Mappings of finite distortion: opimal regularity, applications in random quasiconformal maps

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I Optimal regularity for mappings of finite distortion in plane
(based on joint work with K. Astala, J. Gill, and S. Rohde)

II Lehto method and random weldings
(based on joint work with K. Astala, P. Jones and A. Kupiainen)

Warning:

Many statements and proofs below are purposedly incomplete, even false if taken as face value, assumptions are missing etc: we try to convey the main ideas and not to stick to technicalities. Unfortunately, not all of our mistakes are deliberate...

Part I: Optimal regularity for mappings of finite distortion in plane

1. The basic question:

The classical theory of planar quasiconformal maps considers the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$$

where $|\mu(z)| \leq k < 1$ almost everywhere. Our basic question is what happens if one gives up the condition $\|\mu\|_\infty < 1$, but allows $\|\mu\|_\infty = 1$ instead?

1. The basic question:

The classical theory of planar quasiconformal maps considers the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$$

where $|\mu(z)| \leq k < 1$ almost everywhere (μ is then the *complex dilatation* of f). Our basic question is what happens if one gives up the condition $\|\mu\|_\infty < 1$, but allows $\|\mu\|_\infty = 1$ instead.

Applications include e.g. 2-dimensional elasticity theory, non-hyperbolic dynamics, . . . , (recently) some random phenomena.

Part of a bigger picture: mappings of finite distortion

A mapping $f : \Omega \to \mathbb{R}^n$ defined in a domain $\Omega \subset \mathbb{R}^n$ is a *mapping of finite distortion* if

1. $f \in W^{1,1}_{loc}(\Omega)$,
2. $J(\cdot, f) \in L^1_{loc}(\Omega)$, and
3. there is a measurable function $K(z) \geq 1$, finite almost everywhere, such that

$$|Df(z)|^n \leq K(z)\, J(z, f) \quad \text{almost everywhere in } \Omega$$

The smallest such function is denoted by $K(z, f)$ and is called the distortion function of f.

The more general framework: mappings of finite distortion

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Arguably, the last condition is a minimal requirement for a mapping to carry geometric (conformality related) information.

In two dimension a homeomorphic mapping f is a map of finite distortion exactly if $f \in W^{1,1}_{loc}$ and

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}, \qquad |\mu(z)| < 1 \quad \text{a.e..}$$

Recall that one has

$Df = |f_z| + |f_{\bar{z}}|$ for the operator norm and

$J(z, f) = |f_z|^2 - |f_{\bar{z}}|^2$ for the Jacobian,

whence the distortion has the the formula

$$K(z) = \frac{1 + |\mu|}{1 - |\mu|}.$$

The measurable Riemann mapping theorem: review of the classical case

Theorem. (Morrey, Bojarski, Ahlfors-Bers,...) If $|\mu(z)| \leq k = \frac{K-1}{K+1}$ for $z \in \mathbb{C}$ and for some constant $k < 1$, then the Beltrami equation has a homeomorphic solution, unique up to post-composing with analytic maps.

Understanding the local regularity of the solution reduces to that of the principal solution: a principal solution is a homeomorphism of $\mathbb{C}$ onto itself with the expansion

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

in $\mathbb{C} \setminus K$ (K compact, usually $K = \mathbb{D}$), whence f is conformal in $\overline{\mathbb{C}} \setminus K$.

- The Beurling operator $\mathcal{S}$:

$$\mathcal{S}\varphi(z) := -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\varphi(\tau)}{(z-\tau)^2}\, d\tau$$

acts unitarily on $L^2(\mathbb{C})$.

- One has $\mathcal{S}(h_{\overline{z}}) = h_z$ for every $h \in W^{1,2}(\mathbb{C})$.

Assume that μ is supported in the unit disc. Classical way to look for a solution of the Beltrami eq. is to make the Ansatz

$$f = z + \mathcal{C}(\omega), \qquad\qquad C\omega(z) := \frac{1}{\pi}\iint_{\mathbb{C}} \frac{\varphi(\tau)}{(z-\tau)}\, d\tau$$

where $\mathcal{C}$ is the Cauchy transform.

Then $\omega = f_{\bar{z}}$ is to satisfy the identity

$$\omega(z) = \mu(z)\mathcal{S}\omega(z) + \mu(z) \quad \text{for almost every } z \in \mathbb{C}.$$

Such an ω is easy to find when $\|\mu\|_\infty \le k < 1$. We define

$$\omega := (1 - \mu\mathcal{S})^{-1}\mu = \mu + \mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu\mathcal{S}\mu + \cdots$$

The above series ('the Neumann series') converges in $L^p(\mathbb{C})$ for p close to 2 since

$$\|\mu\mathcal{S}\mu\mathcal{S}\cdots\mu\mathcal{S}\|_{L^p(\mathbb{C})} \;\le\; (k\|S\|_{L^p\to L^p})^n, \quad n \in \mathbb{N},$$

and (by interpolation) $\|S\|_{L^p\to L^p} \to 1$ as $p \to 2$.

