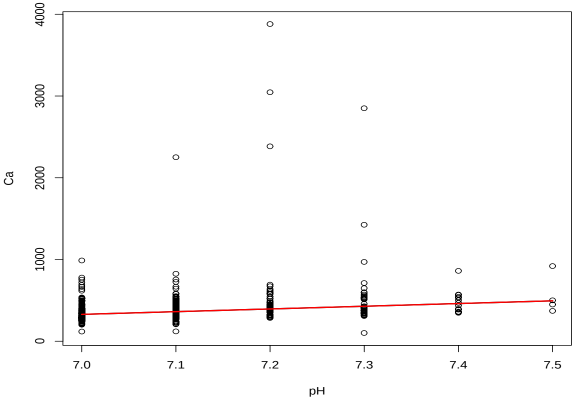

See Goegebeur et al. (2005), Vandewalle, Beirlant, Hubert (2006), Beirlant et al. (2004).

Heavy-tailed data – univariate case

Have E_i i.i.d. random variables. We say

$$E_i \in \mathcal{D}(G_\gamma), G_\gamma = \exp\left(-\left(1+\gamma x\right)^{-1/\gamma}\right)$$

i.e. there exists $a_n > 0$ and $b_n \in \mathbb{R}$

$$P\left(X_{n:n} \leq a_n x + b_n\right) \to G_\gamma(x).$$

for all $x \in \mathbb{R}$.

We are chiefly interested in the heavy-tailed errors ($\gamma > 0$), i.e. $F^{-1}(1-x)$ is regularly varying at zero (RV^0_γ).

$$\lim_{t \searrow 0} \frac{F^{-1}(1-tx)}{F^{-1}(1-t)} = x^{-\gamma}$$

…and as usual (to get more precise asymptotic), suppose we have a constant signed $A(t)$ and the second order approximation with some $\rho \in \mathbb{R}^+$

$$\lim_{t \searrow 0} \frac{\frac{F^{-1}(1-tx)}{F^{-1}(1-t)} - x^{-\gamma}}{A(t)} = x^{-\gamma} \cdot \frac{1-x^\rho}{\rho} =: K_{\gamma,\rho}(x).$$

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Heavy-tailed data – Drees (1998)

Suppose that E_i, $i = 1, \ldots, n$ are i.i.d. random variables fulfilling the second order condition for some $\gamma, \rho > 0$ and $k = k(n)$ is an intermediate sequence. Then we can define a sequence of Wiener processes $W_n(t), t \in [0,1]$ such that for $\varepsilon > 0$ sufficiently small

$$\sup_{0 \le t \le 1} t^{\gamma+1/2+\varepsilon} \left| k^{1/2} \left(\frac{E_{n-[kt],n}}{F^{-1}\left(1 - \frac{k}{n}\right)} - t^{-\gamma} \right) - \gamma t^{-\gamma-1} W_n(t) \right.$$
$$\left. - k^{1/2} A\left(\frac{k}{n}\right) t^{-\gamma} \frac{1 - t^{\rho}}{\rho} \right| \xrightarrow[n \to \infty]{\mathrm{P}} 0.$$

- wide range of applications – consider functional $T(E_{n-[kt],n})$ with T being location and scale invariant smooth functional
- a more complicated version for $\gamma \in \mathbb{R}$ exists
- Q: can something similar be established for linear models?

Simple linear model with heavy tails

$$\mathbf{Y}_{n\times 1} = \mathbf{X}_{n\times d}\boldsymbol{\beta}_{d\times 1} + \mathbf{E}_{n\times 1}$$

- $\mathbf{X}_{n\times d}$ known covariate matrix
- $\mathbf{E}_{n\times 1}$ i.i.d. errors
- $E_i \in \mathcal{D}(G_\gamma)$, $G_\gamma = \exp\left(-\left(1+\gamma x\right)^{-1/\gamma}\right)$

  i.e. there exists $a_n > 0$ and $b_n \in \mathbb{R}$

  $$P\left(X_{n:n} \leq a_n x + b_n\right) \to G_\gamma(x).$$

  for all $x \in \mathbb{R}$.
- $\gamma > 0$.

Regression quantiles

regression quantiles for $\alpha \in (0,1)$ and loss $\rho_\alpha(u) = u(\alpha - I(u<0))$ are defined

$$\widehat{\boldsymbol{\beta}}_n(\alpha) = \widehat{\boldsymbol{\beta}}_n\left(\alpha \middle| \mathbf{Y}, \mathbf{X}\right) := \arg\min_{b \in \mathbb{R}^d} \sum_{i=1}^{n} \rho_\alpha\left(Y_i - \mathbf{x}_i b\right).$$

Extreme regression quantiles

- the largest regression quantile

$$\widehat{\boldsymbol{\beta}}_n(1) = \widehat{\boldsymbol{\beta}}_n\left(1 \middle| \mathbf{Y}, \mathbf{X}\right) := \arg\min_{b \in \mathbb{R}^d} \sum_{i=1}^{n} \left(Y_i - \mathbf{x}_i b\right)^+,$$

cf. Smith (1994), Portnoy and Jurečková (1999), Jurečková (2000), Knight (2002).

