Odhady a testy v modelu s chybami měření

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Consider the linear regression model

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + e_i, \quad i = 1, \ldots, n \tag{1}$$

with observations $Y_1, \ldots, Y_n$,

$e_1, \ldots, e_n$, are independent errors – identically distributed according to an unknown distribution function F;

$\mathbf{x}_{ni} = (x_{i1}, \ldots, x_{ip})'$ is the vector of covariates, $i = 1, \ldots, n$,

$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$, and $\boldsymbol{\beta}^* = (\beta_0, \boldsymbol{\beta}')'$ are unknown parameters.

The measurement of the covariates can be affected by random errors:

We observe
$w_{ij} = x_{ij} + v_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, p,$
where $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})',\ i = 1, \ldots, n$ are random measurement errors.

We assume that random vectors $\mathbf{v}_i,\ i = 1, \ldots, n$ are independent and identically distributed, independent of $e_1, \ldots, e_n$, but their distribution is unknown.

We shall consider two models:

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + e_i, \quad i = 1, \dots, n \tag{2}$$

where we want to test the hypothesis

$$\mathbf{H}: \ \boldsymbol{\beta} = \mathbf{0}, \tag{3}$$

and

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + \mathbf{z}'_{ni}\boldsymbol{\delta} + e_i, \quad i = 1, \dots, n \tag{4}$$

with unknown parameters $\boldsymbol{\beta} \in \mathbb{R}^p, \ \boldsymbol{\delta} \in \mathbb{R}^q$, and the hypothesis

$$\mathbf{H}: \ \boldsymbol{\delta} = \mathbf{0}, \tag{5}$$

considering β_0 and $\boldsymbol{\beta}$ as a nuisance parameters.

Problem: test the hypotheses in such models if the covariates are measured with random errors.

Regression rank score tests

Consider the model

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + \mathbf{z}'_{ni}\boldsymbol{\delta} + e_i, \quad i = 1, \ldots, n \tag{6}$$

with unknown parameters $\beta_0 \in \mathbb{R}^1,\ \boldsymbol{\beta} \in \mathbb{R}^p,\ \boldsymbol{\delta} \in \mathbb{R}^q$, and the hypothesis

$$\mathbf{H}_0 :\ \boldsymbol{\delta} = \mathbf{0},$$

considering β_0 and $\boldsymbol{\beta}$ as a nuisance parameters.

Gutenbrunner et al. (1993) constructed a class of tests of $\mathbf{H}_0$ based on regression rank scores, that are invariant to the nuisance parameters $\beta_0, \boldsymbol{\beta}$.

R. Koenker a G. Basset (1978) defined the α-regression quantile $\widehat{\boldsymbol{\beta}}\,(\alpha)$ $(0 < \alpha < 1)$ for the model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}$ as any solution of the minimization

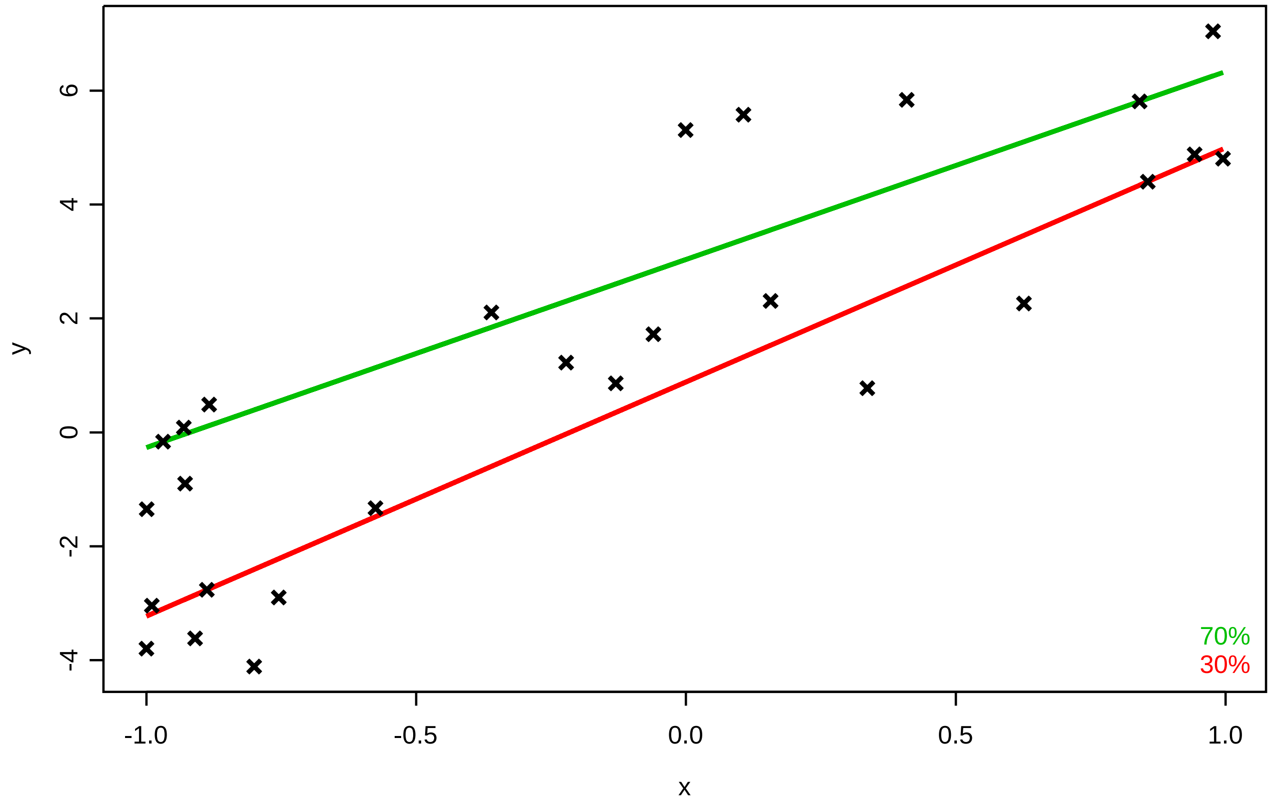
$$\sum_{i=1}^{n} \rho_\alpha(Y_i - \mathbf{x}_i'\mathbf{t}) := \text{min}, \quad \mathbf{t} \in I\!R^{p+1}, \tag{7}$$

where

$$\rho_\alpha(x) = x\psi_\alpha(x),\ x \in I\!R^1 \tag{8}$$

and

$$\psi_\alpha(x) = \alpha - I_{[x<0]},\ x \in I\!R^1.$$



The advantage of this approach is that many aspects of usual quantiles and order statistics are generalized naturally to the linear model.

