TAIL MODELLING IN LINEAR MODELS BY QUANTILE REGRESSION[1]

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Linear model and regression quantiles

$$\boldsymbol{Y}_{n\times 1} = \boldsymbol{X}_{n\times d}\boldsymbol{\beta}_{d\times 1} + \boldsymbol{E}_{n\times 1}$$

- $\boldsymbol{X}_{n\times d}$ known covariate matrix
- $\boldsymbol{E}_{n\times 1}$ vectore of (i.i.d.) errors

Then the regression quantiles for $\alpha \in (0,1)$ are defined

$$\widehat{\boldsymbol{\beta}}_n\left(\alpha \middle| \boldsymbol{Y}, \boldsymbol{X}\right) := \arg\min_{b\in\mathbb{R}^d} \sum_{i=1}^{n} \rho_\alpha\left(Y_i - \mathbf{x}_i b\right),$$

- $\mathbf{x}_i$ is the i-th row of the $\boldsymbol{X}_{n\times d}$
- ρ_α is the loss function $\rho_\alpha(u) := u \cdot \left(\alpha - 1_{\{u<0\}}\right)$
- we assume $\mathbf{x}_{i1} = 1,\ i = 1, \dots, n$

Question:

How much the regression quantiles reflect the properties of F?

Bahadur representation

$$\sup_{\alpha_n^* \le \alpha \le 1-\alpha_n^*} \left| \sigma_\alpha^{-1} (\widehat{\boldsymbol{\beta}}_n\,(\alpha |\, Y, \mathbf{x}) - \boldsymbol{\beta}(\alpha)) \right| = O_P(n^{-1/2} (\log\log n)^{\frac{1}{2}}),$$

and

$$n^{1/2} \sigma_\alpha^{-1} \left(\widehat{\boldsymbol{\beta}}_n\,(\alpha |\, \boldsymbol{Y}, \boldsymbol{X}) - \boldsymbol{\beta}(\alpha) \right) = \\ n^{-1/2} (\alpha(1-\alpha))^{-1/2} \boldsymbol{D}_n^{-1} \sum_{i=1}^{n} \mathbf{x}_i \frac{\rho_\alpha(E_i - F^{-1}(\alpha))}{|E_i - F^{-1}(\alpha)|} + o_P(1)$$

where $\sigma_\alpha := (\alpha(1-\alpha))^{1/2} / f(F^{-1}(\alpha))$ and $\alpha_n^* = n^{-1} (\log\log n)^c$ for any $c > 2$.

Proof Dienstbier (2010) based on Csörgő, Révész (1977), Gutenbrunner et al. (1993), Jurečková (1999).

Notational

- upper right endpoint $x^* = \inf_{x \in \mathbb{R}} \{F(x) = 1\}$
- lower left endpoint $x_* = \sup_{x \in \mathbb{R}} \{F(x) = 0\}$

- quantile function $F^{-1}(\alpha) = \inf \{x, x \geq F(\alpha)\}$

The Assumptions of the Approximation

Distribution function

(F.1) F is absolutely continuous with the positive density on (x_*, x^*). There exists f', the derivative of density f.

(F.2) There exists some $0 < K_\gamma < \infty$ such that

$$\sup_{x_* < x < x^*} F(x)(1 - F(x)) \left| \frac{f'(x)}{f^2(x)} \right| \leq K_\gamma.$$

Covariance matrix

(X.1) $x_{i1} = 1, \quad i = 1, \ldots, n$.

(X.2) $\lim_{n\to\infty} \boldsymbol{D}_n = \boldsymbol{D}$, where $\boldsymbol{D}_n = n^{-1} \boldsymbol{X}_n^\top \boldsymbol{X}_n$ and $\boldsymbol{D}$ is a positive definite $(d \times d)$ matrix.

(X.3) $n^{-1} \sum_{i=1}^{n} |\mathbf{x}_{ni}|^4 = O(1)$ as $n \to \infty$.

(X.4) $\max_{1 \leq i \leq n} |\mathbf{x}_{ni}| = O(1)$ as $n \to \infty$.