- $\alpha_n^* \to 1$ with a given order
  - extreme order regression quantiles $(1-\alpha)n \to k > 0$, $n \to \infty$,
  - intermediate order regression quantiles $(1-\alpha)n \to \infty$, $\alpha \to 0$,

cf. Chernozhukov (2005).

Example: asymptotic for intermediate regression quantiles by Chernozhukov (2005)

$$\frac{\sqrt{\alpha n}}{\mu_{\mathbf{X}}^{\top}(\beta(\alpha) - \beta(m\alpha))}\left(\hat{\beta}(\alpha) - \beta(\alpha)\right) \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, \Omega(\gamma))$$

where $\mu_{\mathbf{X}} = E\mathbf{x}$, $\boldsymbol{\beta}(\alpha) = (\beta_1 + F^{-1}(\alpha), \beta_2, \ldots, \beta_d)$, $m < 1$.

Tail quantile function vs. Regression quantile process

- the tail quantile function

$$Q_{n,k}(t) := F_n^{-1}\left(1 - \frac{kt}{n}\right) = E_{n-[k_n t]:n}, \quad t \in [0,1].$$

- the sample quantile process

$$q_n(\alpha) = n^{1/2}(F_n^{-1}(\alpha) - F^{-1}(\alpha)), \quad 0 < \alpha \leq 1.$$

- the tails of regression quantiles

$$\hat{\mathbf{Q}}_{n,k}(t) := \widehat{\boldsymbol{\beta}}_n\left(1 - \frac{tk}{n}\right), \qquad t \in [0,1],$$

- the process of regression quantiles

$$\hat{\mathbf{q}}_n(\alpha) := n^{\frac{1}{2}} f(F^{-1}(\alpha))\left(\widehat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right), \qquad 0 < \alpha < 1,$$

where $\boldsymbol{\beta}(\alpha) := (\beta_1 + F^{-1}(\alpha), \beta_2, \dots, \beta_d)$.

Tail quantile function vs. Regression quantile process

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$$q_n(\alpha) = n^{1/2}(F_n^{-1}(\alpha) - F^{-1}(\alpha)), \quad 0 < \alpha \leq 1.$$

- the tails of reparametrized regression quantiles

$$\hat{Q}_{n,k}(t) := \overline{\mathbf{x}}^\top \widehat{\boldsymbol{\beta}}_n\left(1 - \frac{tk}{n}\right), \qquad t \in [0,1],$$

- the process of reparametrized regression quantiles

$$\hat{q}_n(\alpha) := n^{\frac{1}{2}} f(F^{-1}(\alpha))\overline{\mathbf{x}}^\top\left(\widehat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right), \qquad 0 < \alpha < 1,$$

where $\boldsymbol{\beta}(\alpha) = (\beta_1 + F^{-1}(\alpha), \beta_2, \ldots, \beta_d)$ and $\overline{\mathbf{x}} := \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i$.

Main results – an outline

1. Approximate $\mathbf{q}_n(\alpha)$, on $[\alpha_n^*, 1-\alpha_n^*]$, $\alpha_n^* \to 0$.
2. Approximate $\hat{q}_n(\alpha)$ on $[1-\alpha_n^*, 1-1/n]$.
3. Approximate $\hat{Q}_{n,k}(t)$ in the same way as $Q_{n,k}(t)$, cf. Drees (1998).
4. Describe estimators of γ as functionals of $Q_{n,k}(t)$.
5. The functionals have same properties on $\hat{Q}_{n,k}(t)$.

1. Approximation of regression quantile process

Under suitable conditions it holds

$$\sup_{\alpha_n^* \le \alpha \le 1-\alpha_n^*} \left| \sigma_\alpha^{-1} (\widehat{\boldsymbol{\beta}}_n\,(\alpha |\, Y, \mathbf{x}) - \boldsymbol{\beta}(\alpha)) \right| = O_P(n^{-1/2} (\log \log n)^{\frac{1}{2}}),$$

and

$$n^{1/2} \sigma_\alpha^{-1} \left(\widehat{\boldsymbol{\beta}}_n\,(\alpha |\, \mathbf{Y}, \mathbf{X}) - \boldsymbol{\beta}(\alpha) \right) =$$

$$n^{-1/2} (\alpha(1-\alpha))^{-1/2} \mathbf{D}_n^{-1} \sum_{i=1}^{n} \mathbf{x}_i \left(\alpha - I[E_i - F^{-1}(\alpha) < 0] \right) + o_P(1)$$

where $\sigma_\alpha := (\alpha(1-\alpha))^{1/2} / f(F^{-1}(\alpha))$ and $\alpha_n^* = (\frac{1}{n} \log^{2+\delta} n)$ for any $\delta > 0$

cf. Gutenbrunner et al. (1993) and Jurečková (1999), where $\alpha_n^* = n^{-1+\varepsilon}$ is used.