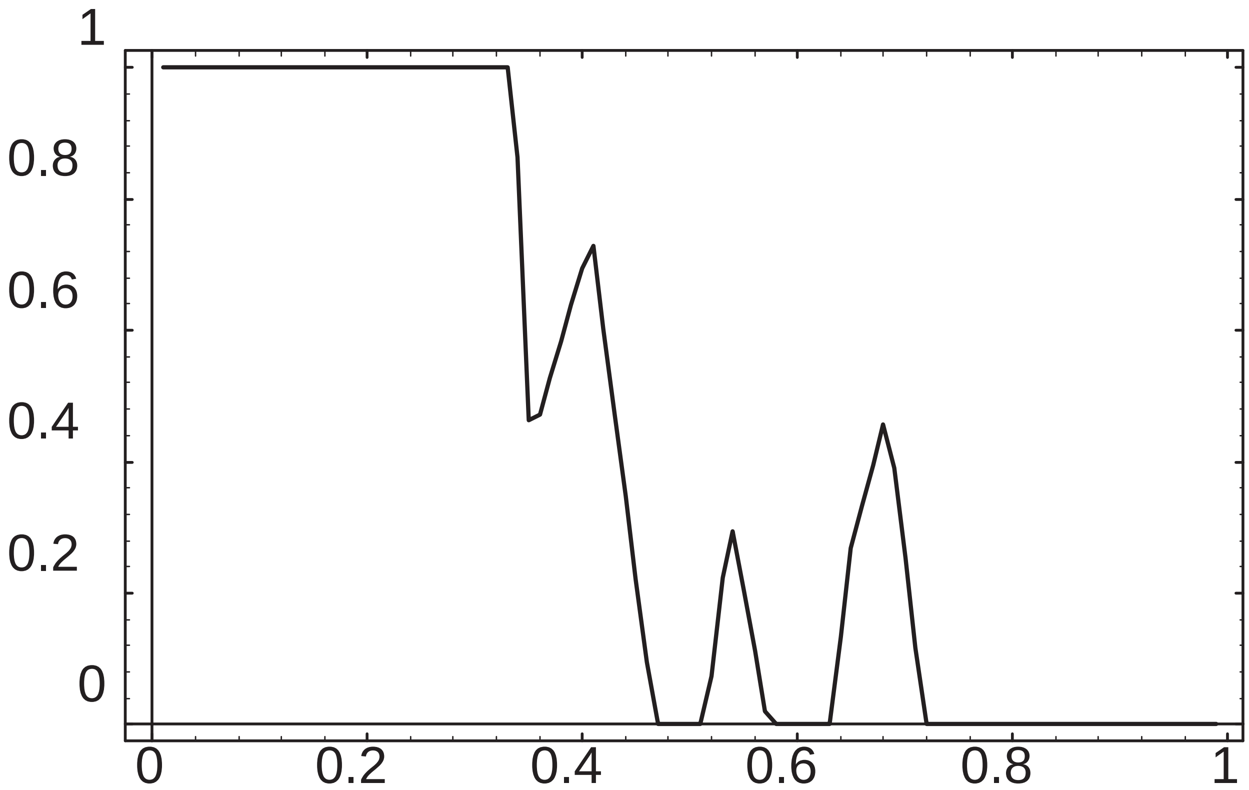
The same authors characterized the α-regression quantile $\widehat{\boldsymbol{\beta}}(\alpha)$ as the component $\widehat{\boldsymbol{\beta}}$ of the optimal solution $(\widehat{\boldsymbol{\beta}}, \mathbf{r}^+, \mathbf{r}^-)$ of the linear program

$$\begin{gathered}
\alpha \mathbf{1}_n' \mathbf{r}^+ + (1-\alpha)\mathbf{1}_n' \mathbf{r}^- := \min \\
\mathbf{X}\widehat{\boldsymbol{\beta}} + \mathbf{r}^+ - \mathbf{r}^- = \mathbf{Y} \\
\widehat{\boldsymbol{\beta}} \in I\!\!R^{p+1}, \mathbf{r}^+, \mathbf{r}^- \in I\!\!R^n_+,\ 0 < \alpha < 1,\ \mathbf{1}_n = (1, \ldots, 1)' \in I\!\!R^n.
\end{gathered} \tag{9}$$

The formal dual program to (9) can be written in the form

$$\begin{gathered}
\mathbf{Y}_n' \hat{\mathbf{a}} := \max \\
\mathbf{X}_n' \hat{\mathbf{a}} = (1-\alpha)\mathbf{X}'\mathbf{1}_n \\
\hat{\mathbf{a}} \in [0,1]^n, \quad 0 < \alpha < 1
\end{gathered} \tag{10}$$

The components of the optimal solutions $\hat{\mathbf{a}}(\alpha) = (\hat{a}_1(\alpha), \ldots, \hat{a}_n(\alpha))'$ are called the regression rank scores. (Gutenbrunner and Jurečková 1992)


1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Motivation:

In the location model with $\mathbf{X} = \mathbf{1}_n$ we have $\hat{a}_{ni}(\alpha) \equiv a_n^*(R_i, \alpha)$, where R_i is the rank of Y_i among $Y_1, \ldots, Y_n$ and

$$a_n^*(R_i, \alpha) = \begin{cases} 1 & \text{if } \alpha \le (R_i - 1)/n \\ R_i - \alpha n & \text{if } (R_i - 1)/n < \alpha \le R_i/n, \\ 0 & \text{if } R_i/n < \alpha \end{cases}$$

The score function $a_n^*(R_i, \alpha)$ were first used by Hájek (1965) as a starting point for various rank tests.