As a consequence of Astala's area distortion theorem one finds the optimal regularity in the classical case in the form:

$$|Df|^p \in L^1_{loc} \qquad p < \frac{2K}{K-1}.$$

Degenerate case: some conditions on μ needed !

Example Consider the *smooth* function $f(z)) := z(z+\bar{z})$. It satisfies Beltrami with

$$\mu(z) = \frac{z}{2z+\bar{z}}.$$

Clearly f is a mapping of finite distortion, but $f(i\mathbb{R}) = 0$, whence the inverse function is not very well defined ! On the other hand, distortion is very large:

$$K(z) \asymp 1 + (y/x)^2,$$

especially $K \notin L^1_{loc}$.

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Clearly f is a mapping of finite distortion, but $f(i\mathbb{R}) = 0$! Especially, the inverse function is not very well defined. On the other hand, distortion is very large:

$$K(z) \asymp 1 + (y/x)^2,$$

especially it is not integrable.

A positive surprise ahead : bare integrability of K alone will help in several nice ways!

Lemma: (modulus of continuity, Gehring, Goldstein and Vodopyanov)
Assume that $f \in W^{1,2}(2\mathbb{D})$ is a homeomorphism. Then, if $z_1, z_2 \in \mathbb{D}$ one has

$$|f(z_1) - f(z_2)|^2 \leq \frac{c \int_{2\mathbb{D}} |\nabla f|^2}{\log(e + 1/|z_1 - z_2|)}.$$

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<u>Proof.</u> Let $B_t := B(z_0, t)$ be the disc with center $z_0 := (z_1 + z_2)/2$. Obtain for a.e. $t > |z_0 - z_1|/2$

$$\frac{|u(z_1) - u(z_2)|}{\pi t} \le \frac{1}{2\pi t} \int_{\partial B_t} |\nabla f||ds|.$$

Do Cauchy-Schwarz and multiply by t :

$$\frac{|u(z_1) - u(z_2)|^2}{\pi^2 t} \le \frac{1}{2\pi} \int_{\partial B_t} |\nabla f|^2|ds|.$$

Simply integrate over $t \in (|z_0 - z_1|/2, 1)$. $\square$

Lemma: (regularity of the inverse, Iwaniec and Sverak, ...)
Assume that $f \in W^{1,2}_{loc}$ be a (homeomorphic) principal solution with distortion K. Then, the inverse map $g := f^{-1}$ satisfies

$$|g(a)) - g(b)|^2 \ \leq \ C\frac{(1+|a|^2+|b|^2)}{\log(e+1/|a-b|)}\int_{\mathbb{D}} K(z).$$

Proof. The basic idea (omitting easy technicalities) is to use reduce to the previous Lemma via the identity

$$\int |Dg|^2 = \int K_g \circ (f \circ g) J_g = \int K_g \circ f = \int K_f.$$

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$\square$

Remark: The right hand side depends only on K. This allows one to (try to) approximate f with classical qc-homeos, while retaining uniformly the above estimate !

Lemma: (limits of approximations solve Beltrami) *Assume that the dilatation μ has compact support and satisfies $\frac{1+|\mu|}{1-|\mu|} = K \in L^1_{loc}$. Let f_n be the principal solution with dilation*

$$\mu_n := (1 - 1/n)\mu.$$

Assume additionally that we know that $f_n \to f$ locally uniformly. Then f is a principal solution of the Beltrami equation corresponding to μ.

Proof. By the previous Lemma the inverses g form an equicontinuous family, whence (by the local uniformity of $f_n \to f$) we deduce that f is a homeomorphism. Observe that standard Koebe-Bieberbach estimates ensure that nothing escapes to infinity.

We have that $|Df_n| \leq (K_{f_n} J_{f_n})^{1/2} (K J_{f_n})^{1/2}$. By basic Koebe- Bieberback theory we see that $\int_{2D} J_{f_n} \leq |f_n(2\mathbb{D})| \leq C$ for all n. Hence the sequence $|Df_n|$ is uniformly integrable, and we obtain weak convergence $Df_n \to Df$ in L^1 locally.

Consider for $\phi \in C_0^\infty$ the identity

$$\int \phi(\frac{\partial f_n}{\partial \bar{z}} - \mu \frac{\partial f_n}{\partial z}) = \int \phi((\mu_n - \mu)\frac{\partial f_n}{\partial z}).$$

The right hand side converges to 0 as $n \to \infty$, again by an application of the uniform integrability of the functions $\frac{\partial f_n}{\partial z}$. This and the weak convergence of the derivatives shows that the limit satisfies Beltrami.