Assumptions

Distribution function

(F.1) F is absolutely continuous with the positive density on (x_*, x^*). There exists f', the derivative of density f.

(F.2) There exists some $0 < K_\gamma < \infty$ such that

$$\sup_{x_* < x < x^*} F(x)(1 - F(x)) \left| \frac{f'(x)}{f^2(x)} \right| \le K_\gamma .$$

(F.3)

$$\limsup_{x \uparrow x^*} \frac{(1 - F(x)) f'(x)}{f^2(x)} = -1 - \gamma^* .$$

for some $\gamma^* > -1/2$ (lower tail index γ_* similarly).

Covariance matrix

(X.1) $x_{i1} = 1, \quad i = 1, \ldots, n$.

(X.2) $\lim_{n\to\infty} \mathbf{D}_n = \mathbf{D}$, where $\mathbf{D}_n = n^{-1} \mathbf{X}_n^\top \mathbf{X}_n$ and $\mathbf{D}$ is a positive definite $(d \times d)$ matrix.

(X.3) $n^{-1} \sum_{i=1}^n |\mathbf{x}_{ni}|^4 = O(1)$ as $n \to \infty$.

(X.4) $\max_{1\le i \le n} |\mathbf{x}_{ni}| = O((\log\log n)^{1/2})$ as $n \to \infty$.

Proof

- Prove that $\sup\left\{|r_n(\mathbf{t},\alpha)| : \alpha_n^* \le \alpha \le 1-\alpha_n^*, \|\mathbf{t}\| \le (\log\log n)^{1/2}\right\} = o_P(1)$,

$$r_n(\mathbf{t},\alpha) := (\alpha(1-\alpha))^{-1/2}\sigma_\alpha^{-1}\sum_{i=1}^n\left[\rho_\alpha\left(E_{i\alpha}-n^{-1/2}\sigma_\alpha\mathbf{x}_i^\top\mathbf{t}\right)-\rho_\alpha(E_{i\alpha})\right]$$
$$+n^{-1/2}(\alpha(1-\alpha))^{-1/2}\mathbf{t}^\top\sum_{i=1}^n\mathbf{x}_i\psi_\alpha(E_{i\alpha})-\frac{1}{2}\mathbf{t}^\top\mathbf{D}_n\mathbf{t}$$

  and $E_{i\alpha} := E_i - F^{-1}(\alpha)$, $i=1,\ldots,n$, $\quad 0<\alpha<1$, $\psi_\alpha(u) := \alpha - I(u<0)$.

  - approximate the mean of $r_n(\mathbf{t},\alpha)$ for any suitable α and $\mathbf{t}$.
  - Bernstein inequality gives a probabilistic bound for any α and $\mathbf{t}$.
  - Chaining arguments give the uniform bound.

- $n^{1/2}\sigma_\alpha^{-1}(\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{\beta}(\alpha))$ minimizes the convex function

$$G_{n\alpha}(\mathbf{t}) = (\alpha(1-\alpha))^{-1/2}\sigma_\alpha^{-1}\sum_{i=1}^n\left[\rho_\alpha(E_{i\alpha}-n^{-1/2}\sigma_\alpha\mathbf{x}_i^\top\mathbf{t})-\rho_\alpha(E_{i\alpha})\right]$$

- use the properties of $r_n(\alpha,\mathbf{t})$ to calculate the solution for $\|\mathbf{t}\| \le (\log\log n)^{1/2}$.
- convexity of $G_{n\alpha}(\mathbf{t})$ implies that the minimum cannot be attained elsewhere.

2. Regression quantile process at the tails

Suppose that $\gamma^* > 0$. Then

$$\sup_{1-\alpha_n^* \le \alpha \le \frac{n-1}{n}} \left|\overline{\mathbf{x}}^\top \mathbf{q}_n(\alpha)\right| = \sup_{1-\alpha_n^* \le \alpha \le \frac{n-1}{n}} \left| n^{1/2} f\left(F^{-1}(\alpha)\right) \overline{\mathbf{x}}^\top \left(\widehat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)\right|$$
$$= \mathcal{O}_P(n^{-1/2}(\log n)^{(2+\delta)(1\vee\gamma^*)}) = o_P(1).$$

and if $\gamma_* > 0$ is the tail index of the lower tail it holds also

$$\sup_{1/n \le \alpha \le \alpha_n^*} \left|\overline{\mathbf{x}}^\top \mathbf{q}_n(\alpha)\right| = \sup_{1/n \le \alpha \le \alpha_n^*} \left| n^{1/2} f\left(F^{-1}(\alpha)\right) \overline{\mathbf{x}}^\top \left(\widehat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)\right|$$
$$= \mathcal{O}_P(n^{-1/2}(\log n)^{(2+\delta)(1\vee\gamma_*)}) = o_P(1),$$