A general class of tests based on regression rank scores, parallel to classical rank tests as the Wilcoxon, normal scores and median, was constructed in Gutenbrunner et al. (1993)

Typically, the test based on regression rank scores apply to our model

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + \mathbf{z}'_{ni}\boldsymbol{\delta} + e_i, \quad i = 1, \ldots, n \tag{11}$$

with unknown parameters $\beta_0 \in \mathbb{R}^1$, $\boldsymbol{\beta} \in \mathbb{R}^p$, $\boldsymbol{\delta} \in \mathbb{R}^q$, and the hypothesis

$$\mathbf{H}_0 : \ \boldsymbol{\delta} = \mathbf{0}, \quad \beta_0 \text{ and } \boldsymbol{\beta} \text{ unspecified}$$

versus the Pitman (local) alternatives

$$H_n : \boldsymbol{\delta} = n^{-1/2}\boldsymbol{\delta}_0.$$

First, calculate the regression rank scores $\hat{\mathbf{a}}(\alpha) = (\hat{a}_1(\alpha), \ldots, \hat{a}_n(\alpha))'$ under H_0, i.e. they correspond to the submodel

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + e_i, \quad i = 1, \ldots, n$$

Then calculate the scores $\hat{b}_{ni}$ generated by φ in the form

$$\hat{b}_{ni} = -\int_0^1 \varphi(t)\ d\hat{a}_{ni}(t),\ \ i = 1, \ldots, n \tag{12}$$

and the q-dimensional vector of *linear regression rank scores statistics*

$$\mathbf{S}_n = n^{-1/2}(\mathbf{Z}_n - \widetilde{\mathbf{Z}}_n)'\widehat{\mathbf{b}}_n, \quad \widehat{\mathbf{b}}_n = (\hat{b}_{n1}, \ldots, \hat{b}_{nn})' \tag{13}$$

where

$$\widetilde{\mathbf{Z}}_n = \widetilde{\mathbf{H}}_n\mathbf{Z}_n,\ \widetilde{\mathbf{H}}_n = \widetilde{\mathbf{X}}_n(\widetilde{\mathbf{X}}'_n\widetilde{\mathbf{X}}_n)^{-1}\widetilde{\mathbf{X}}'_n$$

is the projection of $\mathbf{Z}_n$ on the space spanned by the columns of $\widetilde{\mathbf{X}}_n$. For the simplicity of notation:

$$\widetilde{\mathbf{X}}_n = \left[\mathbf{1}_n \vdots\ \mathbf{X}_n\right] \tag{14}$$

The test criterion for the hypothesis $\mathbf{H}_0$ is

$$\mathcal{T}_n^2 = (A(\varphi))^{-2}\ \mathbf{S}_n'\ \widetilde{\mathbf{D}}_n^{-1}\ \mathbf{S}_n \tag{15}$$

where

$$\widetilde{\mathbf{D}}_n = n^{-1}(\mathbf{Z}_n - \widetilde{\mathbf{Z}}_n)'(\mathbf{Z}_n - \widetilde{\mathbf{Z}}_n). \tag{16}$$

The asymptotic distribution of $\mathcal{T}_n^2$ under $\mathbf{H}_0$ is the χ^2 distribution with q degrees of freedom.

Consider the situation that we still want to test the hypothesis $\mathbf{H}_0$, in the model

$$Y_i = \beta_0 + \mathbf{x}'_{ni}\boldsymbol{\beta} + \mathbf{z}'_{ni}\boldsymbol{\delta} + e_i, \quad i = 1, \ldots, n \tag{17}$$

the vectors $\mathbf{z}_{ni}$ can be only determined with additive errors, i.e. we observe $\mathbf{w}_{ni} = \mathbf{z}_{ni} + \mathbf{v}_{ni}$, $i = 1, \ldots, n$, where the $\mathbf{v}_{ni} \in \mathbb{R}^q$ are random errors. Denote

$$\mathbf{W}_n = \begin{bmatrix} \mathbf{w}_{n1} \\ \ldots \\ \mathbf{w}_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{V}_n = \begin{bmatrix} \mathbf{v}_{n1} \\ \ldots \\ \mathbf{v}_{nn} \end{bmatrix} \tag{18}$$

the $n \times q$ matrices and $\widetilde{\mathbf{W}}_n = \widetilde{\mathbf{H}}_n \mathbf{W}_n$ and $\widetilde{\mathbf{V}}_n = \widetilde{\mathbf{H}}_n \mathbf{V}_n$ their projections on the space spanned by the columns of $\widetilde{\mathbf{X}}_n$,

$$\widetilde{\mathbf{H}}_n = \widetilde{\mathbf{X}}_n (\widetilde{\mathbf{X}}'_n \widetilde{\mathbf{X}}_n)^{-1} \widetilde{\mathbf{X}}'_n$$

Test:

Calculate the regression rank scores $\hat{\mathbf{a}}(\alpha)$, then the scores $\hat{b}_{ni}$ generated by φ.
We replace the statistic $\mathbf{S}_n$ by

$$\widetilde{\mathbf{S}}_n = n^{-1/2}(\mathbf{W}_n - \widetilde{\mathbf{W}}_n)'\widehat{\mathbf{b}}_n \tag{19}$$

and the test criterion $\mathcal{T}_n^2$ is:

$$\widetilde{\mathcal{T}}_n^2 = (A(\varphi))^{-2}\ \widetilde{\mathbf{S}}'_n\ (\widetilde{\mathbf{D}}_n + \mathbf{G}_n)^{-1}\ \widetilde{\mathbf{S}}_n, \tag{20}$$

where

$$\widetilde{\mathbf{D}}_n + \mathbf{G}_n = n^{-1}(\mathbf{W}_n - \widetilde{\mathbf{W}}_n)'(\mathbf{W}_n - \widetilde{\mathbf{W}}_n). \tag{21}$$

Under $\mathbf{H}_0$, it has asymptotic χ^2 distribution with q degrees of freedom.

