Power tessellation as a tool for estimating parameters in a model of a random set

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based on joint work with Jesper Møller, David Dereudre and Frederic Lavancier

1st February 2010

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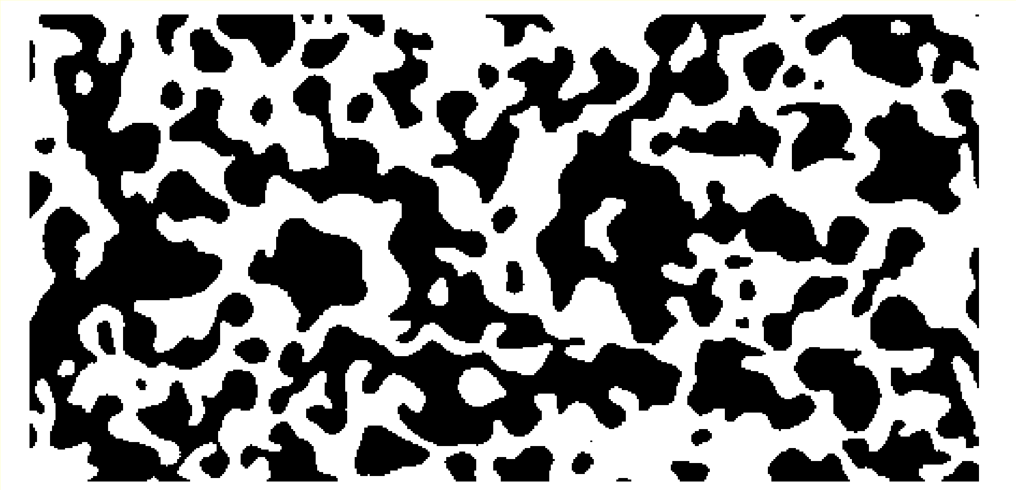
Motivation



Heather dataset first presented by Peter Diggle in 1981. The image shows the presence of heather (indicated by black) in a 10×20 m region at Jädraås, Sweeden.

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1. Motivation
2. Describing the model
3. Simulations
4. Power tessellation of a union of discs
5. Estimating the parameters by MCMC MLE
6. Estimating the parameters using integral characterization

Point processes

Definition Consider N the system of locally finite subsets of $\mathbb{R}^d$ with the σ-algebra $\mathcal{N} = \sigma(\{\mathbf{x} \in N : \sharp(\mathbf{x} \cap A) = m\} : A \in \mathcal{B}, m \in \mathbf{N}_0)$. *A point process* X defined on $\mathbb{R}^d$ is a measurable mapping from some probability space $(\Omega, \mathcal{F}, P)$ to $(N, \mathcal{N})$.

Definition A locally finite, diffusion measure μ on $\mathcal{B}$ satisfying $\mu(A) = \mathrm{E}X(A)$ for all $A \in \mathcal{B}$ is called *the intensity measure*.

Definition If there exists a function $\rho(x)$ for $x \in \mathbb{R}^d$ such that $\mu(A) = \int_A \rho(x)dx$, then $\rho(x)$ is called *the intensity function*.

Definition If $\rho(x) = \rho$ is constant then the constant ρ is called *intensity*.

Poisson point process

Definition *The Poisson process* Y is the process which satisfies:

- for any finite collection $\{A_n\}$ of disjoint sets in $\mathbb{R}^d$, the numbers of points in these sets, $Y(A_n)$, are independent random variables,
- for each $A \subset \mathbb{R}^d$ such that $\mu(A) < \infty$, $Y(A)$ has Poisson distribution with parameter $\mu(A)$, i.e. $P[Y(A) = k] = \frac{\mu(A)^k}{k!} e^{-k}$, where μ is the intensity measure.

Point process given by the density with respect to Poisson process

Let Y be the Poisson process with an intensity measure μ.

For $F \in \mathcal{N}$, denote $\Pi(F) = P(Y \in F)$.