Proof

- $\overline{\mathbf{x}}^\top \boldsymbol{\beta}_n(\alpha_1) \leq \overline{\mathbf{x}}^\top \boldsymbol{\beta}_n(\alpha_2)$ iff $\alpha_1 \leq \alpha_2$,
- similarly as in Portnoy and Jurečková (1999) get

$$P_{\boldsymbol{\beta}}\left(\sum_{i=1}^{n} \mathbf{x}_i^\top \left(\boldsymbol{\beta}(1) - \boldsymbol{\beta}\right) \geq nt\right) \quad \leq \quad P\left(E_{n:n} \geq t\right),$$

- assuming $\gamma = \gamma^* > 0$ it follows

$$P\left(\frac{E_{n:n}}{F^{-1}\left(1 - 1/n\right)} \geq \zeta\right) \xrightarrow[n\to\infty]{\mathcal{D}} 1 - \exp\left(-\zeta^{-\frac{1}{\gamma}}\right),$$

- use von Mises condition and Lemma 4.5.2. of Csörgő and Révész (1977) for transition from $f(F^{-1}(1 - kt/n))$ to $f(F^{-1}(1 - k/n))$.

3. Tails of regression quantiles

Assume

- model with i.i.d. errors fulfilling the second order condition, $\gamma, \rho > 0$,
- $k = k(n) \to \infty$, $k/n \to 0$ and $\sqrt{k}A(k/n) = \lambda$,
- $k \geq \log^{\Delta(1\vee\gamma)}(n)$, $\Delta > 4 + 2\delta$.
- $\|z\|_{\gamma,\varepsilon} := \sup_{t\in[0,1]} \left|t^{1/2+\gamma+\varepsilon} z(t)\right|$, $z \in D[0,1]$.

There are Wiener processes $W_n(t)$, $\tilde{W}_n(t)$, and $\mathbf{W}(t)$ such that for any $\varepsilon > 0$.

$$\left\| k^{1/2} \left(\frac{\overline{\mathbf{x}}^\top \left(\widehat{\boldsymbol{\beta}}_n \left(1 - \frac{kt}{n}\right) - \boldsymbol{\beta} \right)}{F^{-1}\left(1 - \frac{k}{n}\right)} - t^{-\gamma} \right) - \gamma t^{-\gamma-1} W_n(t) \right.$$
$$\left. - k^{1/2} A\left(\frac{k}{n}\right) t^{-\gamma} \frac{1 - t^\rho}{\rho} \right\|_{\gamma,\varepsilon}$$
$$\leq \left\| \gamma t^{-\gamma} \overline{\mathbf{x}}^\top \mathbf{D}^{-1} \mathbf{W}(t) \right\|_{\gamma,\varepsilon} + \left\| \gamma t^{-\gamma} \tilde{W}_n(t) \right\|_{\gamma,\varepsilon} + o_P(1),$$

...which is an analogy to Drees(1998).

$$\sup_{0\le t\le 1} t^{\gamma+1/2+\varepsilon}\left|k^{1/2}\left(\frac{E_{n-[kt],n}}{F^{-1}\left(1-\frac{k}{n}\right)}-t^{-\gamma}\right)-\gamma t^{-\gamma-1}W_n(t)\right.$$
$$\left.-k^{1/2}A\left(\frac{k}{n}\right)t^{-\gamma}\frac{1-t^{\rho}}{\rho}\right|\xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Proof

- combination of the previous results on approximations of regression quantiles,
- von Mises condition and Lemma 4.5.2. of Csörgő and Révész (1981) used for transition from $f(F^{-1}(1-kt/n))$ to $f(F^{-1}(1-k/n))$,
- direct procedure – just a rough approximation.

4. Functionals of tail quantile functions

Have $\gamma \in \mathbb{R}$ and a functional $T : span(\mathcal{H}_\mathcal{M}, 1) \to \mathbb{R}$ satisfying

1. $\mathcal{H}_\mathcal{M}$ is semimetric space, where the tail quantile function and its relatives live
2. $T(az+b) = T(z), \qquad$ for all $z \in \mathcal{H}_\mathcal{M}$, $a > 0$, $b \in \mathbb{R}$,
3. $T(z_\gamma) = T(\frac{x^{-\gamma}-1}{\gamma}) = \gamma$
4. $T_{|\mathcal{H}_\mathcal{M}}$ is Hadamard differentiable tangentially to suitable continuous $\mathcal{C}_\mathcal{M} \subset \mathcal{H}_\mathcal{M}$, at z_γ with a derivative T'_γ, i.e. for some signed measure $\nu_{T,\gamma}$ it holds for all $0 < \varepsilon_n \to 0$ and all $y_n \in \mathcal{H}_\mathcal{M}$ such that $y_n \to y \in \mathcal{C}_\mathcal{M}$

$$\lim_{\varepsilon_n \to 0} \frac{T(z_\gamma - \varepsilon_n y_n) - T(z_\gamma)}{\varepsilon_n} = T'_\gamma(y) = \int_0^1 y \mathrm{d}\nu_{T,\gamma}.$$

Then $T(Q_{n,k}) \to \gamma$ and for intermediate sequence k_n with the rate parameter $\lambda = func.(\gamma, \rho, k_n)$ it holds

$$\mathcal{L}(k_n^{1/2}(T(Q_{n,k}) - \gamma)) \to \mathcal{N}(\lambda\nu_{T,\gamma,\rho}, \sigma_{T,\gamma}),$$

c.f. Drees (1998).

