Modelling with Jump Processes and Optimal Control

ROBUST 2010

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January 31 - February 5, 2010, Králíky

Are continuous-time financial models satisfactory?

Empirical facts of financial time series:

- Sudden large movements, heavy tails
  - Diffusion models (DM): extremely large volatility term needs to be added, events cannot be accounted for
  - Jump models (JM): generic property

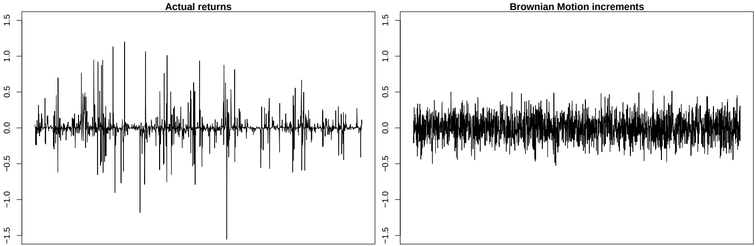

Figure: **Left:** Asset returns observed every 6 seconds. **Right:** Brownian motion increments with the same mean and variance.

Are continuous-time financial models satisfactory?

Empirical facts cont.:

- Volatility clustering
  - partially overcome by time change
- Asymmetric returns (respectively log returns)
  - JM: selecting appropriate model

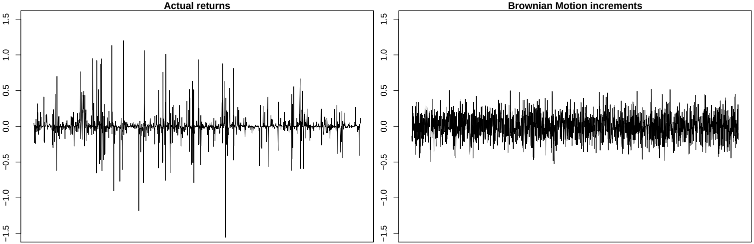

Figure: **Left:** Asset returns observed every 6 seconds. **Right:** Brownian motion increments with the same mean and variance.

Optimal Control - Model set-up

An agent invests in two assets

- A riskfree bond that pays interest rate r,
- A risky asset with dynamics

$$\mathrm{d}S(t) = S(t^-)\left(\alpha \mathrm{d}t + \sigma \mathrm{d}W_t + \int_{-1}^{\infty} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)\right). \qquad (2.1)$$

An investor controls

- the number of stocks Δ_t in his portfolio,
- consumption $C_t \geq 0$,

for $t \geq 0$.

Model set-up cont.

Notation

- $\theta_t = \frac{\Delta_t S_{t-}}{X_{t-}}$ is the proportion of capital invested in risky asset at time t,
- $c_t = \frac{C_t}{X_{t-}}$ denotes the consumption proportion.

Objective

$$v(x) = \sup_{(\Delta_t, C_t) \in \mathcal{A}(x)} \int_0^\infty e^{-\beta t} \mathrm{E}\, U(C_t) \mathrm{d}t, \qquad (2.2)$$

where $\mathcal{A}(x)$ is the set of admissible strategies, β is a discount factor and U denotes a power utility function of the form

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0,\ p \neq 1.$$

Optimal Proportion and Consumption

Assume the wealth and consumption portfolio, $p > 1$ and the objective (2.2). Let

$$\theta_p^* = \operatorname*{argmin}_{\theta_p} h(\theta_p) = \operatorname*{argmin}_{\theta_p} \Big\{ \alpha\theta_p(1-p) - \frac{1}{2}\sigma^2\theta_p^2 p(1-p) + \int_{-1}^{\infty} \left((1+\theta_p z)^{1-p} - 1 - \theta_p z(1-p) \right) \nu(\mathrm{d}z) \Big\}.$$

If $\beta - r(1-p) - h(\theta_p^*) > 0$ (finiteness of the value function) then

- θ_p^* is the optimal proportion,
- $c^* = (K(1-p))^{-1/p}$ is the optimal consumption,
- $v(z) = Kz^{1-p}$ is the value function,

where

$$K = \frac{1}{1-p} \left(\frac{\beta - r(1-p) - h(\theta_p^*)}{p} \right)^{-p}.$$

Optimal proportion as a function of moments

Proposition

Let

$$\int_{-1}^{\infty} \sum_{k=2}^{\infty} \left| \binom{1-p}{k} \theta_p^k z^k \right| dz < \infty$$

and $p > 1$. *Then*

$$\theta_p^* = \underset{\theta_p}{argmin} \left\{ \alpha\theta_p(1-p) + \binom{1-p}{2} \theta_p^2 \sigma_J^2 + \sum_{k=3}^{\infty} \binom{1-p}{k} \theta_p^k \kappa_k \right\},$$

where

$$\sigma_J^2 = \sigma^2 + \int_{-1}^{\infty} z^2 \nu(dz),$$

σ^2 *is the volatility of the diffusion part, and* κ_k *is the k-th cumulant.*

Conclusion

Influence of two major moments on optimal behavior

- ☺ As skewness grows θ_p^* ascends (odd cumulants have negative sign for $p > 1$). Positive skewness is good for the investor.
- ☹ Effect of heavy tails: As kurtosis grows θ_p^* descends (even cumulants have positive sign).
- ? Is it possible to construct jump process with positive skewness so that $\theta_p^* > {\theta_p^*}^M$ (Merton proportion)?
  - $\Rightarrow$ See my poster!

$\Rightarrow$ Numerical study still needs to be finished.

Thank you for attention