Aligned rank tests

Let us turn back to the model

$$Y_i = \beta_0 + \mathbf{x}_{ni}'\boldsymbol{\beta} + \mathbf{z}_{ni}'\boldsymbol{\delta} + e_i, \quad i = 1, \ldots, n \tag{22}$$

Consider the problem that the $\mathbf{x}_{ni}$ are observed only with errors, and we observe $\mathbf{w}_{ni} = \mathbf{x}_{ni} + \mathbf{v}_{ni}$ instead of $\mathbf{x}_{ni},\ i = 1, \ldots, n$. We want again to test the hypothesis

$$\mathbf{H}_0 : \ \boldsymbol{\delta} = \mathbf{0},$$

considering β_0 and $\boldsymbol{\beta}$ as a nuisance parameters.

Unfortunately, the parallel test based on regression rank scores is not asymptotically distribution free without additional conditions on the model.

A possible construction of a distribution free test in this line is still an open problem.

Solution:
Replace the nuisance slope parameter $\boldsymbol{\beta}$ with an estimator $\widehat{\boldsymbol{\beta}}_n$ and then construct the test based on aligned ranks of the residuals.

The rank estimator of $\boldsymbol{\beta}$:

$$\widehat{\boldsymbol{\beta}}_n = \arg\min\{\|\mathbf{L}_n(\mathbf{b})\| : \ \mathbf{b} \in \mathbb{R}^p\} \tag{23}$$

where $\|\cdot\|$ can be the $L_1,\ L_2$ norms, or eventually the sup-norm.

The vector of aligned rank statistics

$$\mathbf{L}_n(\mathbf{b}) = n^{-1/2} \sum_{i=1}^{n} (\mathbf{w}_{ni} - \bar{\mathbf{w}}_n) a_n(\tilde{R}_{ni}(\mathbf{b})) \tag{24}$$

where $\tilde{R}_{ni}(\mathbf{b})$ stands for the aligned rank of $Y_i - \mathbf{w}'_{ni}\mathbf{b}$ among $Y_1 - \mathbf{w}'_{n1}\mathbf{b}, \ldots, Y_n - \mathbf{w}'_{nn}\mathbf{b},\ \mathbf{b} \in \mathbb{R}^p$.

The test criterion for the hypothesis $\mathbf{H}_0 : \ \boldsymbol{\delta} = \mathbf{0}$ is based on the vector:

$$\mathbf{S}_n^*(\widehat{\boldsymbol{\beta}}_n) = n^{-1/2} \sum_{i=1}^{n} \Big(\mathbf{z}_{ni} - \widehat{\mathbf{z}}_{ni} - \frac{1}{n} \sum_{j=1}^{n} (\mathbf{z}_{nj} - \widehat{\mathbf{z}}_{nj}) \Big) a_n(\tilde{R}_{ni}(\widehat{\boldsymbol{\beta}}_n)) \quad (25)$$

and the test criterion is

$$\mathcal{T}_n^* = (A(\varphi))^{-2} \left[(\mathbf{S}_n^*(\widehat{\boldsymbol{\beta}}_n))' \mathbf{C}_n^{-1} \mathbf{S}_n^*(\widehat{\boldsymbol{\beta}}_n) \right], \quad (26)$$

where

$$\mathbf{C}_n = n^{-1} \sum_{i=1}^{n} \Big(\mathbf{z}_{ni} - \widehat{\mathbf{z}}_{ni} - \bar{\mathbf{z}}_n \Big) \Big(\mathbf{z}_{ni} - \widehat{\mathbf{z}}_{ni} - \bar{\mathbf{z}}_n \Big)' \quad (27)$$

with $\bar{\mathbf{z}}_n = \frac{1}{n} \sum_{j=1}^{n} (\mathbf{z}_{nj} - \widehat{\mathbf{z}}_{nj})$,

Under $\mathbf{H}_0$, the asymptotic distribution of $\mathcal{T}_n^*$ is central χ^2 with q degrees of freedom.

Numerical illustration:

The power of the tests: means of the frequency of rejections

$\beta_0 = 1$ and $\beta_1 = 1$.

aligned Wilcoxon test proposed ($\varphi(u) = u - \frac{1}{2}$)

The regressors $x_{n1}, \ldots, x_{nn}, z_{n1}, \ldots, z_{nn}$ were simulated from the uniform distribution, independently of the errors, for $n = 30$ and 500.

The measurement errors $v_i,\ i = 1, \ldots, n$ were generated from the normal distribution with various parameters.

10 000 replications were simulated and the aligned Wilcoxon test with the Wilcoxon R-estimator of β_1 was performed every time on level $\alpha = 0.05$; the mean power was then calculated.

	δ	$v_i \equiv 0$	$v_i \sim \mathcal{N}(0, 0.25)$	$v_i \sim \mathcal{N}(0, 1)$
$n = 30$	$\delta = -0.4$	0.745	0.706	0.644
	$\delta = -0.3$	0.632	0.608	0.604
	$\delta = -0.2$	0.412	0.389	0.367
	$\delta = -0.1$	0.168	0.138	0.123
	$\delta = 0.0$	0.059	0.064	0.066
	$\delta = 0.2$	0.413	0.389	0.369
	$\delta = 0.4$	0.743	0.705	0.650
$n = 500$	$\delta = 0.0$	0.049	0.048	0.046
	$\delta = 0.1$	0.891	0.872	0.855
	$\delta = 0.2$	0.997	0.981	0.958
	$\delta = 0.3$	1.000	0.999	0.976
	$\delta = 0.4$	1.000	1.000	0.998
	$\delta = 0.5$	1.000	1.000	1.000

Empirical power of the aligned test, e_i's have $\mathcal{N}(0, 1)$

	δ	$v_i \equiv 0$	$v_i \sim \mathcal{N}(0, 0.25)$	$v_i \sim \mathcal{N}(0, 1)$
$n = 30$	$\delta = 0.5$	0.858	0.832	0.794
	$\delta = 0.4$	0.674	0.656	0.610
	$\delta = 0.3$	0.542	0.521	0.506
	$\delta = 0.2$	0.367	0.345	0.332
	$\delta = 0.1$	0.139	0.117	0.092
	$\delta = 0.0$	0.057	0.061	0.063
$n = 500$	$\delta = 0.0$	0.052	0.054	0.058
	$\delta = 0.1$	0.821	0.792	0.705
	$\delta = 0.2$	0.937	0.901	0.858
	$\delta = 0.3$	0.995	0.959	0.906
	$\delta = 0.4$	1.000	0.993	0.968
	$\delta = 0.5$	1.000	1.000	0.997

