

ESTIMATION OF PARAMETERS OF A CLIPPED MA(1) PROCESS

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Představení modelu

- ↪ Nechť $\{X_t, t \in \mathbb{Z}\}$ jsou iid náhodné veličiny s absolutně spojitým rozdělením a $a > 0$, $c \in \mathbb{R}$ jsou konstanty.
- ↪ Uvažujeme stacionární posloupnost 1-závislých veličin $\{\xi_t, t \in \mathbb{Z}\}$, kde

$$\xi_t = \begin{cases} 1 & \text{pokud } X_t - aX_{t-1} < c, \\ 0 & \text{jinak.} \end{cases} \quad (1)$$

- ↪ Zajímají nás odhadování některých charakteristik v modelu (1).

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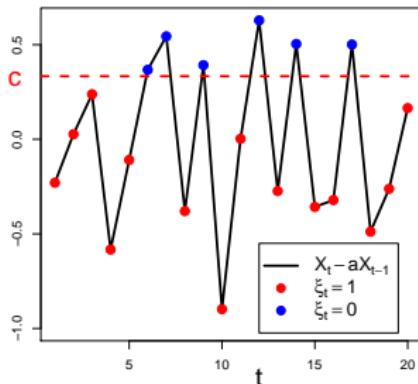
Na základě pozorování ξ_1, \dots, ξ_n chceme odhadnout

- pravděpodobnost úspěchu $p = P(\xi_t = 1)$,
- korelací $r_1 = \text{corr}(\xi_t, \xi_{t+1})$,
- parametry a a c .

Motivace

Model (1) transformuje MA(1) proces $X_t - aX_{t-1}$ na posloupnost 0-1 veličin $\{\xi_t, t \in \mathbb{Z}\}$ pomocí operace zvané **clipping** (nebo **hard limiting**)

- ξ_t indikuje, zda $X_t - aX_{t-1}$ leží pod hladinou c v čase t
- vlastnosti obecných “useknutých” procesů ↪ rozsáhlá literatura
- diskretizace spojité veličiny se často objevuje v biologii, inženýrství a dalších **aplikacích**



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- “useknutý” MA(1) proces ↪ určité speciální aplikace
- **odhadování parametrů** původního procesu na základě “useknutého” ↪ v literatuře pouze pro Gaussovský případ



Více informací na posteru

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Introduction
Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of iid random variables with distribution $CDF F$. Let $a > 0$ and $c \in \mathbb{R}$ be some constants. Define

$$Y_t = \begin{cases} a & \text{if } X_t < -a \\ c & \text{if } -a \leq X_t \leq c \\ a & \text{if } X_t > c \end{cases} \quad (1)$$

We deal with the estimation problem in model (1).

Why such processes?
There are many reasons why we study processes from MA(1) family. One reason is that they can be reduced into a sequence of independent random variables. Another reason is that they are often used in time series analysis [1]. We also mention the specific properties of the process, for example the fact that it has a short memory and that it is a linear process. The last reason is that there are many statistical methods available for the estimation of parameters of such processes. There are many papers on this topic, which have been studied extensively in the literature, see [6].

Q is an indicator function. Both the process and the distribution of the process are continuous if the parameter a is continuous. Such distributions are called processes in a continuous sense. It is interesting to note that the process is a continuous phenomenon — Model (1) with $a = 0$ is proposed in [2] as a phenomenon of a continuous nature.

[1] Vassiliou, G. (2002) Adaptive ARMA models. $\hat{\alpha}_1$ are negatively correlated with $\hat{\alpha}_2$. See their paper here: <http://www.sciencedirect.com/science/article/pii/S0378375802000010>.

Several authors have been concerned with an estimation of the clipped signal only, see [6], [3], [2], [1].

The properties of the model

The variables $\{y_t, t \in \mathbb{Z}\}$ are **stationary** and the process $\{X_t, t \in \mathbb{Z}\}$ is **non-stationary**. The probability $P(X_t < -a) = p$, the probability $P(X_t > c) = q$ and the last autocorrelation $P(Y_t Y_{t-1})$ depend on (X_t, X_{t-1}) and (Y_t, Y_{t-1}) — see [6].

Define $Q(a, c) = E[\int_{-\infty}^{\infty} \delta(y-aX_t) \delta(y-cX_t)]$. Then

$$p(a, c) = Q(a, c) / (1 - p(a, c))$$

Properties of p and r_{11} are studied in more detail in [6].

Estimation of p
 $Q(a, c) = P(X_t < -a) + P(X_t > c) - 2P(-a < X_t < c)$.
Then $R = r_{11} = \sum_{t=1}^{\infty} r_{11}(t)$ is the **observed autocorrelation** of the process $\{Y_t\}$ and R is asymptotically normal, AN(0, 1), see [6].