Variance and bias of the estimation

Provided that $\sqrt{k}A(k/n) \to \lambda$ it holds

(i) $T(Q_{n,k}) \to \gamma$

(ii) $\mathcal{L}(k_n^{1/2}(T(Q_{n,k}) - \gamma)) \to \mathcal{N}(\lambda\nu_{T,\gamma,\rho}, \sigma_{T,\gamma})$, where

$$
\begin{aligned}
\mu_{T,\gamma,\rho} &:= \int_0^1 t^{-\gamma}\frac{1-t^\rho}{\rho}\mathrm{d}\,\nu_{T,\gamma} \\
\sigma_{T,\gamma} &:= Var\left(\int_0^1 t^{-\gamma-1}W(t)\,\mathrm{d}\nu_{T,\gamma}(t)\right) \\
&= \int_0^1\int_0^1 (st)^{\gamma-1}\min(s,t)\mathrm{d}\,\nu_{T,\gamma}(s)\,\mathrm{d}\nu_{T,\gamma}(t)
\end{aligned}
$$

5. $T(\hat{Q}_{n,k}(t))$ is the estimator

Suppose that T fulfills the given assumptions. For $T(\hat{Q}_{n,k}(t))$ it follows:

1. consitency:
   - follows immediately from continuity of T and approximations given previously.
2. asymptotic normality:
   - requires Hadamard differentiability,
   - as we have an extra random remainder (with zero mean), asymptotic variance can be only roughly estimated,
   - asymptotic bias is the same one as in the i.i.d. case.

Available estimators

Examples:

- Pickands estimator

$$T_{\text{Pick}}(z) := \frac{1}{\log 2} \log\left(\frac{z(1/4) - z(1/2)}{z(1/2) - z(1)}\right) I\left[\frac{z(1/4) - z(1/2)}{z(1/2) - z(1)} > 0\right].$$

- Probability weighted moments estimator

$$T_{\text{PWM}}(z) := \frac{\int_0^1 (z(t) - z(1))(1 - 4t)dt}{\int_0^1 (z(t) - z(1))(1 - 2t)dt} I\left[\int_0^1 (z(t) - z(1))(1 - 2t)dt > 0\right].$$

- Maximum likelihood estimator – generated by an implicitly given functional, see Drees (1998).

Functionals on $\overline{\mathbf{x}}^\top \widehat{\boldsymbol{\beta}}_n(1-tk/n)_{t\in[0,1]}$

$T(\overline{\mathbf{x}}^\top \hat{\mathbf{Q}}_{n,k}) = T(\overline{\mathbf{x}}^\top \widehat{\boldsymbol{\beta}}_n(1-tk/n)_{t\in[0,1]})$ are consistent and asymptotically normal estimators of γ.

- ML-estimator of γ based on the k largest unique estimates of $\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau), \quad \tau \in (0,1)$, i.e. the estimator fits generalized Pareto distribution (GPD) on the exceedances of $\{\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_j)\}_{j=m-k,\ldots,m}$ over $\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-k-1})$.
- Probability weighted moments estimator (PWM)

$$\hat{\gamma}_{m,k}^{\text{RQ,PWM}} = \frac{\frac{1}{k}\sum_{j=1}^{k}\left(4\frac{j}{k+1}-3\right)\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-i+1})}{\frac{1}{k}\sum_{j=1}^{k}\left(2\frac{j}{k+1}-1\right)\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-i+1})}$$

- Pickands estimator

$$\hat{\gamma}_{m,k}^{\text{RQ,P}} = \frac{1}{\log 2}\log\left(\frac{\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-[k/4]}) - \overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-[k/2]})}{\overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-[k/2]}) - \overline{\mathbf{x}}^\top \hat{\boldsymbol{\beta}}_n(\tau_{m-k})}\right)$$

Where $\tau_1,\ldots,\tau_m$ are such that $\widehat{\boldsymbol{\beta}}_n\tau_i, i=1,\cdots,m$ are m unique solution of minimazation problem $\arg\min_{b\in\mathbb{R}^d}\sum_{i=1}^{n}\rho_\alpha(Y_i-\mathbf{x}_i\mathbf{b})$ for $\alpha\in[0,1]$.

Reparametrization

Have

$$\begin{aligned}\tilde{x}_{i,1} &= 1, \\ \tilde{x}_{i,j} &= x_{i,j} - \frac{1}{n}\sum_{i=1}^{n} x_{i,j}, \qquad j = 2, \ldots, p.\end{aligned}$$

Hence, after reparametrization $\overline{\mathbf{x}} = (1, 0, \ldots, 0)$.

Functionals on $\hat{\beta}_{n,1}(1-tk/n)_{t\in[0,1]}$

$T(\hat{Q}_{n,k}) = T(\hat{\beta}_{n,1}(1-tk/n)_{t\in[0,1]})$ are consistent and asymptotically normal estimators of γ.