Empirical power of the aligned test, e_i's have Laplace distribution

	δ	$v_i \equiv 0$	$v_i \sim \mathcal{N}(0, 0.25)$	$v_i \sim \mathcal{N}(0, 1)$
$n = 30$	$\delta = 0.5$	0.580	0.432	0.354
	$\delta = 0.4$	0.467	0.265	0.190
	$\delta = 0.3$	0.299	0.178	0.106
	$\delta = 0.2$	0.172	0.098	0.076
	$\delta = 0.1$	0.068	0.057	0.056
	$\delta = 0.0$	0.037	0.031	0.029
$n = 500$	$\delta = 0.0$	0.055	0.058	0.059
	$\delta = 0.1$	0.510	0.320	0.205
	$\delta = 0.2$	0.747	0.501	0.384
	$\delta = 0.3$	0.901	0.742	0.556
	$\delta = 0.4$	0.997	0.890	0.693
	$\delta = 0.5$	1.000	0.987	0.876

Empirical power of the aligned test, e_i's have Cauchy distribution

R-Estimation

Consider the multiple linear regression model with measurement errors

$$\begin{aligned} y_i &= \beta_0 + \boldsymbol{\beta}_1^T \mathbf{x}_i + e_i, \quad i = 1, \ldots, n, \\ \mathbf{x}_i^0 &= \mathbf{x}_i + \mathbf{u}_i, \quad i = 1, \ldots, n, \end{aligned} \tag{28}$$

where e_i is the response error and $\mathbf{u}_i$ is the vector of measurement errors on the regressor vector $\mathbf{x}_i$, which is unobservable while $\mathbf{x}_i^0$ is the corresponding observed vector. Here $(\beta_0, \boldsymbol{\beta}_1^T)^T$ are the unknown intercept and slope parameters of the model (28). Our problem is the rank estimation of the parameters $(\beta_0, \boldsymbol{\beta}_1^T)^T$.

Let $R_{ni}(\mathbf{b})$ be the rank of $y_i - \mathbf{b}^T\mathbf{x}_i^0$ among $y_1 - \mathbf{b}^T\mathbf{x}_1^0, \ldots, y_n - \mathbf{b}^T\mathbf{x}_n^0$. We consider scores $a_n^\varphi(1), \ldots, a_n^\varphi(n)$ generated by a non-decreasing, square integrable functions $\varphi : (0,1) \to \mathbb{R}$:

$$a_n^\varphi(i) = \varphi\left(\frac{i}{n+1}\right),$$

Define the linear rank statistics (LRS) for the slope parameter $\boldsymbol{\beta}_1$ as

$$\mathbf{L}_n(\mathbf{b}) = n^{-1/2} \sum_{i=1}^{n} (\mathbf{x}_i^0 - \bar{\mathbf{x}}_n^0) a_n^\varphi(R_{ni}(\mathbf{b})). \tag{29}$$

We define the R-estimator of $\boldsymbol{\beta}_1$ as

$$\hat{\boldsymbol{\nu}}_{1n}^{(R)} = \{\mathbf{b};\ ||\mathbf{L}_n(\mathbf{b})|| \text{ is minimum}\}.$$

R-estimator of the slope consistently estimate $\boldsymbol{\kappa}_x \boldsymbol{\beta}_1$ not $\boldsymbol{\beta}_1$, where $\boldsymbol{\kappa}_x = (\mathbf{C}^2 + \mathbf{D}^2)^{-1}\mathbf{C}^2$ and

$$\mathbf{C}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}}_n)(\mathbf{x}_i - \bar{\mathbf{x}}_n)^T, \quad \lim_{n\to\infty} \mathbf{C}_n^2 = \mathbf{C}^2$$

$$\mathbf{D}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{u}_i - \bar{\mathbf{u}}_n)(\mathbf{u}_i - \bar{\mathbf{u}}_n)^T \to \mathbf{D}^2.$$

$$\hat{\boldsymbol{\beta}}_{1n}^{(R)} = \hat{\boldsymbol{\kappa}}_x^{-1} \hat{\boldsymbol{\nu}}_{1n}^{(R)}$$

and

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_{1n}^{(R)} - \boldsymbol{\beta}_1) \sim \mathsf{N}_p \left(\mathbf{0},\ \frac{A_\varphi^2}{\gamma^2(\varphi, f)} (\mathbf{C}^2 \boldsymbol{\kappa}_x)^{-1} \right).$$

We considered the model

$$y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + e_i, \quad i = 1, \ldots, n, \qquad (30)$$
$$x^0_{i,j} = x_{i,j} + u_{i,j}, \quad i = 1, \ldots, n, \; j = 1, 2$$

where the errors $e_i,\ i = 1, \ldots, n$, were simulated from the normal $N(0, 1)$, the Laplace $L(0, 1)$ and the Cauchy distributions. The measurement errors $u_{i,j},\ i = 1, \ldots, n;\ j = 1, 2$, were generated independently from the normal $N(0, 0.5)$, $N(0, 1)$ and the uniform $U(-\sqrt{3}, \sqrt{3})$ distributions. Concerning the design points we consider vectors, where $x_{1,1}, \ldots, x_{n,1}$ were generated from the uniform distribution on the interval (-2,10) and $x_{1,2}, \ldots, x_{n,2}$ from the uniform distribution on the interval (0,30). Firstly, the values were generated but then they stayed fixed for all simulations.

