Change point in trending regression

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Outline

1. Introduction

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2. Regression model-formulation
3. Some asymptotics

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• Procedures on stability of statistical models - structural breaks, disorder, stability, segmented regression, switching regression, change point problem, etc.

• Observations $Y_1, \ldots, Y_n$ obtained at the ordered time points $t_1 < \cdots < t_n$ such that
$Y_1, \ldots, Y_{k^*}$ —- model I
$Y_{k^*}, \ldots, Y_n$ —- model II
k^* — **change point** — unknown

• The problem: to detect (to test H_0: no change & H_1: there is a change), to identify k^* (to estimate k^*) and to estimate the model before and after the change.

• Many variants – multiple changes, abrupt changes, gradual changes, changes in various parameters, changes in distributions, independent observations, dependent observations.

• Construction of tests and estimators — various approaches as in most of the statistical problems.

• Theoretical problems and theoretical problems.

• Applications (meteorology, climatology, hydrology or environmental studies, econometric time series, statistical quality control, etc.)

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Regression model- formulation

$Y_1, \ldots, Y_n$ are observed at time points $t_1 < \cdots < t_n$:

$$
\begin{aligned}
Y_i \quad &= \mathbf{x}_i^T \boldsymbol{\beta} + e_i, \qquad & i = 1 \ldots, k^* \\
&= \mathbf{x}_i^T \boldsymbol{\beta} + \mathbf{x}_i^T \boldsymbol{\delta} + e_i, \qquad & i = k^* + 1 \ldots, n,
\end{aligned}
$$

$e_1, \ldots, e_n$ — random errors – i.i.d., zero mean, nonzero variance σ^2 and finite $E|e_i|^{2+\Delta}$ with some $\Delta > 0$

$\boldsymbol{\beta}, \boldsymbol{\delta} \neq \mathbf{0}$ — parameters

k^* .... change point

$\mathbf{x}_1, \ldots, \mathbf{x}_n$ – p-dim. design points (random or nonrandom):

nontrending regression: $\frac{1}{n}\sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^T \approx \frac{k}{n}\mathbf{C}, \ k \leq n$

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trending regression: $\mathbf{x}_i = \mathbf{h}(i/n), i = 1, \ldots, n$, h smooth nonconstant vector function

Main problems:

H_0 : no jump in regression & H_1: at most one jump

estimators of change points

Test statistics

$$T_n = \max_{p \le k < n-p} \left\{ \left(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0 \right)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} \left(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0 \right) \right\}$$

- $\widehat{\boldsymbol{\beta}}_k$ — LSE of $\boldsymbol{\beta}$ based on $Y_1, \dots, Y_k$
- $\widehat{\boldsymbol{\beta}}_k^0$ — LSE of $\boldsymbol{\beta}$ based on $Y_{k+1}, \dots, Y_n$
- $\widehat{\boldsymbol{\Sigma}}_k^{-1}$ is an estimator of the variance matrix of $\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0$

Equivalently

$$T_n = \max_{p \le k < n-p} \left\{ \mathbf{S}_k^T \mathbf{C}_k^{-1} \mathbf{C}_n (\mathbf{C}_k^0)^{-1} \mathbf{S}_k \frac{1}{\widehat{\sigma}_n^2} \right\},$$

$\mathbf{S}_k = \sum_{i=1}^k \mathbf{x}_i \widehat{e}_i, \quad k = 1, ..., n,$
$\widehat{e}_i = Y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_n, \ i = 1, \dots, n$ – - residuals

$\mathbf{C}_k = \sum_{i=1}^k \mathbf{x}_i \mathbf{x}_i^T, \quad \mathbf{C}_k^0 = \mathbf{C}_n - \mathbf{C}_k$

Test statistics

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- $\widehat{\beta}_k$ — LSE of β based on $Y_1, \dots, Y_k$
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$\mathbf{C}_k = \sum_{i=1}^k \mathbf{x}_i \mathbf{x}_i^T, \quad \mathbf{C}_k^0 = \mathbf{C}_n - \mathbf{C}_k$
$\widehat{\sigma}_n^2$ is an estimator of $var\, e_i = \sigma^2$