- ML-estimator of γ based on the k largest unique estimates of $\hat{\beta}_{n,1}(\tau), \quad \tau \in (0,1)$, i.e. the estimator fits generalized Pareto distribution (GPD) on the exceedances of $\{\hat{\beta}_{n,1}(\tau_j)\}_{j=m-k,\dots,m}$ over $\hat{\beta}_{n,1}(\tau_{m-k-1})$.
- Probability weighted moments estimator (PWM)

$$\hat{\gamma}_{m,k}^{\text{RQ,PWM}} = \frac{\frac{1}{k}\sum_{j=1}^{k}\left(4\frac{j}{k+1}-3\right)\hat{\beta}_{n,1}(\tau_{m-i+1})}{\frac{1}{k}\sum_{j=1}^{k}\left(2\frac{j}{k+1}-1\right)\hat{\beta}_{n,1}(\tau_{m-i+1})}$$

- Pickands estimator

$$\hat{\gamma}_{m,k}^{\text{RQ,P}} = \frac{1}{\log 2}\log\left(\frac{\hat{\beta}_{n,1}(\tau_{m-[k/4]}) - \hat{\beta}_{n,1}(\tau_{m-[k/2]})}{\hat{\beta}_{n,1}(\tau_{m-[k/2]}) - \hat{\beta}_{n,1}(\tau_{m-k})}\right)$$

Where $\tau_1,\dots,\tau_m$ are such that $\hat{\beta}_{n,1}(\tau_i), i = 1,\cdots,m$ are m unique intercepts of regression quantiles in reparametrized model.

Summary of previous

Achievements:

- improvements of older approximations of regression quantiles
  - wider interval $[\alpha_n^*, 1-\alpha_n^*]$
  - at least a rough approximation for $[1-\alpha_n^*, 1-1/n]$
- general approximation methodology of γ based on regression quantiles

Open questions:

- further improvements of approximations
  - use Hungarian construction instead of Bahadur representation
  - improve approximation of regression quantile process in $[1-\alpha_n^*, 1-1/n]$
- dependency of errors

Two-step regression quantiles

Step 1: Calculate R-estimate of the slope i.e. invert the rank statistics in Hodges-Lehmann manner. Have

- $R_{ni}(\mathbf{Y} - \mathbf{X}\mathbf{b})$ be the rank of $Y_i - \mathbf{x}_i^\top \mathbf{b}$ among $(Y_1 - \mathbf{x}_1^\top \mathbf{b}, \ldots, Y_n - \mathbf{x}_n^\top \mathbf{b}), \mathbf{b} \in \mathbb{R}^p$
- $\varphi_\alpha = \alpha - I[x < 0], x \in \mathbb{R}$
- $\mathbf{x}_i$ be the i-th row of the $\mathbf{X}_{n \times d}$

Minimize the Jaeckel's measure of rank dispersion.

$$\widehat{\boldsymbol{\beta}}_{nR} = \operatorname{argmin}_{\mathbf{b} \in \mathbb{R}^p} \sum_{i=1}^{n} (Y_i - \mathbf{x}_i' \mathbf{b}) \varphi_\alpha \left(\frac{R_{ni}(\mathbf{Y} - \mathbf{X}\mathbf{b})}{n+1} \right)$$

Step 2: Get the ordered residuals $\tilde{\beta}_{n0} = Y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_{nR}(\alpha)$

$\Rightarrow$ two-step regression quantiles $\left(\tilde{\beta}_{n0}, \hat{\boldsymbol{\beta}}_{nR}(\alpha) \right)$

Two step r.q. process and its tail approximation

$$\hat{E}_{k:n} := \left(\{Y_1 - \mathbf{x}_1^\top \widehat{\boldsymbol{\beta}}_{nR}, \ldots, Y_n - \mathbf{x}_n^\top \widehat{\boldsymbol{\beta}}_{nR}\}\right)_{k:n}$$

and

$$\tilde{Q}_{n,k}(t) := F_n^{-1}\left(1 - \frac{k_n}{n} t\right) = \hat{E}_{n-[k_n t]:n}, \quad t \in [0,1],$$

Have again model with $E_i \sim F$, with F satisfying the second order condition for some $\gamma \in \mathbb{R}$ and $\rho \leq 0$. Then under suitable conditions on F and $\mathbf{X}$ we can define a sequence of Wiener processes $\{W_n(t)\}_{t \geq 0}$ such that for suitable chosen functions A and a and each $\varepsilon > 0$,

$$\sup_{t \in (0,1]} t^{\gamma + \frac{1}{2} + \varepsilon} \left| \frac{\tilde{Q}_{n,k}(t) - F^{-1}\left(1 - \frac{k}{n}\right) - \beta_0}{a(k/n)} - \left(z_\gamma(t) - k^{-\frac{1}{2}} t^{-(\gamma+1)} W_n(t) \right.\right.$$
$$\left.\left. + A\left(\frac{k}{n}\right) H(t)\right) \right| = o_P\left(k^{-1/2} + |A(k/n)|\right)$$

$n \to \infty$, provided $k = k(n) \to \infty$, $k/n \to 0$ and $\sqrt{k} A(k/n) = O(1)$ and $z_\gamma(t) = \frac{t^{-\gamma} - 1}{\gamma}$, cf. Picek and Dienstbier (2010).