The following parameter values of model were used:

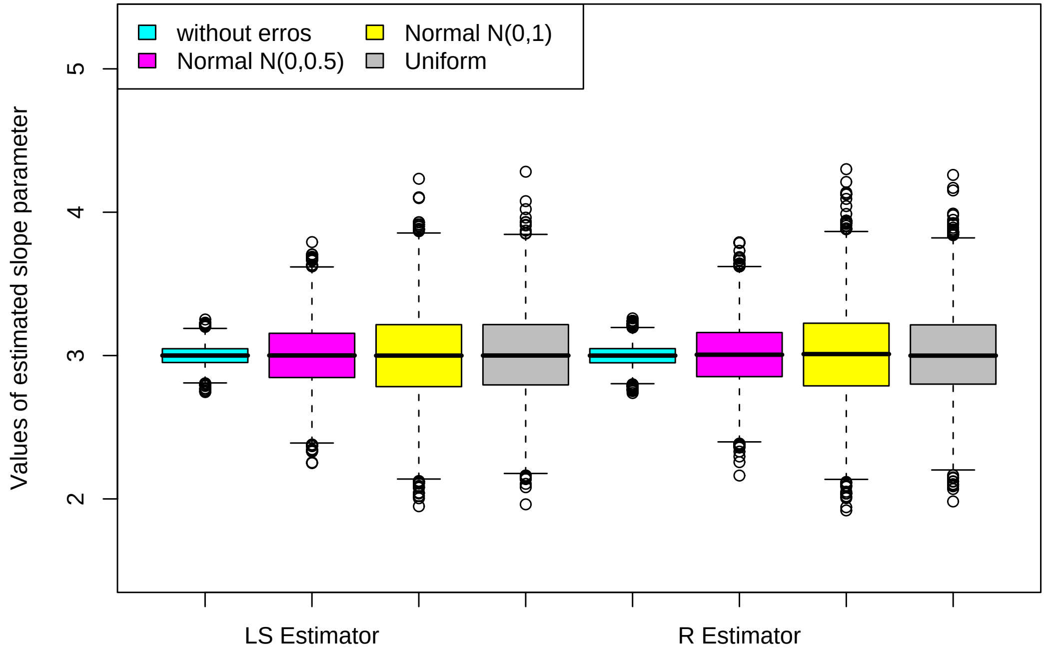
- sample sizes $n = 20,\ 100,\ 500$;
- $\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = -1$;
- Wilcoxon score function $\varphi(u) = 2u - 1,\ 0 \le u \le 1$ and $A^2(\varphi) = \frac{1}{12}$;
- L_2 norm;
- L_1 estimates as the initial values for the optimization problem

10 000 replications of the model were simulated for each combination of the parameters and the distribution measurement errors and the least square of estimator, the R-estimator of slope parameters were computed. For the sake of comparison, the mean square error (MSE), mean, median, 2.5%– and 97.5%–sample quantiles were computed.

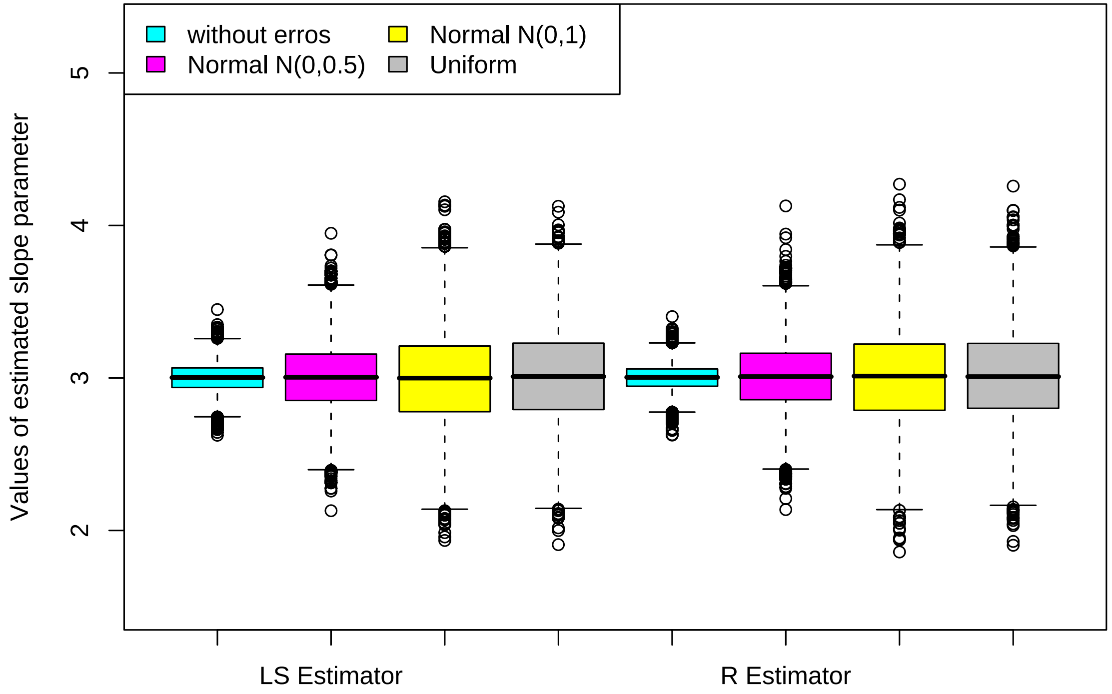
Sample statistics of 10 000 values of estimated slope parameters in the model (30) for the Least Squares and the R-estimator under the various distributions of the measurement error $u_j,\ j = 1, \dots, n$, the sample size $n = 20$, the standard normal distribution of the errors $e_i, i = 1, \dots, n$:

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
β_1						
LSE	–	0.00484	2.99951	2.86354	3.00038	3.13573
	$N(0, 0.5)$	0.05016	3.00132	2.55915	3.00074	3.43186
	$N(0, 1)$	0.09685	3.00096	2.40490	2.99965	3.61651
	$U(-\sqrt{3}, \sqrt{3})$	0.09375	3.00192	2.40661	3.00040	3.60667
R	–	0.00523	2.99931	2.85895	2.99988	3.13877
	$N(0, 0.5)$	0.05182	3.00641	2.56010	3.00583	3.45059
	$N(0, 1)$	0.09951	3.01159	2.41556	3.01038	3.63535
	$U(-\sqrt{3}, \sqrt{3})$	0.09364	3.00506	2.41158	2.99957	3.60526