Critical regions

$$T_n > c_n(\alpha)$$

$c_n(\alpha)$ — critical value

α — level

Approximation of the critical values:

(i) limit distribution of T_n under H_0;

(ii) resampling methods (bootstrap)

Estimator $\widehat{k}^*$ of the change point k^* defined as such $\widehat{k}^*$ it maximizes w.r.t. k

$$\left\{(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} (\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0)\right\}$$

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Test statistics:

$$T_n = \max_{p \le k < n-p} \left\{ \left(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0 \right)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} \left(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0 \right) \right\}$$

$$T_n(\varepsilon) = \max_{\varepsilon n \le k \le (1-\varepsilon)n} \left\{ \dots \right\}, \quad 0 < \varepsilon < 1/2$$

nontrending and polynomial regression:

$$\lim_{n\to\infty} P(a(\log n)(T_n)^{1/2} \le t + b_p(\log n))$$

$$= \exp\{-2\exp\{-t\}\},\ t \in R^1,$$

$a(y) = (2\log y)^{1/2},$

$b_p(y) = 2\log y + \frac{p}{2}\log\log y - \log(2\Gamma(p/2)), y > 1,$

$\mathbf{x}_i = (1, i/n, (i/n)^2, \dots, (i/n)^{p-1})^T, \quad i = 1, \dots, n$

nontrending regression

$$T_n(\varepsilon) \to^d \sup_{\varepsilon<t<1-\varepsilon} \left\{ \frac{\sum_{i=1}^p B_i^2(t)}{t(1-t)} \right\}$$

$\{B_j(t); t \in (0,1)\}$, $j = 1, ..., p$, — independent Brownian bridges

trending regression $\mathbf{x}_i = \mathbf{h}(i/n)$

$$T_n(\varepsilon) \to^d \sup_{\varepsilon<t<1-\varepsilon} \mathbf{S}^T(t)\mathbf{C}(t)\mathbf{C}^{-1}(1)\mathbf{C}^0(t)\mathbf{S}(t)$$

$$\mathbf{S}(t) = \int_0^t \mathbf{h}(x)dB(x) - \mathbf{C}(t)\mathbf{C}^{-1}(1)\int_0^1 \mathbf{h}(x)dB(x), \quad t \in [0,1]$$

with $\{B(x),\ x \in [0,1]\}$ being a Brownian bridge,
$\mathbf{C}(t) = \lim_{n\to\infty} \frac{1}{n}\mathbf{C}_{\lfloor nt \rfloor}$.

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Trending regression

Assumptions

- (A.1) The sequence $(e_i : i \geq 1)$ satisfy
$$E[e_i] = 0, \qquad E[e_i^2] = \sigma^2 > 0 \tag{1}$$
- (A.2) There are independent standard Brownian motions $(W_{1,n}(t) : t \geq 0)$ and $(W_{2,n}(t) : t \geq 0)$ such that
$$\max_{1 \leq k \leq n/2} \frac{1}{k^{1/\nu}} \left| \sum_{i=1}^{k} e_i - \tau W_{1,n}(k) \right| = \mathcal{O}_P(1) \qquad (n \to \infty) \tag{2}$$
and
$$\max_{n/2 < k < n} \frac{1}{(n-k)^{1/\nu}} \left| \sum_{i=k+1}^{n} e_i - \tau W_{2,n}(n-k) \right| = \mathcal{O}_P(1) \qquad (n \to \infty) \tag{3}$$
with some $\nu > 2$ and $\tau > 0$.