Definition A point process X is given by density f with respect to the Poisson process Y if

$$P(X \in F) = \int_F f(\mathbf{x})\Pi(d\mathbf{x}).$$

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Model

Denoting $b = b(u, r)$ a disc with centre in $u \in \mathbb{R}^2$ and radius $r \in (0, \infty)$, we have a process of discs $\cup b_i = \cup b(u_i, r_i)$. Then, we identify b with the point $x = (u, r)$ in $\mathbb{R}^2 \times (0, \infty)$ and the process of discs $\cup b_i = \cup b(u_i, r_i)$ with a point process in $\mathbb{R}^2 \times (0, \infty)$.

Model

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The reference process: A Poisson point process Y (so that the reference Boolean model is the random set given by the union of discs in Y) with intensity measure $\rho(u)\,\mathrm{d}u\,Q(\mathrm{d}r)$ on $\mathbb{R}^2 \times (0, \infty)$.

Model: The process of discs $\mathbf{X}$ such that the corresponding point process is absolutely continuous with respect to the reference Poisson process Y, and given by density $f(\mathbf{x})$ for a finite configurations $\mathbf{x} = \{x_1, \ldots, x_n\}$.

Assumption: $\mathbf{X}$ is a finite point process defined on $S \times (0, R)$, where S denotes a given bounded planar region such that $\int_S \rho(u)\,\mathrm{d}u > 0$ and $R < \infty$.

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Exponential family density

General form of the density:

$$f_\theta(\mathbf{x}) = \exp\left(\theta \cdot T(\mathcal{U}_\mathbf{x})\right)/c_\theta$$

Set $T = (A, L, N_{\mathrm{cc}}, N_{\mathrm{h}})$, where
$A = A(\mathcal{U}_\mathbf{x})$...the area
$L = L(\mathcal{U}_\mathbf{x})$...the perimeter
$N_{\mathrm{cc}} = N_{\mathrm{cc}}(\mathcal{U}_\mathbf{x})$...the number of connected components
$N_{\mathrm{h}} = N_{\mathrm{h}}(\mathcal{U}_\mathbf{x})$...the number of holes,

i.e. the density is of the form

$$f_\theta(\mathbf{x}) = \frac{1}{c_\theta} \exp\left(\theta_1 A(\mathcal{U}_\mathbf{x}) + \theta_2 L(\mathcal{U}_\mathbf{x}) + \theta_3 N_{\mathrm{cc}}(\mathcal{U}_\mathbf{x}) + \theta_4 N_{\mathrm{h}}(\mathcal{U}_\mathbf{x})\right).$$

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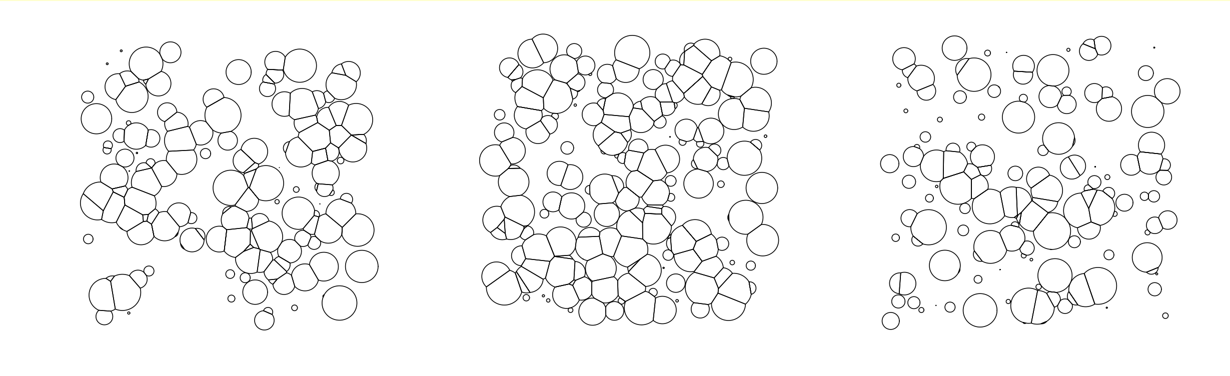
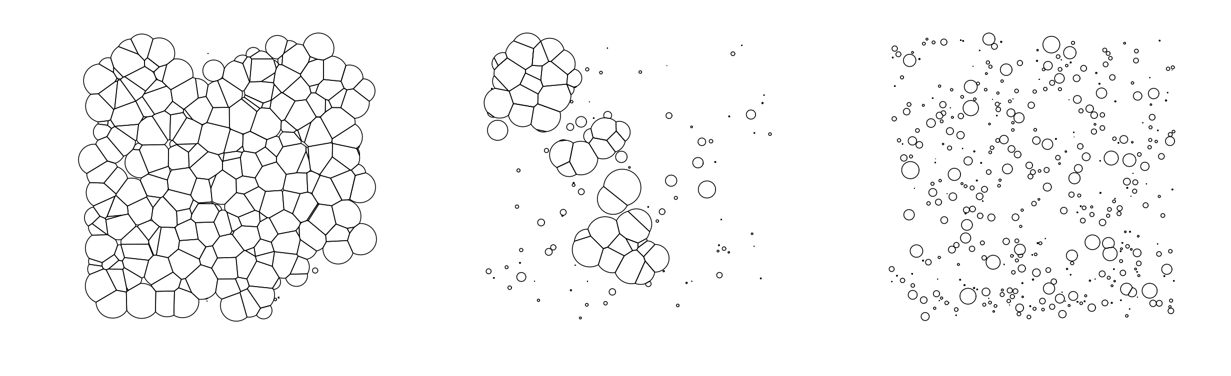
Example of simulations



