Maximal inequality for stochastic convolution integral

Jan Kaluža

kaluza@karlin.mff.cuni.cz

Department of Probability, Charles University, Prague

ÚTIA, Academy of Sciences, Czech Republic

Introduction

This contribution dealt with stochastic convolution integral $\int_0^{\cdot} S(.-s)\psi(s)\,\mathrm{d}M_s$ driven by local martingale in Hilbert space with contraction C_0-semigroup $S(t)$. The maximal inequality and tail estimate are presented.

The introduction is devoted to basic notation and a review of necessary objects (more information can be found in [2] and [4]). In the second section is presented the maximal inequality of Burkholder-Davis-Gundy type and the tail estimate for stochastic convolution. And the last section contains the Szekőfalvi-Nagy theorem about semigroup representation by unitary dilatations and the Burkholder-Davis-Gundy inequality.

We are working on stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ satisfying usual conditions. $\mathcal{O}$ is the $\sigma-$algebra generated by real continuous adapted processes ($\sigma-$algebra of well measurable sets).

Let H and G are (real) separable Hilbert spaces. For $x \in H$ and $y \in G$ we define $x \otimes y \in \mathcal{L}(H,G)$ by the following formula ($x^{\otimes 2} = x \otimes x$)

$$x \otimes y : H \longrightarrow G; h \longmapsto \langle x, h\rangle_H\, y.$$

We denote $(\mathcal{J}_1(H,G), \|.\|_1)$ the Banach space of Nuclear operators and $(\mathcal{J}_2(H,G), \langle .,.\rangle_2)$ the Hilbert space of Hilbert-Schmidt operators. The space of bounded linear operators is denoted $\mathcal{L}(H,G)$.

The family $(S_t)_{t\ge 0}$ of bounded linear operators on G is called C_0-semigroup on G, if $S_0 = I$, $S_{t+s} = S_t \circ S_s$ for every $s,t \ge 0$ and if $\mathbb{R}_+ \longrightarrow G : t \longmapsto S_t x$ is continuous for every $x \in G$. Then there always exist $M < \infty$ and $w \in \mathbb{R}$ such that $\|S_t\| \le M\mathrm{e}^{wt}$ for all $t \ge 0$. The C_0-semigroup $(S_t)_{t\ge 0}$ is contraction C_0-semigroup if $\|S_t\| \le 1$ for all $t \ge 0$. Every C_0-semigroup $(S_t)_{t\ge 0}$ has its generator $(A, \mathcal{D}(A))$ and we write $S_t \equiv \mathrm{e}^{At}$. If A is bounded, then $\mathrm{e}^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$, otherwise it is just a symbol for S_t.

The family $(U_t)_{t\in\mathbb{R}}$ is unitary C_0-group on G if $U_0 = I$, $U_{t+s} = U_t \circ U_s$ for every $s,t \in \mathbb{R}$, for every $x \in G$ is $\mathbb{R} \longrightarrow G : t \longmapsto U_t x$ continuous and if U_t are unitary operators for every $t \in \mathbb{R}$.

We denote $\mathcal{M}^2_{loc}(H)$ respective $\mathcal{M}^2_{\infty}(H)$ the set of $H-$valued local L_2-martingales respective square integrable martingales with *càdlàg* paths. Note that $\mathcal{M}^2_{loc}(H) = \mathcal{M}^2_{\infty,loc}(H)$.

Let $M \in \mathcal{M}^2_{\infty}(H)$. Then there exists a real *càdlàg* process with paths of finite variation, denoted by $[M]$ and called quadratic variation of M and a $\mathcal{J}_1(H)-$valued *càdlàg* process, denoted by $[\![M]\!]$ and called the tensor quadratic variation of M, both are uniquely defined (up to $\mathbb{P}-$equality) with property: For every increasing sequence $(\Pi_n) = \left(\left\{0 = t_0^n < t_1^n < \ldots \nearrow \infty\right\}\right)$ of subdivisions of $\mathbb{R}_+$ with $\sup_{i\ge 0}|t_{i+1}^n - t_i^n| \xrightarrow[n\to\infty]{} 0$ and for every $t \ge 0$ holds