Remark

- under suitable condition, the method of proof can be used for any convergent estimate of β and its ordered residuals
- however, is not “suitable condition” = “neglecting real data structures”?

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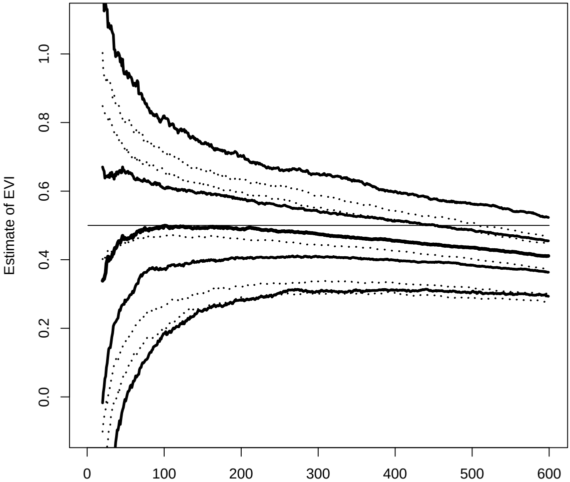
Simulations

Example: $Y_i = 1 + 5x_i + e_i$, where $x_i \sim U(0,1)$, e_i have Burr distribution with shape $\gamma = 0.5$



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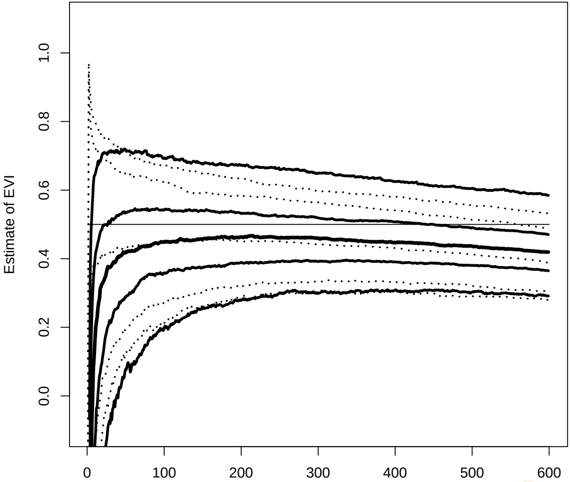

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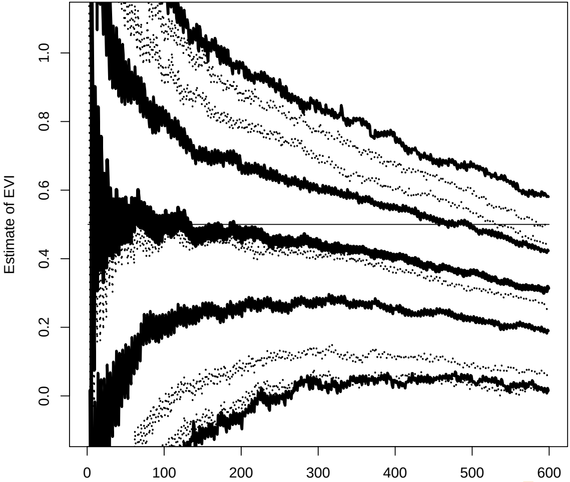
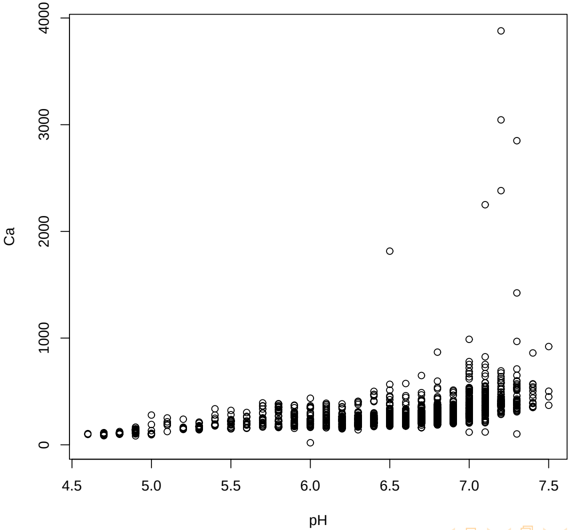