A realization of the reference Poisson process with Q the uniform distribution on the interval $[0, 2]$, $\rho(u) = 0.2$ on a rectangular region $S = [0, 30] \times [0, 30]$, and $\rho(u) = 0$ outside S (left) and A-interaction model with parameters $\theta_1 = 0.1$ (middle), resp. $\theta_1 = -0.1$ (right).

Example of simulations



(A, L, N_{cc})-interaction process, where $N_{\mathrm{cc}}(\mathcal{U}_{\mathbf{x}})$ is the number of connected components, with parameters $(0.6, -1, 1)$ (left), $(0.6, -1, 2)$ (middle) and $(0.6, -1, 5)$ (right).

Papangelou conditional intensity

Definition For finite $\mathbf{x} \subset S \times (0, \infty)$ and $v \in S \times (0, \infty) \setminus \mathbf{x}$, *Papangelou conditional intensity* is defined as

$$\lambda_\theta(\mathbf{x}, v) = f_\theta(\mathbf{x} \cup \{v\}) / f_\theta(\mathbf{x}).$$

Denoting

$$\begin{aligned} A(\mathbf{x}, v) =& A(\mathbf{x} \cup v) - A(\mathbf{x}), \\ L(\mathbf{x}, v) =& L(\mathbf{x} \cup v) - L(\mathbf{x}), \\ &\vdots \end{aligned}$$

we get

$$\lambda_\theta(\mathbf{x}, v) = \exp\left(\theta_1 A(\mathbf{x}, v) + \theta_2 L(\mathbf{x}, v) + \theta_3 N_{cc}(\mathbf{x}, v) + \theta_4 N_h(\mathbf{x}, v)\right).$$

MCMC algorithm

1. Suppose that in time t, we have a configuration $\mathbf{x_t} = \{x_1, \ldots, x_n\}$
2. Proposal in time $t+1$:
   (a) with probability 1/2, the proposal is $\mathbf{x_t} \cup \{x_{n+1}\}$
      i. we accept the proposal with probability $min\{1; H(\mathbf{x_t}, x_{n+1})\}$ and set $\mathbf{x_{t+1}} = \mathbf{x_t} \cup \{x_{n+1}\}$
      ii. else we set $\mathbf{x_{t+1}} = \mathbf{x_t}$
   (b) else, the proposal is $\mathbf{x_t} \backslash \{x_i\}$
      i. we accept the proposal with probability $min\{1; 1/H(\mathbf{x_t} \backslash \{x_i\}, x_i)\}$ and set $\mathbf{x_{t+1}} = \mathbf{x_t} \backslash \{x_i\}$
      ii. else $\mathbf{x_{t+1}} = \mathbf{x_t}$

where $H(\mathbf{x_t}, x_{n+1}) = \lambda_\theta(\mathbf{x_t}, x_{n+1}) \frac{|S|}{\rho(x_{n+1}) \cdot (n+1)}$
and $H(\mathbf{x_t} \backslash \{x_i\}, x_i) = \lambda_\theta(\mathbf{x_t} \backslash \{x_i\}, x_i) \frac{|S|}{\rho(x_i) \cdot n}$.