$$[M]_t = \lim_{n\to\infty} (L^1) \sum_{i=0}^{\infty} \left\| M_{t_{i+1}^n\wedge t} - M_{t_i^n \wedge t}\right\|_H^2 \quad \text{and} \quad [\![M]\!]_t = \lim_{n\to\infty} (\mathbb{P}) \sum_{i=0}^{\infty} \left(M_{t_{i+1}^n\wedge t} - M_{t_i^n \wedge t}\right)^{\otimes 2},$$

where the convergency in probability is the convergency of $\mathcal{J}_2(H)-$valued random variables. There exists q_M $\mathcal{J}_1(H)-$valued $\mathcal{O}-$measurable stochastic process such that: q_M take its values in positive selfadjoint elements of $\mathcal{J}_1(H)$, $\mathrm{Tr}\, q_M(t,\omega) = 1$ for $\mu_{[M]}-$ a.e. $t \ge 0$ $\mathbb{P}-$a.s. and $[\![M]\!]_t = \int_{(0,t]} q_M \,\mathrm{d}[M]$ $\forall t \ge 0$ $\mathbb{P}-$a.s. The process q_M is unique in this sense: if q is another process with the above properties, than $q_M(t,\omega) = q(t,\omega)$ for $\mu_{[M](\omega)}-$a.e. $t \ge 0$ $\mathbb{P}-$a.s. For $M \in \mathcal{M}^2_{loc}(H)$ we define the processes $[M]$ and $[\![M]\!]$ by localisation.

$L^*(M)$ is the space of stochastic processes X with values in linear operators from H to G such that:

i) $\forall (t,\omega) \in \mathbb{R}^+ \times \Omega \quad q_M^{\frac{1}{2}}(t,\omega)(H) \subseteq \mathcal{D}X(t,\omega)$,

ii) $\forall h \in H$ the process $X \circ q_M^{\frac{1}{2}}(h)$ is $\mathcal{O}-$measurable ($G-$valued),

iii) $\forall (t,\omega) \in \mathbb{R}^+ \times \Omega \quad X(t,\omega) \circ q_M^{\frac{1}{2}}(t,\omega)$ is Hilbert-Schmidt operator and $\mathbb{E}\int_0^{\infty} \left\| X \circ q_M^{\frac{1}{2}}\right\|_2^2 \mathrm{d}[M] < \infty$.

We understand the space L^* as the space of classes of equivalence, where $X \sim Y$ iff $\mathbb{E}\int_0^{\infty}\left\|(X-Y)\circ q_M^{\frac{1}{2}}\right\|_2^2 \mathrm{d}[M] = 0$. The mapping $(X,Y) \longmapsto \mathbb{E}\int_0^{\infty} \mathrm{Tr}\,(X \circ q_M \circ Y^*)\ \mathrm{d}[M]$ is a scalar product on L^*.

Let $M \in \mathcal{M}^2_{\infty}(H)$ and $X \in \mathcal{E}(\mathcal{L}(H,G))$, i.e.

$$X = \sum_{i=1}^{n} x_i \mathbb{I}_{(s_i,t_i]\times F_i} + x_0 \mathbb{I}_{\{0\}\times F_0}, \text{ where } n \in \mathbb{N},\ x_i \in \mathcal{L}(H,G),\ 0 \le s_i \le t_i \text{ and } F_i \in \mathcal{F}_{s_i}\ \forall i = 0,\ldots,n.$$

We define the stochastic integral of process X with respect to M as follows

$$\int X\,\mathrm{d}M := \left\{\sum_{i=1}^{n} \mathbb{I}_{F_i} x_i \left(M_{t_i\wedge t} - M_{s_i \wedge t}\right)\right\}_{t\ge 0}. \tag{1}$$

Clearly $\mathcal{E}(\mathcal{L}(H,G)) \subseteq L^*(M)$ and $\Lambda^2(M)$ denote the space of processes from $L^*(M)$ which can be aproximated by processes from $\mathcal{E}(\mathcal{L}(H,G))$ in $L^*(M)$. The stochastic integral of $X \in \Lambda^2(M)$ is defined as a limit (in $\mathcal{M}^2_{\infty}$) of stochastic integrals of the aproximating sequence. We are working with the standard generalisation of this stochastic integral by localisation.