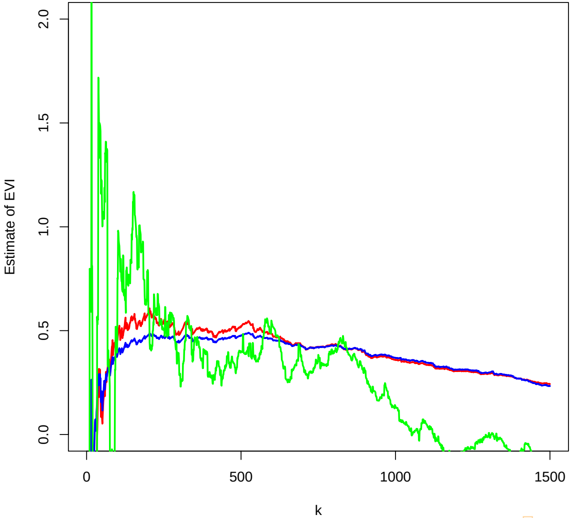
Condroz

Example: Condroz dataset again



Condroz

Example: Condroz dataset again, estimator plots



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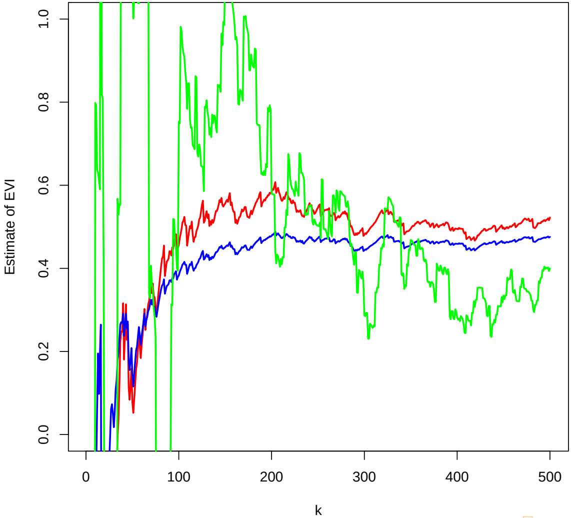
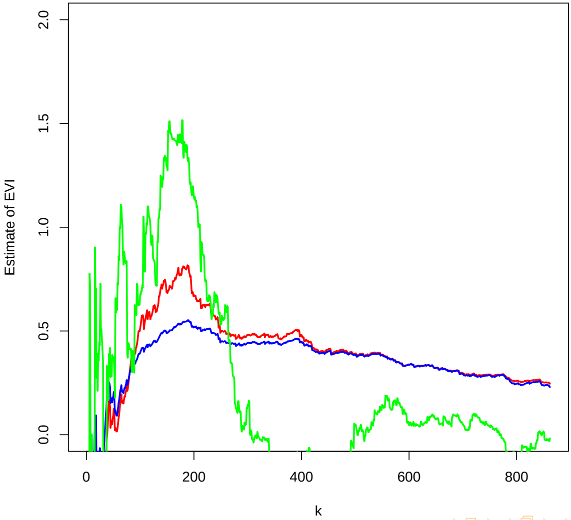


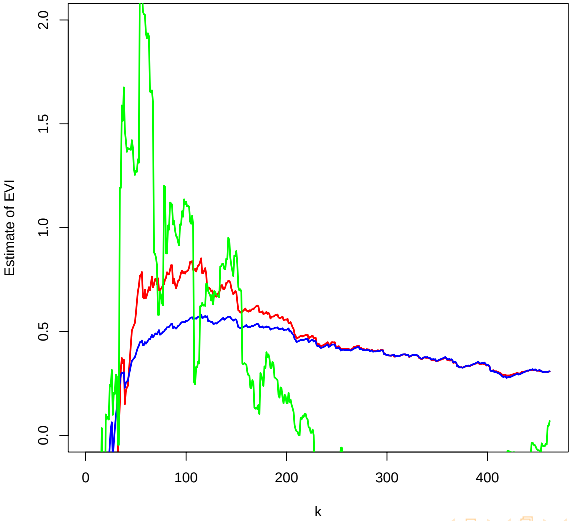
Condroz

Example: Condroz dataset again, estimator plots $6.6 \le pH \le 7.5$



Condroz

Example: Condroz dataset again, estimator plots $6.6 \leq pH \leq 7.3$



Remarks

- theory shows that if "thinks goes well" estimates based on quantile regression works
  - i.e. if the model is same (or simpler) as we suppose
- one additional interpretation of Condroz data *(hurray!)*

Something structural?

- extremes in predictors matters
- we often do not have the same number of observations for different predictors
  - problem, if nice model for all responses desired
  - get a nice model = root out enough data as outliers!
- linear models tend not to be linear
  - at least, if we want to work with all responses
- other possible problems. . .

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Further remarks & hypocrisy continued

- in EVT linear models, the choice of model matters, not the data
- which model is better than others depends strictly on the exact data settings and not the theory
- EVT can be a dangerous drug (do not abuse)

cf.

hypocrite *n. One who, professing virtues that he does not respect, secures the advantage of seeming to be what he despises.*

story *n. A narrative, commonly untrue.*

politeness *n. The most acceptable hypocrisy.*

– Ambrose Bierce, *Devil's Dictionary*

COVARIATE EFFECTS IN EXTREMES – REMARKS AND THEORY[1]

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