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Power tessellation of a union of discs

Assume a union of discs $\mathcal{U} = \cup_I b_i$ in the general position.

For each disc b_i $(i \in I)$ with ghost sphere s_i, let $s_i^+ = \{(y_1, y_2, y_3) \in s_i : y_3 \geq 0\}$ denote the corresponding upper hypersphere.

For $u \in b_i$, let $y_i(u)$ denote the unique point on s_i^+ those orthogonal projection on $\mathbb{R}^2$ is u.

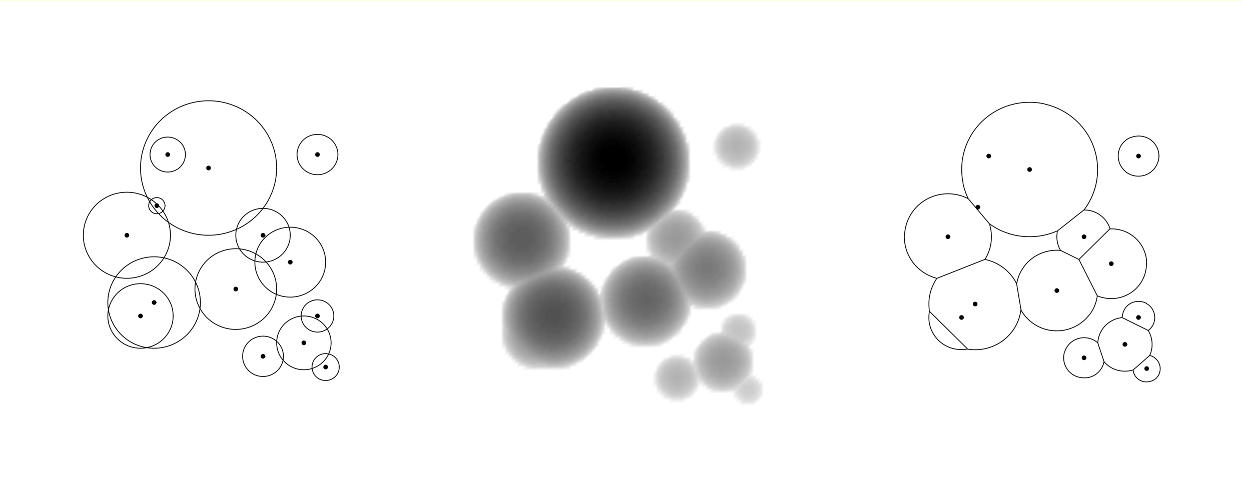
Define

$$C_i = \{y_i(u) : u \in b_i,\ \|u - y_i(u)\| \geq \|u - y_j(u)\| \text{ for } u \in b_j,\ j \in I\}.$$

Denote B_i the orthogonal projection of C_i on $\mathbb{R}^2$.

Definition The system $\mathcal{B}$ of all sets B_i is called a *power tessellation of a union of discs*.

Power tessellation of a union of discs



Left: A configuration of discs in general position. Middle: The upper hemispheres as seen from above. Right: The power tessellation of the union of discs.

Usefulness of power tessellation in MCMC algorithm

1. Calculation of $A(\mathcal{U}_{\mathbf{x}})$: instead of

$$A(\mathcal{U}_{\mathbf{x}}) = \sum_i A(b_i) - \sum_{\{i_1,i_2\}} A(b_{i_1} \cap b_{i_2}) + \dots$$

