Consistent and equivariant estimation in errors-in-variables models with dependent errors

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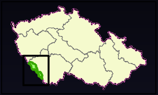
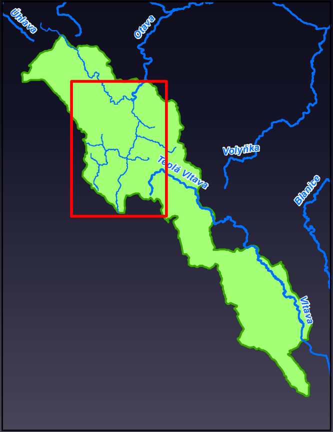
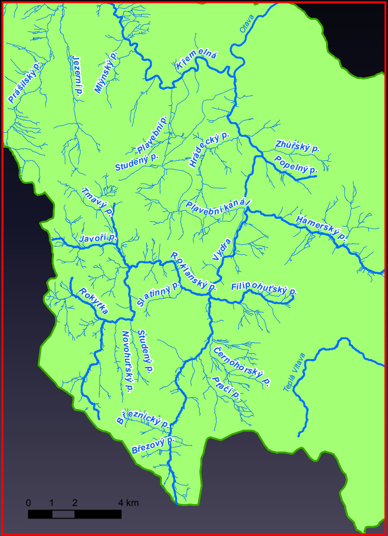
Salmo trutta morpha *fario*; L., 1758

Šumava




Úhlava
Otava
Volyňka
Teplá Vltava
Blanice
Vltava


Prášilský p.
Jezerní p.
Mlýnský p.
Křemelná
Otava
Plavební p.
Studený p.
Hrádecký p.
Zhůřský p.
Popelný p.
Tmavý p.
Plavební kanál
Hamerský p.
Javoří p.
Roklanský p.
Vydra
Slatinný p.
Filipohuťský p.
Rokytka
Novohuťský p.
Studený p.
Černohorský p.
Ptačí p.
Teplá Vltava
Březnický p.
Březový p.
0 1 2 4 km

Vydra (Javoří potok)

Pitfalls, problems, and our approach

- length and weight

Pitfalls, problems, and our approach

- covariate or response

Pitfalls, problems, and our approach

- covariate or response
- least squares or least absolute distance or ... ML

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- water depth and changing conditions

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- data dependence

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- data dependence
- measurement units

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- data dependence
- equivariant estimate (consistency, asymptotic normality)

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- data dependence
- equivariant estimate (consistency, asymptotic normality)
- unknown quantities

Pitfalls, problems, and our approach

- covariate or response
- invariant penalization
- data dependence
- equivariant estimate (consistency, asymptotic normality)
- computational feasibility

Outline

Errors-in-variables estimation
- EIV Model
- Equivariant estimate

Inference
- Assumptions of the EIV model
- Asymptotic properties of the estimate

Bootstrapping
- Moving block bootstrap

Conclusions

Errors-in-Variables (EIV) Model

$$\underset{n\times 1}{\mathbf{Y}} = \underset{n\times p}{\mathbf{Z}}\,\underset{p\times 1}{\boldsymbol{\beta}} + \underset{n\times 1}{\boldsymbol{\varepsilon}}$$

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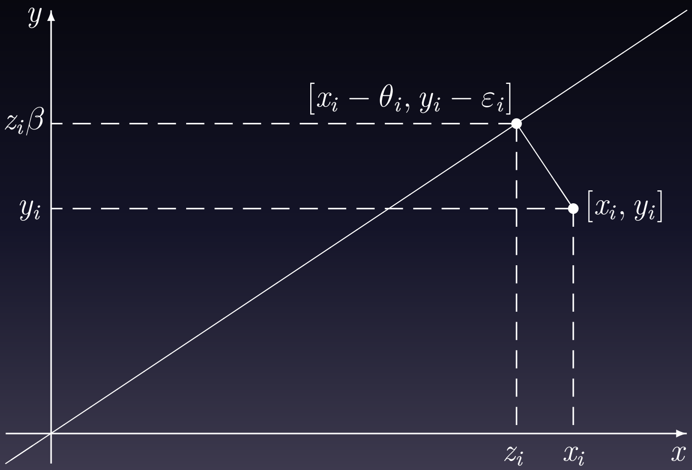
- $\boldsymbol{\varepsilon}$ and $\boldsymbol{\Theta}$ . . . random errors
- $\boldsymbol{\beta}$ . . . regression parameters (to be estimated)

Illustration



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- for a unitary invariant matrix norm

$$\min \|[\mathbf{\Theta}, \varepsilon]\| \quad \text{s.t.} \quad \mathbf{Y} - \varepsilon = (\mathbf{X} - \mathbf{\Theta})\boldsymbol{\beta}$$

Class of the UI matrix norms

- Schatten norms

$$\|\mathbf{A}\|_q = \left(\sum_{i,j} a_{ij}^q\right)^{1/q} = \left(\sum_i \sigma_i^q\right)^{1/q}, \; q \geq 1$$

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$$\|\mathbf{A}\|_q^{(k)} = \left(\sum_{i=1}^{k} \sigma_i^q \right)^{1/q}, \ q \geq 1$$

operator norm ($k = 1$), Schatten norms ($k =$last)

Estimate

- solution with desired properties for any UI MN

$$\boxed{\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X} - \lambda \mathbf{I})^{-1} \mathbf{X}^\top \mathbf{Y}}$$

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λ is the $(p+1)$-st largest eigenvalue of $[\mathbf{X}, \mathbf{Y}]^\top [\mathbf{X}, \mathbf{Y}]$

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Weak dependence

- strong mixing (α-mixing)

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)|$$

$$\alpha(n) = \sup_{k \in \mathbb{N}} \alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0, \; n \to \infty$$

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- uniformly strong mixing $\Rightarrow$ strong mixing

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Assumptions of the EIV model

- rows $[\mathbf{\Theta}_{i,\bullet}, \varepsilon_i]$ are α- or φ-mixing
- rows $[\mathbf{\Theta}_{i,\bullet}, \varepsilon_i]$ with zero mean and non-singular covariance matrix $\sigma^2\mathbf{I}$, where σ^2 is unknown (for simplicity)
- exists a positive definite matrix

$$\mathbf{\Delta} := \lim_{n\to\infty} n^{-1}\mathbf{Z}^\top\mathbf{Z}$$

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Asymptotic properties

- consistency under uniformly strong mixing

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- asymptotic normality under stationary strong mixing and finite $(4+\delta)$-th moment of errors

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{\mathscr{D}} \mathscr{N}(\mathbf{0}, \cdot), \quad n \to \infty$$

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- ex.: $H_0: \beta = 2$ vs $H_1: \beta \neq 2$

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- consistency and asymptotic normality

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- EIV with weakly dependent errors
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- consistency and asymptotic normality
- MBB correctness

Bibliography

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