- (A.3) The components of $\mathbf{h}(.)$ are continuous on $[0,1]$. The matrices $\int_0^t \mathbf{h}(x)\mathbf{h}^T(x)dx$ and $\int_t^1 \mathbf{h}(x)\mathbf{h}^T(x)dx$ are regular for all $t \in (t^0, 1-t^0)$ for all $t_0 \in (0, 1/2)$.
- (A.4) There are p linearly independent p-dimensional vectors $\mathbf{a}_{01}, \ldots, \mathbf{a}_{0p}$ and nonnegative $0 \le \gamma_{01} < \ldots < \gamma_{0p}$ such that

$$\lim \sup_{t \to 0_+} \frac{1}{t^{\gamma_{0p}+1}} \left| \mathbf{h}(t) - \sum_{\ell=1}^{p} \mathbf{a}_{0\ell} t^{\gamma_{0\ell}} \right| < \infty. \tag{4}$$

- (A.5) There are p linearly independent p-dimensional vectors $\mathbf{a}_{11}, \ldots, \mathbf{a}_{1p}$ and nonnegative $0 \le \gamma_{11} < \ldots < \gamma_{1p}$ such that

$$\lim \sup_{t \to 1_-} \frac{1}{t^{\gamma_{1p}+1}} \left| \mathbf{h}(t) - \sum_{\ell=1}^{p} \mathbf{a}_{1\ell} t^{\gamma_{1\ell}} \right| < \infty. \tag{5}$$

Particular cases:

• *polynomial regression*

$$\mathbf{h}(t) = (t^{\gamma_1}, \ldots, t^{\gamma_p})^T,\ t \in [0,1], \quad 0 \le \gamma_1 < \ldots < \gamma_p$$

• *harmonic regression*

$$\mathbf{h}(t) = (\cos(2\pi t\omega_1), \sin(2\pi t\omega_1), \ldots, \cos(2\pi t\omega_p), \cos(2\pi t\omega_p))^T,\ t \in [0,1]$$

$\omega_1, \ldots, \omega_p$ known.

• *independent* $e_1, \ldots, e_n$: $\tau = \sigma$.

• *dependent* $e_1, \ldots, e_n$: the estimator with flat top kernel used:

$$\widehat{\tau}_n^2 = \frac{1}{n}\sum_{j=1}^{n}(\widehat{e}_j - \overline{e}_n)^2 + \frac{2}{n}\sum_{j=1}^{q_n} w_j \sum_{i=1}^{n-j}(\widehat{e}_i - \overline{e}_n)(\widehat{e}_{i+j} - \overline{e}_n)$$

$\widehat{e}_j = Y_j - \mathbf{x}_j^T\widehat{\beta}_n, \quad \overline{e}_n = \frac{1}{n}\sum_{i=1}^n \widehat{e}_i$

$$w_j = 1I\{1 \le j \le q_n/2\} + 2(1 - j/q_n)I\{q_n/2 < j \le q_n\}$$

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Proof

Recall notation:

$T_n = \max_{p \le k < n-p} V_k(\tau^2/\widehat{\tau}_n^2)$

$V_k = \mathbf{S}_k^T \mathbf{C}_k^{-1} \mathbf{C}_n (\mathbf{C}_k^0)^{-1} \mathbf{S}_k / \tau^2$

$\mathbf{S}_k = \sum_{i=1}^k \mathbf{x}_i \widehat{e}_i, \quad k = 1, ..., n,$

$\widehat{e}_i = Y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_n, \ i = 1, \ldots, n$ – - residuals

$\mathbf{C}_k = \sum_{i=1}^k \mathbf{x}_i \mathbf{x}_i^T, \quad \mathbf{C}_k^0 = \mathbf{C}_n - \mathbf{C}_k$

$\mathbf{x}_i = \mathbf{h}(i/n)$

Main step of the proof

- The limit behavior of T_n is the same as $\max(T_{n0}, T_{n1})$

$$T_{n0} = \max_{p \le k < ns_n} V_k$$

$$T_{n1} = \max_{p \le n-k < ns_n} V_k$$

for some $s_n \to 0_+$, T_{n0} and T_{n1} are asymptotically independent.