$$+(-1)^{n+1} \sum_{\{i_1,\dots,i_n\}} A(b_{i_1} \cap \dots \cap b_{i_n})$$

we use

$$A(\mathcal{U}_{\mathbf{x}}) = \sum_i A(B_i).$$

2. Analogously we calculate $L(\mathcal{U}_{\mathbf{x}})$.

Usefulness of power tessellation in MCMC algorithm

3. For calculation of $N_{\mathrm{h}}(\mathcal{U}_{\mathbf{x}})$, we use Euler-Poincaré characteristic $\chi(\mathcal{U}_{\mathbf{x}})$ satisfying $\chi(\mathcal{U}_{\mathbf{x}}) = N_{\mathrm{cc}}(\mathcal{U}_{\mathbf{x}}) - N_{\mathrm{h}}(\mathcal{U}_{\mathbf{x}})$:
   from its definition $\chi(K_i) = 1$ for K_i compact convex and $\chi(K) = \sum_{k=1}^{N} (-1)^{k+1} \sum_{\{i_1,\dots,i_k\}} \chi(K_{i_1} \cap \dots \cap K_{i_k})$ for $K = \cup_{i=1}^{N} K_i$, we have that

$$\chi(\mathcal{U}_{\mathbf{x}}) = N_{\mathrm{c}}(\mathcal{U}_{\mathbf{x}}) - N_{\mathrm{ie}}(\mathcal{U}_{\mathbf{x}}) + N_{\mathrm{iv}}(\mathcal{U}_{\mathbf{x}}),$$

   where N_{c} is the number of cells, N_{ie} the number of interior edges and N_{iv} the number of interior vertices.

4. All the calculations are local.

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Estimating parameters

Denote $f_\theta(\mathbf{x}) = h_\theta(\mathbf{x})/c_\theta$ (i.e. $h_\theta(\mathbf{x}) = \exp(\theta \cdot T(\mathcal{U}_\mathbf{x}))$ is the unnormalized density).

For an observation $\mathbf{x}$, the log likelihood function is given by

$$l(\theta) = \log h_\theta(\mathbf{x}) - \log c_\theta = \theta \cdot T(\mathcal{U}_\mathbf{x}) - \log c_\theta.$$

Problem: c_θ has no explicit expression.

Estimating parameters

For fixed θ_0, the log likelihood ratio

$$l(\theta) - l(\theta_0) = \log(h_\theta(\mathbf{x})/h_{\theta_0}(\mathbf{x})) - \log(c_\theta/c_{\theta_0})$$

can be approximated by

$$l(\theta) - l(\theta_0) = \log(h_\theta(\mathbf{x})/h_{\theta_0}(\mathbf{x})) - \log \frac{1}{n} \sum_{m=0}^{n-1} h_\theta(Y_m)/h_{\theta_0}(Y_m),$$

where Y_m are realizations from $f_{\theta_0}(\mathbf{x})$ obtained from MCMC simulations.

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Integral characterization

Assume for simplicity that all the discs have the same radii r and denote B_r the set of all such discs.

If the set of the discs centers $S = \mathbb{R}^2$ and the reference process $\mathbf{Y}$ as well as the disc process $\mathbf{X}$ are stationary then for an arbitrary measurable function $g : N \times B_r \to \mathbb{R}$ it holds that

$$\mathrm{E}\sum_{x\in\mathbf{X}} g(\mathbf{X}\setminus x, x) = \rho\mathrm{E}\int_{\mathbf{R}^2} g(\mathbf{X}, y)\lambda_\theta(\mathbf{X}, y)\mathrm{d}u,$$

where u is the center of the disc y.

Possible approximations

If the observation window W for the data $\mathbf{x}$ is large enough then we can use approximation

$$\sum_{x \in \mathbf{x}} g(\mathbf{x} \setminus x, x) = \rho \sum_{u \in W_{grid}} g(\mathbf{x}, y) \lambda_\theta(\mathbf{x}, y) \tag{1}$$

$$= \rho \sum_{u \in W_{grid}} g(\mathbf{x}, y) \exp\left(\theta_1 A(\mathbf{x}, y) + \ldots + \theta_4 N_h(\mathbf{x}, y)\right).$$

where W_{grid} is a discretization of W.

Choosing suitable function(s) g and solving (1), we obtain estimations of the parameters.

For calculating λ_θ in (1), the power tessellation is used again.

Thank you for your attention!