Index of heaviness γ - MDA$_\gamma$ (of the upper tail)

Have the real constants $a_n > 0$ and b_n such that for all x with $1 - \gamma x > 0$ holds

$$\lim_{n\to\infty} F^n(a_n x - b_n) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right)$$

$$\Updownarrow$$

Suppose, there is a function $a > 0$ such that for all $x > 0$

$$\lim_{t\to\infty} \frac{F^{-1}(1 - x/t) - F^{-1}(1 - 1/t)}{a(t)} = \frac{x^{-\gamma} - 1}{\gamma}$$

$$\Updownarrow$$

(if F is differentiable enough)

$$\lim_{x\uparrow x^*} (1 - F(x)) \left(\frac{f'(x)}{f^2(x)}\right) = -1 - \gamma.$$

Approximations of the tails

Assume it holds the first (& also the second) order approximation of extremes and $n \to \infty$, $k = k(n) \to \infty$, $k/n \to 0$, and $\sqrt{k}A_0(n/k) = O(1)$. Then for any $\varepsilon > 0$

$$\sup_{\frac{(\log\log n)^c}{k} < s \le 1} s^{\gamma+1/2+\varepsilon} \left| \sqrt{k} \left(\frac{\widehat{\boldsymbol{\beta}}_{n,1}\left(1 - \frac{ks}{n} \,\middle|\, \boldsymbol{Y}, \boldsymbol{X}\right) - \boldsymbol{\beta}_1\left(1 - \frac{ks}{n}\right)}{a_0(n/k)} - \frac{s^{-\gamma} - 1}{\gamma} \right) \right.$$

$$\left. -\gamma s^{-\gamma-1} W_n(s) - \sqrt{k}A_0\left(\frac{n}{k}\right) s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

where

- $W_n(s)$ are Wiener processes (Brownian motions)
- $\gamma \in \mathbb{R}, \rho \le 0$ are the first and the second order extreme value indices
- a_0 and A_0 are known functions related to the first and second order
- and $\beta_1(\alpha) = \beta_1 - F^{-1}(\alpha)$, where $\boldsymbol{\beta}$ comes from the basic model

i.i.d. analogy – Drees (1998)

Analogy to i.i.d. case result of Drees (1998): Provided that $n \to \infty$, $k = k(n) \to \infty$, $k/n \to 0$, and $\sqrt{k}A_0(n/k) = O(1)$ it holds for any $\varepsilon > 0$

$$\sup_{k^{-1}<s\leq 1} s^{\gamma+1/2+\varepsilon} \left| \sqrt{k} \left(\frac{X_{n-[ks],n} - F^{-1}\left(1-\frac{k}{n}\right)}{a_0(n/k)} - \frac{s^{-\gamma}-1}{\gamma} \right) - \gamma s^{-\gamma-1} W_n(s) - \sqrt{k} A_0\left(\frac{n}{k}\right) s^{-\gamma} \frac{s^{-\rho}-1}{\rho} \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

It allows to

- build class of consistent and asymptotically normal location and scale invariant estimators of γ as smooth functionals of $F^{-1}(1-1/t)$
- and other aims of extreme value theory - high conditional quantiles, return periods, tests etc.

Smooth tail functionals

Have an estimate $\hat{\gamma}_{n,k} = T(Q_n, k)$ defined as a smooth functional of empirical tail quantile function (i.e. $Q(t) = F^{-1}(1 - \frac{1}{t})$ and $Q_n(t) := X_{n-[k_n t]:n}$). Suppose

- $T(\gamma^{-1}(x^{-\gamma} - 1)) = \gamma$
- $T(az + b) = T(z),\ \forall a > 0, b \in \mathbb{R}$
- T is Hadamard differentiable in $z_\gamma(x) = \frac{x^{-\gamma}-1}{\gamma}$, i.e. for some signed measure $\nu T, \gamma$ holds

$$\frac{T(z_\gamma + \varepsilon y) - T(z_\gamma)}{\varepsilon} \to T'_\gamma(y) = \int_0^1 y d\nu_{T,\gamma}$$

Then

$$k_n^{1/2}(T(\hat{Q}_n) - \gamma) \to N(\lambda\mu_{T,\gamma,\rho}, \sigma^2_{T,\gamma})$$

if $k_n^{1/2} A_0(n/k) \to \lambda \in \mathbb{R}$

Bias and variance of the estimators

where

$$\begin{aligned} \mu_{T,\gamma,\rho} &:= \int_0^1 K_{\gamma,\rho} d\nu_{T,\gamma} \\ \sigma^2_{T,\gamma} &:= Var\left(\int_0^1 t^{\gamma-1} W(t) d\nu_{T,\gamma}(t)\right) \\ &= \int_0^1 \int_0^1 (st)^{\gamma-1} \min(s,t) \nu_{T,\gamma}(s) \nu_{T,\gamma}(d) \end{aligned}$$

with

$$K_{\gamma,\rho} := \begin{cases} z_{\gamma-\rho}(x) & \gamma \neq 0 \neq \rho \\ x^{-\gamma} z_0(x) & \gamma \neq 0 = \rho \\ z_0^2(x) & \gamma = 0 = \rho \end{cases}$$

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