- T_{n0} has the same limit distribution as

$$T_{n0}^0 = \max_{p \le k < ns_n} \mathbf{Z}_k^T \mathbf{M}_k^{-1} \mathbf{Z}_k / \tau^2$$

$\mathbf{Z}_k = \sum_{i=1}^k \mathbf{x}_i e_i, \quad \mathbf{M}_k = \sum_{i=1}^k \mathbf{x}_i \mathbf{x}_i^T.$

- T_{n0}^0 will not change if $\mathbf{x}_i$ is replaced by $\mathbf{x}_i^0 = \mathbf{A}\mathbf{x}_i$ with arbitrary nonsingular $p \times p$ matrix $\mathbf{A}$.

- $\max_{1 \le k \le n/2} |\mathbf{Z}_k - \tau \sum_{j=1}^k \mathbf{x}_j (W_{1,n}(j) - W_{1,n}(j-1))| = o_P(n^{-\kappa})$ for some $\kappa > 0$.

- After some steps we get that T_{n0}^0 has the same distribution as

$$\max_{k_n \le k < ns_n} \frac{1}{k\tau} \mathbf{Q}_k^T \mathbf{R}_k^{-1} \mathbf{Q}_k$$

with $\mathbf{Q}_k = \sum_{i=1}^k ((i/k)^{\gamma_{01}}, \dots, (i/k)^{\gamma_{0p}})^T e_i$
$\mathbf{R}_k = \left(\frac{1}{k}\sum_{j=1}^k (j/k)^{\gamma_{0v}+\gamma_{0r}}\right)_{v,r=1}^p.$

- Finally, we get that T_{n0}^0 has the same distribution as

$$\sup_{k_n \le t \le ns_n} \mathbf{Q}(t)^T \mathbf{V}(t)^{-1} \mathbf{Q}(t)$$

$$\mathbf{Q}(t) = \left(\frac{1}{t^{\gamma_{0\ell}+1/2}} \int_0^t x^{\gamma_{0\ell}} dW_{n,1}(x);\ \ell = 1, \dots, p\right)$$

$$\mathbf{V}(t) = \left(\frac{1}{t^{\gamma_{0\ell}+\gamma_{0\ell'}+1}} \int_0^t x^{\gamma_{0\ell}+\gamma_{0\ell'}} dx;\ \ell, \ell' = 1, \dots, p\right)$$

- After exponential transformation $\mathbf{Q}(t)^T \mathbf{V}(t)^{-1} \mathbf{Q}(t)$ is a norm of stationary Gaussian process results of Albin, Piterbarg, Jaruskova etc. are applied.

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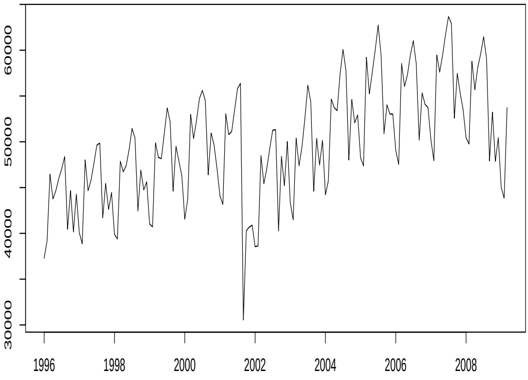

Figure 1: Monthly air carrier traffic in the United Stated from January 1996 to March 2009.

Simulation

$$Y_j = 4 + \frac{6j}{n} + \frac{3}{2}\cos(\frac{2\pi j}{12n}) + \frac{3}{2}\sin(\frac{2\pi}{12n}) + \frac{9}{10}\cos(\frac{2\pi}{4n} + \frac{1}{2}\sin(\frac{2\pi}{4n}) + e_j,$$

$j = 1, \ldots, 200$

e_j –either AR(1) or MA(1), normal distr.