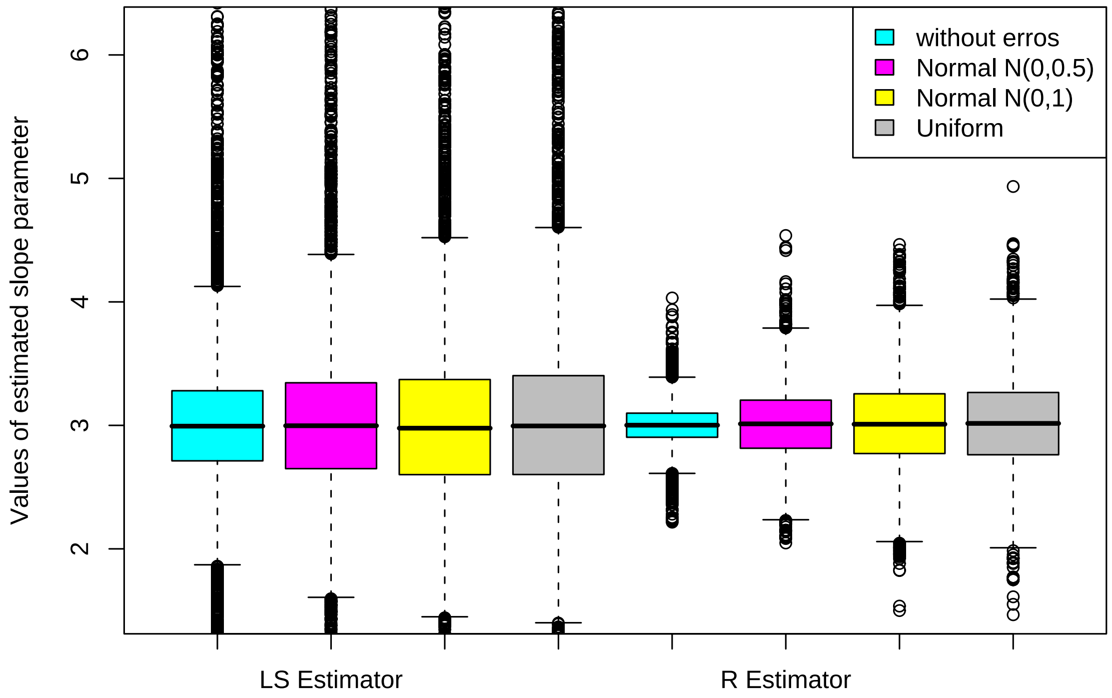
Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
β_2						
LSE	–	0.00072	-1.00087	-1.05359	-1.00081	-0.94804
	$N(0, 0.5)$	0.00200	-1.00057	-1.08714	-0.99997	-0.91208
	$N(0, 1)$	0.00325	-1.00063	-1.11189	-1.00050	-0.89034
	$U(-\sqrt{3}, \sqrt{3})$	0.00312	-1.00047	-1.11025	-1.00059	-0.89249
R	–	0.00077	-1.00103	-1.05594	-1.00071	-0.94634
	$N(0, 0.5)$	0.00217	-1.00041	-1.09071	-0.99973	-0.90777
	$N(0, 1)$	0.00350	-1.00019	-1.11611	-0.99922	-0.88555
	$U(-\sqrt{3}, \sqrt{3})$	0.00350	-1.00119	-1.11819	-1.00131	-0.88784



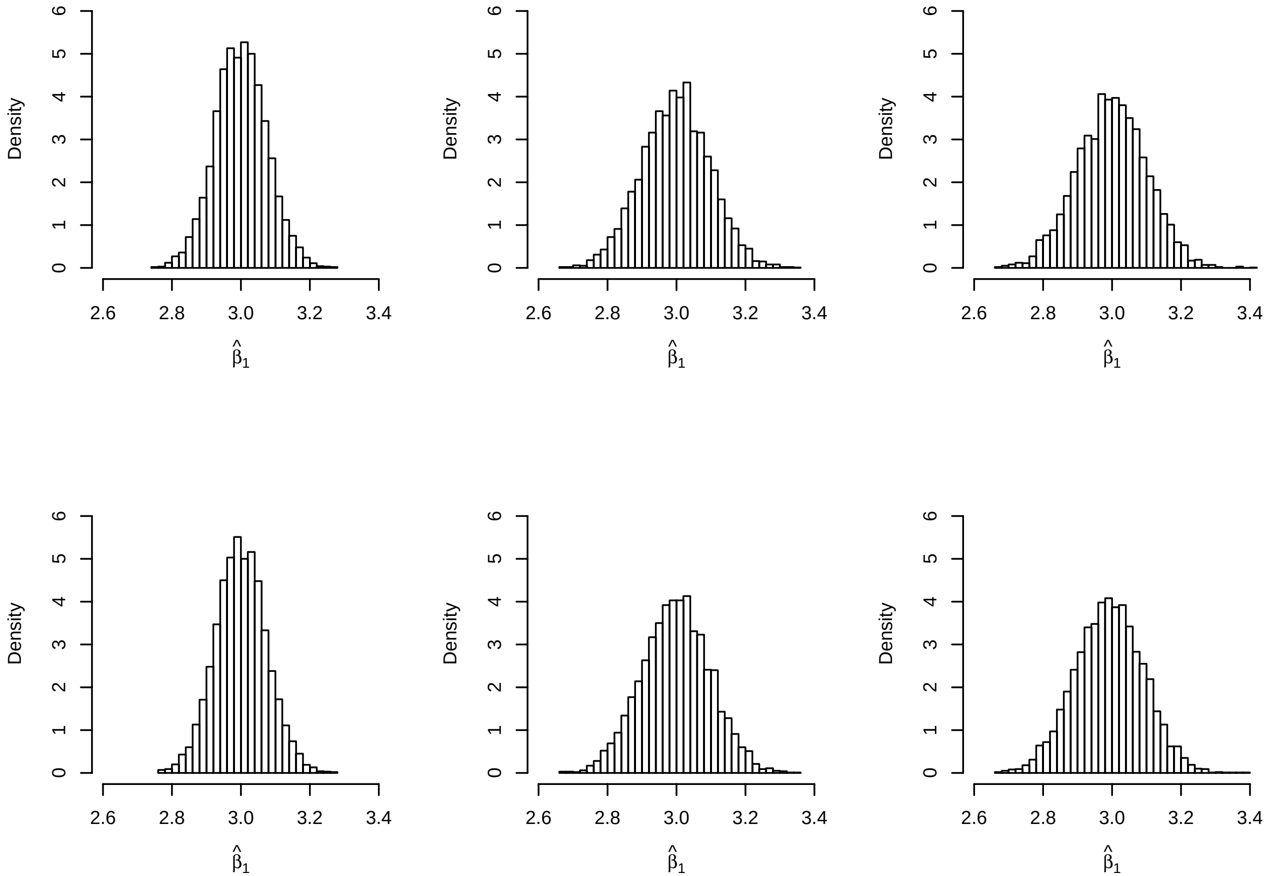
Box-plots of the 10 000 estimated values of slope parameter β_1 for the Least Squares and the R-estimator under the various distributions of the measurement error $u_j,\ j = 1, \ldots, n$, and the standard normal distribution of errors $e_i,\ i = 1, \ldots, n$, the sample size $n = 20$.



