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# REFORMULATION OF CHANCE CONSTRAINED PROBLEMS

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## SUMMARY

**We explore reformulation of stochastic programs with chance constraints by stochastic programs with suitably chosen penalty-type objectives. We show that the two problems are asymptotically equivalent. The obtained problems with penalties and with a fixed set of feasible solutions are much simpler to solve and analyze then the chance constrained programs.**

## PROBLEMS FORMULATION

Let $g_{ji}(x,\omega), i = 0,\dots,k_j, j = 1,\dots,m$, be real functions on $\mathbb{R}^n \times \Omega$ measurable in $\omega$ for all $x \in X$. Then the multiple chance constrained problem can be formulated as follows:

$$\begin{aligned} \psi_\epsilon \ = \ & \min_{x\in X} f(x), \\ & \text{s.t.} \\ & P\big(g_{11}(x,\omega) \le 0,\dots,g_{1k_1}(x,\omega) \le 0\big) \ \ge \ 1-\varepsilon_1, \\ & \vdots \\ & P\big(g_{m1}(x,\omega) \le 0,\dots,g_{mk_m}(x,\omega) \le 0\big) \ge 1-\varepsilon_m, \end{aligned} \qquad (1)$$

with optimal solution $x_\epsilon$, where $\epsilon = (\varepsilon_1,\dots,\varepsilon_m)$, with the levels $\varepsilon_j \in (0,1)$.

At the time of preparing this poster we do not know any solver which could be used for solving previous multiple jointly chance constrained problems. On the other hand, stochastic programs with penalties and fixed set of feasible solution can be solved much simpler. Thus, the recourse reformulation which is stated bellow may be very useful.

In [3], asymptotic equivalence between problem with one joint chance constraint and problem with simple recourse penalty function is shown. The approach by [3] can be extended to a whole class of penalty functions with desirable properties which was done in [2]. We propose further extension to multiple jointy chance constrained problems (1).

Below, we will consider penalty functions $\vartheta_j : \mathbb{R}^{k_j} \to \mathbb{R}_+$, $j = 1,\dots,m$, which are continuous nondecreasing in their components, equal to 0 on $\mathbb{R}^{k_j}_-$ and positive otherwise. We denote $g_j(x,\omega) = (g_{j1}(x,\omega),\dots,g_{jk_j}(x,\omega)) : \mathbb{R}^n \times \Omega \to \mathbb{R}^{k_j}$ and set $p_j(x,\omega) = \vartheta_j(g_j(x,\omega))$. Our choice is appropriate, because it holds

$$P\big(g_{ji}(x,\omega) \le 0,\ i = 1,\dots,k_j\big) \ge 1-\varepsilon_j \iff P\big(p_j(x,\omega) > 0\big) \le \varepsilon_j.$$

The corresponding penalty function problem can be formulated as follows

$$\varphi_N = \min_{x\in X} \Big[f(x) + N \cdot \sum_{j=1}^m \mathbb{E}[p_j(x,\omega)]\Big] \qquad (2)$$

with $N$ a positive parameter. We denote $x_N$ an optimal solution of (2).

## ASYMPTOTIC EQUIVALENCE

Consider the two problems (1) and (2) and assume:

(i) $X \neq \emptyset$ compact, $f(x) = \mathbb{E}[g_0(x,\omega)]$ a finite continuous function of $x$;

(ii) $g_{ji}(\cdot,\omega),\ i = 1,\dots,k_j, j = 1,\dots,m$, are almost surely continuous;

(iii) there exists a nonnegative random variable $C(\omega)$ with $\mathbb{E}[C^{1+\kappa}(\omega)] < \infty$ for some $\kappa > 0$, such that $|p_j(x,\omega)| \le C(\omega),\ j = 1,\dots,m$, for all $x \in X$;

(iv) $\mathbb{E}[p_j(x',\omega)] = 0,\ j = 1,\dots,m$, for some $x' \in X$;

(v) $P(g_{ji}(x,\omega) = 0) = 0,\ i = 1,\dots,k_j, j = 1,\dots,m$, for all $x \in X$.

Denote $\gamma = \kappa/(2(1+\kappa))$, and for arbitrary $N > 0$ and $\epsilon \in (0,1)^m$ put

$$\begin{aligned} \varepsilon_j(x) &= P\big(p_j(x,\omega) > 0\big), \quad j = 1,\dots,m, \\ \alpha_N(x) &= N \cdot \sum_{j=1}^m \mathbb{E}[p_j(x,\omega)], \\ \beta_\epsilon(x) &= \varepsilon_{max}^{-\gamma} \sum_{j=1}^m \mathbb{E}[p_j(x,\omega)], \end{aligned}$$

where $\varepsilon_{max}$ denotes maximum of the vector $\epsilon = (\varepsilon_1,\dots,\varepsilon_m)$ and $[1/N^{1/\gamma}] = (1/N^{1/\gamma},\dots,1/N^{1/\gamma})$ is the vector of length $m$.

THEN for any prescribed $\epsilon \in (0,1)^m$ there always exists $N$ large enough so that minimization (2) generates optimal solutions $x_N$ which also satisfy the chance constraints (1) with the given $\epsilon$.

Moreover, bounds on the optimal value $\psi_\epsilon$ of (1) based on the optimal value $\varphi_N$ of (2) and vice versa can be constructed:

$$\begin{aligned} \varphi_{1/\varepsilon^\gamma_{max}(x_N)} - \beta_{\epsilon(x_N)}(x_{\epsilon(x_N)}) \le \psi_{\epsilon(x_N)} &\le \varphi_N - \alpha_N(x_N), \\ \psi_{\epsilon(x_N)} + \alpha_N(x_N) \le \varphi_N &\le \psi_{[1/N^{1/\gamma}]} + \beta_{[1/N^{1/\gamma}]}(x_{[1/N^{1/\gamma}]}), \end{aligned}$$

with

$$\lim_{N\to+\infty} \alpha_N(x_N) = \lim_{N\to+\infty} \varepsilon_j(x_N) = \lim_{\varepsilon_{max}\to 0_+} \beta_\epsilon(x_\epsilon) = 0$$

for any sequences of optimal solutions $x_N$ and $x_\epsilon$.

## CONCLUSION AND FUTURE PLANS

Reformulation of chance constrained programs by incorporating a suitably chosen penalty function into the objective helps to arrive at problems with expectation in objective and a fixed set of feasible solutions. The obtained problems are much simpler to solve and analyze then the chance constrained programs. The recommended form of the penalty function follows the basic ideas of penalty methods and its suitable properties follow by generalization of results of [1, 3]. The questions for future research are how to choose the parameter $N$ so that the probability levels $\epsilon$ are ensured and to find bounds which can be simply evaluated.

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