Change point $k^* = 100$ either in the intercept or in one harmonic regressor

1000 repetitions

critical value obtained through circular block bootstrap

$H_A^{(1)}$ change in intercept, AR (1) or MA(1)
$H_A^{(1)}$ change in one regressor, AR (1) or MA(1)

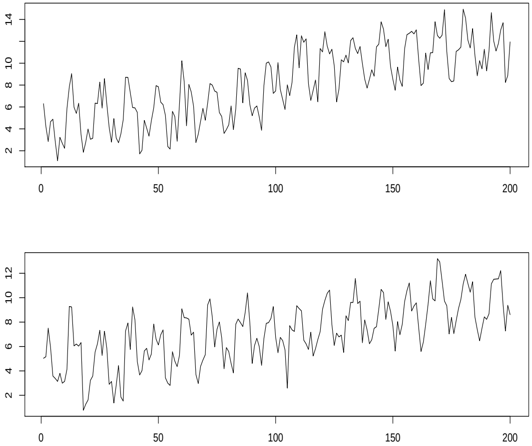

Figure 4: Time series plots of the processes under $H_A^{(1)}$ with $\delta = 2$ (upper panel) and $H_A^{(2)}$ (lower

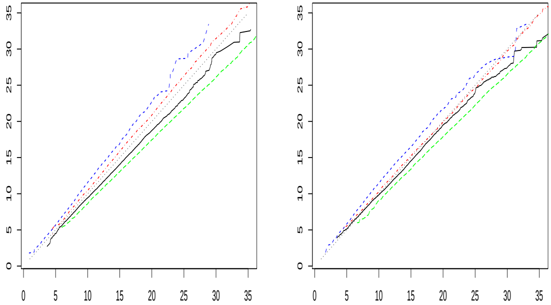

Figure 2: Circular bootstrap distribution versus finite sample distribution for AR(1) innovations (left) and MA(1) innovations (right) with scalings $\varphi^2 = 2$ (- - -), 1.5 (·-·-·), 1.0 (—), 0.5 (- - -).

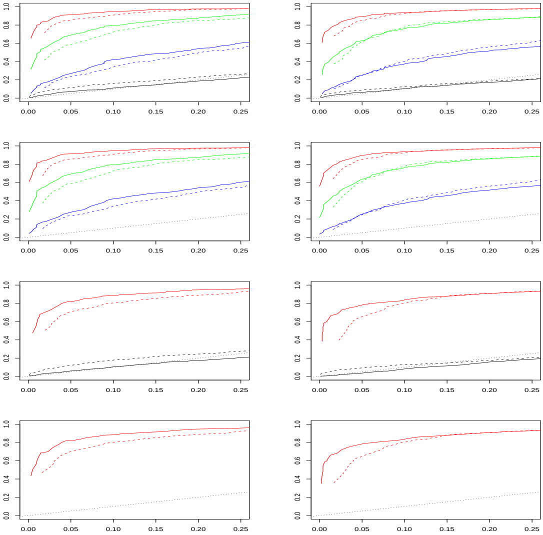

Figure 3: Size-power curves for $H_A^{(1)}$ (upper half) and $H_A^{(2)}$ (lower half) for the asymptotic modification (first and third line) and the the circular bootstrap (second and fourth line) with AR(1) innovations (left) and MA(1) innovations (right).