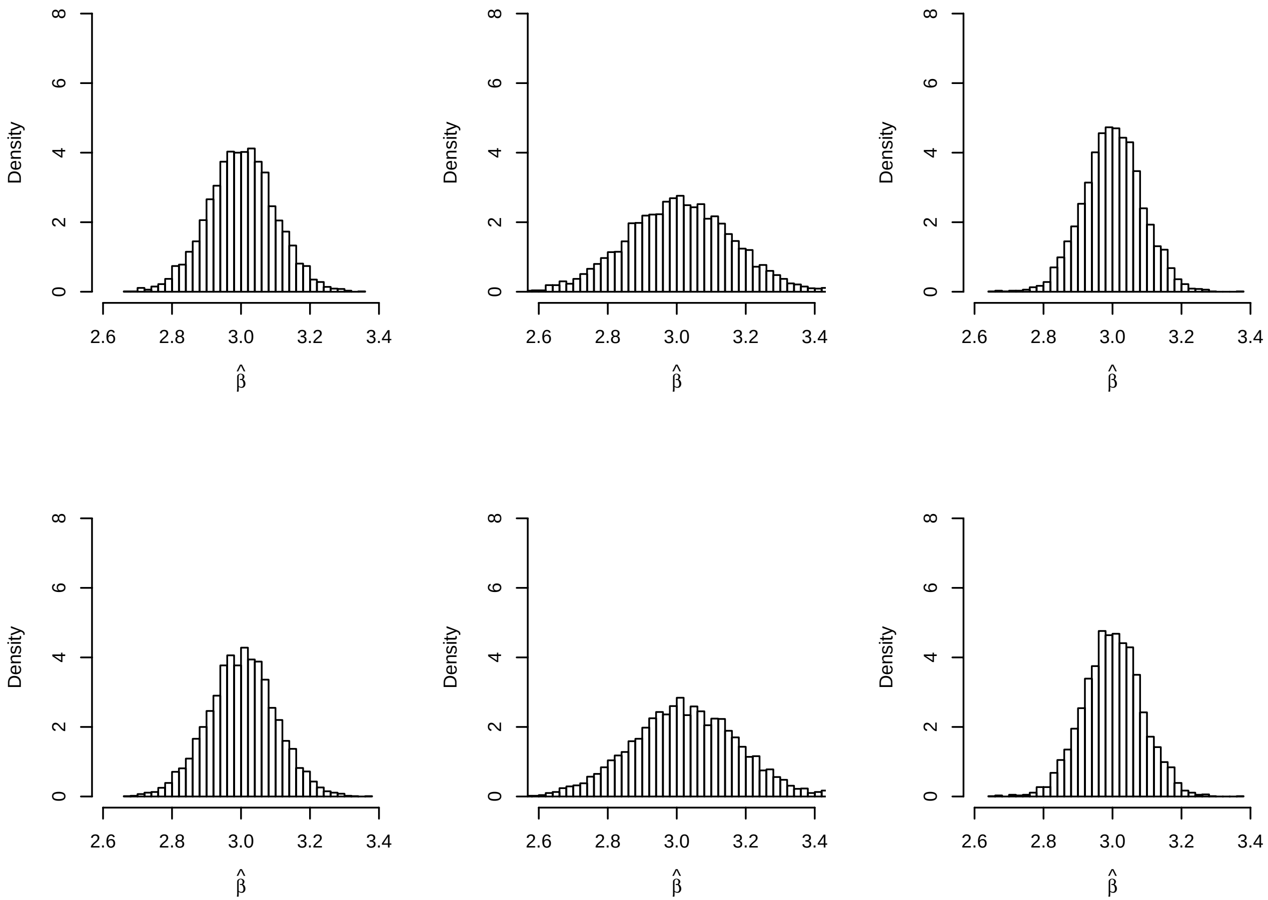
Box-plots of the 10 000 estimated values of slope parameter β_1 for the Least Squares and the R-estimator under the various distributions of the measurement error $u_j,\ j = 1, \ldots, n$, and the Laplace distribution of errors $e_i,\ i = 1, \ldots, n$, the sample size $n = 20$.



Box-plots of the 10 000 estimated values of slope parameter β_1 for the Least Squares and the R-estimator under the various distributions of the measurement error u_j, $j = 1, \ldots, n$, and the Cauchy distribution of errors e_i, $i = 1, \ldots, n$, the sample size $n = 20$.



LS-estimator (top) and the R-estimator (bottom) under the normal $N(0, 0.5)$ (left), the normal $N(0, 1)$ (middle) and the uniform $U(-\sqrt{3}, \sqrt{3})$ (right) distributions of the measurement error $u_j,\ j = 1, \ldots, n$, Laplace distribution of errors, $n = 100$.



LS-estimator (top) and the R-estimator (bottom) under the normal $N(0, 0.5)$ (left), the normal $N(0, 1)$ (middle) and the uniform $U(-\sqrt{3}, \sqrt{3})$ (right) distributions of the measurement error $u_j,\ j = 1, \ldots, n$, Cauchy distribution of errors, $n = 100$.