The asymptotic variance of R can be written as

$$U = P(X_t < -a) + P(X_t > c) - 2P(-a < X_t < c),$$

and U tends to 0 as c approaches zero. If $a = 0$ it is hard to find R because then $R = P(Y_t = 0) - P(Y_t = a)$ as $a \rightarrow 0$. If $a = 0$ and $c = a$, then $R = 0$. Finally, we have $U(0, 1) = U(0, 0)$.

Example 1. Let $\{X_t, t \in \mathbb{Z}\}$ be iid random variates with $F(x) = 1 - e^{-x}$ and $P(X_t = 0) = 0.5$. Then $p(a, c) = 1 - e^{-c/a}$. Consider the case $a = 2.0$. (The results for $a = 2.0$ can be obtained from the symmetry $p(a, c) = 1 - p(c, a)$) and $c = 2.0$. We have

$$\begin{aligned} p(a, c) &= 1 - e^{-c/a} \\ r_{11}(t) &= \frac{e^{-2t} - e^{-2(t+1)} - (1 - e^{-2})^2}{(1 - e^{-2})^2 - (1 - e^{-2})^2} \end{aligned}$$

Then $\sqrt{n}(\hat{p}(a, c) - p(a, c))$ is AN(0, 1), where

$$U = \frac{e^{-2} - e^{-2} - (1 - e^{-2})^2}{(1 - e^{-2})^2 - (1 - e^{-2})^2} = \frac{2e^{-2} - 2 - 2e^{-4}}{(1 - e^{-2})^2 - (1 - e^{-2})^2} = \frac{2e^{-2} - 2 - 2e^{-4}}{2e^{-2} - 2e^{-4}}$$

Estimation of r_{11}
 $E[Y_t Y_{t-1}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_t y_{t-1} f_{X_t, X_{t-1}}(x_t, x_{t-1}) dx_t dx_{t-1}$. Then

$$E[Y_t Y_{t-1}] = \begin{cases} a^2 & \text{if } x_t < -a, x_{t-1} < -a \\ ac & \text{if } -a \leq x_t \leq c, -a \leq x_{t-1} \leq c \\ c^2 & \text{if } x_t > c, x_{t-1} > c \end{cases}$$

It is a consistent estimator of r_{11} and $\sqrt{n}(E[Y_t Y_{t-1}] - r_{11})$ is asymptotically normal.

Example 2. (Example 1, continued)
It is $a = 2.0$, $c = 2.0$.

$$\begin{aligned} \hat{r}_{11} &= \frac{1}{n} \sum_{t=1}^n Y_t Y_{t-1} = \left(\frac{1}{n} \sum_{t=1}^n Y_t \right)^2 = 0.52 \approx 0.50 \\ \hat{r}_{11} &= \frac{1}{n} \sum_{t=1}^n Y_t Y_{t-1} = \frac{1}{n} \sum_{t=1}^n \left(\frac{1}{2} \sum_{x_t=-2}^2 \sum_{x_{t-1}=-2}^2 \delta(x_t) \delta(x_{t-1}) \right) = 0.52 \approx 0.50. \end{aligned}$$

The estimator is consistent and asymptotically normal, when $\sqrt{n}(E[Y_t Y_{t-1}] - r_{11})$ is approximately zero.

If $a = 1.2$, $c = 0.5$, $n = 100$, $r_{11} = 0.15$ corresponds to the distribution of a random variable $Z(n)$, where $Z(n)$ follows the distribution of a binomial random variable with parameters $n = 100$ and $p = 0.15$, i.e. $Z(n) \sim \text{Bin}(100, 0.15)$.

$$d = \sqrt{n} \left(\frac{E[Y_t Y_{t-1}] - r_{11}}{\sqrt{r_{11}}} \right) = \sqrt{100} \left(\frac{0.15 - 0.15}{\sqrt{0.15}} \right) = 0.00 \approx 0.$$

The estimator is consistent and asymptotically normal, because $d \approx 0$.

Numerical Study
For given $a \in (0, 2]$ and the distribution F , we generate $n = 100$ observations of the process $\{Y_t\}$ and estimate $\hat{p}(a, c)$ ($c = a$, respectively). For each sample size n this procedure was repeated 1000 times. The samples were generated by the R command `rexp` and the estimates were generated by the R command `mean`. We summarize the obtained results.

- very good properties (almost) regardless the value of a and c .
- converges to the true probability as n increases.
- the asymptotic normal approximation gives rise to a normal distribution with mean 0 and standard deviation $1/\sqrt{n}$.
- behavior more sensitive to the values of a and c .
- converges to the true parameter quite fast.
- the estimator is unbiased for $a = 0$ and $c = a$ and is not “biased” (such that it points to 0).
- if $a = c$ “unbiased”, then the moment approximation is not unbiased.
- the χ^2 approximation for $\hat{p}(a, c) - p(a, c)$ is quite good for large sample sizes ($n = 100$, $a = 0.5$ “unbiased” for $c = 1.0$).

Acknowledgements
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