Maximal inequality and tail estimate

The integrand of stochastic convolution integral (2) depends on the upper limit of the integral and in general it is not a martingale. If the semigroup can be extended to a C_0-group, we can take the factor e^{At} in front of the integral in (2) and the convolution is then an image of stochastic integral by a bounded linear operator e^{At}. We follow this idea and we use the representation theorem 4, as in [3]. This restricts results to contraction semigroups.

The first question is what can we tell about paths of stochastic convolutions?

Proposition 1 *Let $M \in \mathcal{M}^2_{loc}(H)$, $\psi \in \Lambda^2_{loc}(M)$ and let $(\mathrm{e}^{At})_{t\ge 0}$ is a contractive C_0-semigroup on Hilbert space G. Then the stochastic convolution integral*

$$M_{A,\psi}(t) \equiv \int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s, \quad t \ge 0, \tag{2}$$

has càdlàg paths in G. If M is continuous then $M_{A,\psi}$ is continuous as well.

Sketch of proof: We apply Sz.-Nagy Theorem 4:

$$\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s = PU_t \int_0^t U_{-s}\pi\psi(s)\,\mathrm{d}M_s =: PU_t N_t, \tag{3}$$

where $N \in \mathcal{M}^2_{loc}(\widetilde{G})$. Than we use the right continuity of N and strong continuity of group $(U_t)_{t\in\mathbb{R}}$. □

Now we prove the inequality of Burgholder-Davis-Gundy type for convolution integrals, but still only in the case of contraction semigroup $(\mathrm{e}^{At})_{t\ge 0}$. The result is already known for continuous integrator M (because in this case quadratic variation coincide with compensator). There already exist a weaker version for M general and stochastic integral based on compensator (see inequality (1.10) in [3]). The reason to define stochastic integral via quadratic variation is to know its quadratic variation. This allow us to use the BDG-inequality (7).

Theorem 2 *For every $p \ge 1$ there exists a constant $C_p < \infty$ such that for every $M \in \mathcal{M}^2_{loc}(H)$, $\psi \in \Lambda^2_{loc}(M)$, τ stopping time and every contractive C_0-semigroup $(\mathrm{e}^{At})_{t\ge 0}$ on Hilbert space G is*

$$\mathbb{E}\left(\sup_{0\le t\le\tau}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s\right\|_G^p\right) \le C_p^p \mathbb{E}\left(\left(\int_0^{\tau}\left\|\psi(s)q_M^{\frac{1}{2}}(s)\right\|^2_{\mathcal{J}_2(H,G)} \mathrm{d}[M]_s\right)^{\frac{p}{2}}\right). \tag{4}$$

It is possible to set $C_p = 2p$ for $p \ge 2$ (the constant is the same as the one in the BDG-inequality (7)). If we limit ourself to continuous martingales, we can choose C_p such that $C_p = O(\sqrt{p})$ as $p \longrightarrow \infty$.

Proof: The assertion results from (3) and Burkholder-Davis-Gundy inequality (Theorem 5). □

If we consider only continuous $M \in \mathcal{M}^2_{loc}(H)$, we can use the Burkholder-Davis-Gundy inequality with $C_p = O(\sqrt{p})$ as $p \longrightarrow \infty$ and the stochastic convolution is exponential L^2-integrable (in our setting with contraction C_0-semigroup). See Proposition 2.1 in [3].

If M is generally not continuous, we prove only the exponential L^1-integrability of stochastic convolutions. The reason is that for generall M the BDG-inequality holds only with $C_p = O(p)$ as $p \longrightarrow \infty$.