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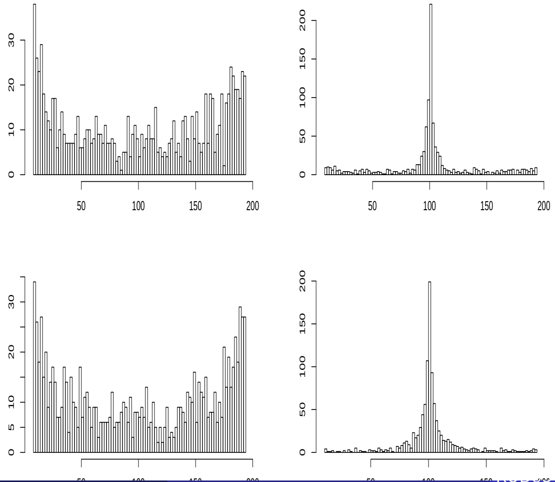
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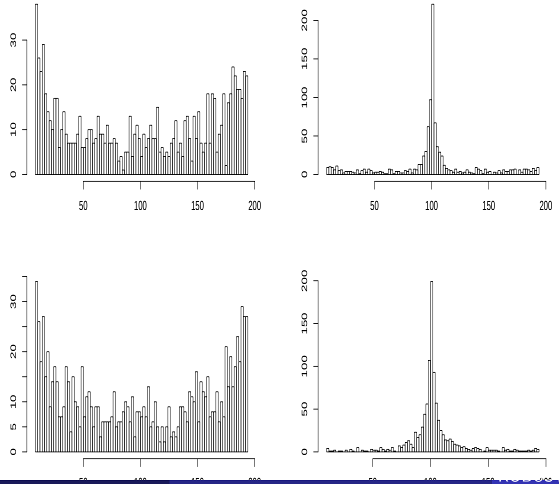
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Data

Monthly air traffic data

model through the root of data, $n = 159$

$$Y_j = \beta_0 + \beta_1 j/n + \sum_{\ell=1}^{q} \Big(\beta_\ell^c \cos(2\pi\omega_{ell} j/n) + \beta_\ell^{(s)} \sin(2\pi\omega_{ell} j/n) \Big) + e_j$$

$q = 4$, $\omega_1 = 2/160$, $\omega_3 = 13/160$, $\omega_3 = 40/160$, $\omega_1 = 80/160$

ω_2 – annual cycle

ω_3 – quarterly cycle

ω_4 —two months cycle

$\widehat{k}^* = 69$

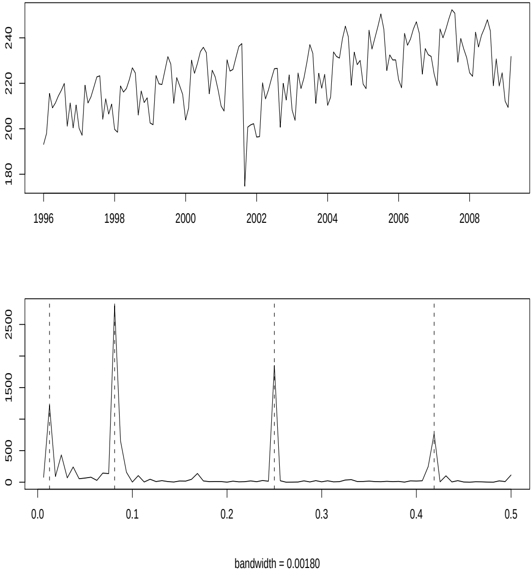

Figure 6: Square root transformation of the monthly air carrier traffic data (upper panel) and its

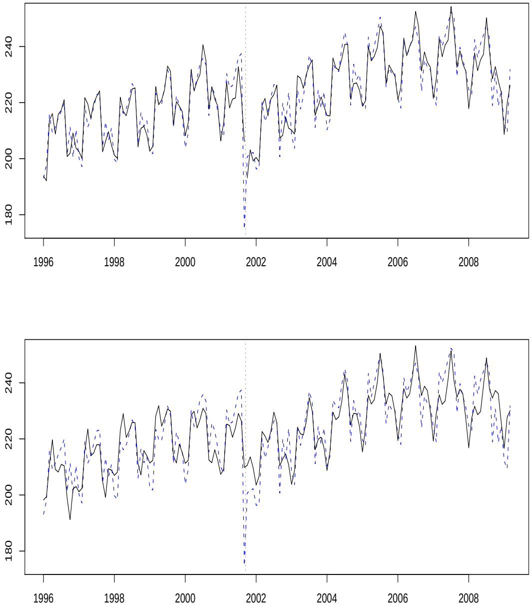

Figure 7: The fitted model based on the proposed data segmentation procedure (upper panel) and