Proposition 3 *There exist constants $\lambda > 0$ and $C_\lambda < \infty$ such that for every local martingale $M \in \mathcal{M}^2_{loc}(H)$, each stopping time τ and any $\kappa \in (0,\infty)$ the estimate*

$$\mathbb{E}\exp\left\{\frac{\lambda}{\sqrt{\kappa}}\sup_{0\le t\le\tau}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s\right\|_G\right\} \le C_\lambda$$

holds for all processes $\psi \in \Lambda^2_{loc}(M)$ satisfying

$$\operatorname*{ess\,sup}_{\Omega}\int_0^{\tau}\left\|\psi(s)q_M^{\frac{1}{2}}(s)\right\|^2_{\mathcal{J}_2(H,G)} \mathrm{d}[M]_s \le \kappa. \tag{5}$$

In particular, an exponential tail estimate

$$\mathbb{P}\left\{\sup_{0\le t\le\tau}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s\right\| \ge \epsilon\right\} \le C_\lambda \exp\left\{-\frac{\lambda\epsilon}{\sqrt{\kappa}}\right\}$$

holds for all $\epsilon > 0$.

Sketch of proof: We use the Theorem 2 and the series representation of an exponential. □

C_0-semigroup e^{At} is quasi-contractive if there exist $w \in \mathbb{R}$ such that $\left\|\mathrm{e}^{At}\right\| \le \mathrm{e}^{wt}$ for all $t \ge 0$. Let $w > 0$. Then $\widetilde{S}_t := \mathrm{e}^{-wt}\mathrm{e}^{At}$ is a contraction C_0-semigroup and the maximal inequality (4) has the form

$$\mathbb{E}\left(\sup_{0\le t\le\tau}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}M_s\right\|_G^p\right) \le C_p^p(\mathrm{e}^{w\ \mathrm{ess\,sup}\ \tau})^p \mathbb{E}\left(\left(\int_0^{\tau}\left\|\psi(s)q_M^{\frac{1}{2}}(s)\right\|^2_{\mathcal{J}_2(H,G)} \mathrm{d}[M]_s\right)^{\frac{p}{2}}\right). \tag{6}$$

Similarly, for every $\alpha > 1$ the Proposition 3 is true if we consider only stopping times τ satisfying $\mathrm{e}^{w\ \mathrm{ess\,sup}\ \tau} \le \alpha$. The constant C_λ then depends on α and λ.

Apendix

Theorem 4 (Szekőfalvi-Nagy) *Let $(\mathrm{e}^{At})_{t\ge 0}$ is a contractive C_0-semigroup on a Hilbert space G, then there exist a Hilbert superspace $\widetilde{G}$ of G and an unitary C_0-group $(U_t)_{t\in\mathbb{R}}$ on $\widetilde{G}$ such that $\pi^* \circ U_t \circ \pi = \mathrm{e}^{At}$, $\forall t \ge 0$, where $\pi : G \longrightarrow \widetilde{G}$ is an isometric embedding. $P = \pi^*$ is an orthogonal projection from $\widetilde{G}$ on G. If G is separable, then the space $\widetilde{G}$ can be chosen separable.*

Proof: [1], section 7.2. □

Theorem 5 (Burkholder-Davis-Gundy inequality) $\forall p \ge 1 \quad \exists\, c_p, C_p \in \mathbb{R}$ *such that for any $M \in \mathcal{M}_{loc}(H)$ and any τ stopping time is*

$$c_p \mathbb{E}\,[M]_{\tau}^{\frac{p}{2}} \le \mathbb{E}\sup_{t\ge 0}\|M_{t\wedge\tau}\|^p \le C_p^p \mathbb{E}\,[M]_{\tau}^{\frac{p}{2}}. \tag{7}$$

It is possible to set $C_p = 2p$. If we limit ourself to continuous martingales, we can choose C_p such that $C_p = O(\sqrt{p})$ as $p \longrightarrow \infty$.

For more information on this result see [3], pages 104-105.

Acknowledgements. I would like to express my thanks to Jan Seidler for his advices and consultations.

References

[1] E. B. Davies. *Quantum Theory of Open Systems.* Academic Press, London, 1976.

[2] Klaus-Jochen Engel and Rainer Nagel. *A short course on operator semigroups.* Universitext, Springer, New York, 2006.

[3] Erika Hausenblas and Jan Seidler. Stochastic convolutions driven by martingales: Maximal inequalities and exponential integrability. *Stochastic Anal. Appl.*, 26(1): 98-119, 2008.

[4] Michel Métivier. *Semimartingales,* volume 2 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1982. A course on stochastic processes.