Remark: Part of the general research on mappings on finite distortion use results like the above one, where the uniform continuity of f is deduced from a suitable condition on K via modulus methods (e.g. Ryazanov, Srebro and Yakubov).

Some conditions on μ needed - part 2. !

Example. Consider the radial map: $f(z) = \frac{z}{|z|}\rho(|z|)$, where $\rho(r) \to 0$ slowly as $r \to 0^+$. One computes that (for reasonable ρ) in this case $K(|z|) = \rho(|z|)/(|z|\rho'(|z|)$. Choose

$f(z) := \frac{z}{|z|}\Big(1+\log^{-\varepsilon}(e+1/|z|)\Big)$, which solves Beltrami with μ satisfying

$$K_f = \frac{1+|\mu|}{1-|\mu|} \asymp \log^{1+\varepsilon}(e+1/|z|).$$

The distortion has quite nice integrability properties:

$$\exp(K^{1-2\varepsilon}) \in L^1_{loc}.$$

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However, the image of f misses completely the ball $\mathbb{D}$! Danger of cavitation present even if K is almost exponentially integrable !

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The right class of Beltrami equations try to solve is G. David's class of *mapping of exponentially integrable distortion*, i.e for some $p > 0$ one has

$$(*) \qquad e^{pK(z)} \in L^1_{loc} \qquad ,\text{with} \quad K := \frac{1 + |\mu|}{1 - |\mu|}$$

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Here are some fundamental questions:

- can we solve the Beltrami $f_{\bar{z}} = \mu f_z$ with $Df \in L^2_{loc}$?
- given $p > 0$, what is the precise regularity for f?
- Does usual uniqueness (Stoilow etc.) hold?

Guessing the optimal regularity result... Consider the following example (modification of one due to Kovalev)

$$g_p(z) = \quad \frac{z}{|z|}\left[\log\left(e+\frac{1}{|z|}\right)\right]^{-p/2}\left[\log\log\left(e+\frac{1}{|z|}\right)\right]^{-1/2} \qquad \text{for } |z|<1,$$
$$g_p(z) = \quad c_0 \qquad\qquad \text{for } |z|>1.$$

For each $p>0$ one computes that

$$e^{K(z,g_p)} \in L^p(\mathbb{D})$$

If $0<\beta<p$, we obtain that

$$e^{\beta K(z,g_p)} \in L^1_{loc}(\mathbb{C}) \quad \text{with} \quad |Dg_p|^2 \log^{\beta-1}|Df| \in L^1_{loc}(\mathbb{C}),$$

while the latter inclusion fails for $\beta = p$.

Especially, if $p=1$, $g_1 \notin W^{1,2}_{loc}(\mathbb{C})$.

Our main result

The following theorem gives a positive answer to a conjecture due to Iwaniec and Sbordonne, Iwaniec, Koskela and Martin ($\sim$ 2001).

Theorem (Astala, Gill, Rohde, S. (2010)) *Let $\Omega \subset \mathbb{R}^2$ be a domain. Suppose the distortion function $K(z,f)$ of a mapping of finite distortion $f \in W^{1,1}_{loc}(\Omega)$ satisfies*

$$e^{K(z,f)} \in L^p_{loc}(\Omega) \qquad \textit{for some } p > 0$$

Then we have for every $0 < \beta < p$,

$$J(z,f)\log^{\beta}(e + J(z,f)) \in L^1_{loc}(\Omega) \qquad \textit{and}$$

$$|Df|^2 \log^{\beta-1}(e + |Df|) \in L^1_{loc}(\Omega)$$

The result is optimal.

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Note: Earlier Faraco, Koskela and Zhong (2005) had the result in $\mathbb{R}^n$ with the condition $0 < \beta < p/c$ with some constant $c = c(n) > 1$. Optimality for $n = 2$ correponds to $c = 1$. It is conjectured that $c = 1$ should work for all n.

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The result is optimal.

<u>Note 2:</u> Case $\beta = 1$ shows that that the optimal condition for $Df \in L^2_{loc}$ is $p > 1$. This is also open in higher dimensions.

Plan for the rest of Part I: we try to

- sketch two different approaches to obtain L^2-estimates: the Iwaniec, Koskela and Martin method, as well David's original method.

- refine David's approach in order to obtain the above theorem.

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- sketch two approaches to obtaineing L^2-estimates: the Iwaniec, Koskela and martin method, as well David's original method.

- refine David's approach in order to obtain the above theorem.

- skip proofs of certain things, like uniqueness via Stoilow etc.

L^2-regularity of the principal solution a la Iwaniec, Koskela and Martin

Recall that $\omega = f_{\bar{z}}$ satisfies

$$\omega(z) = \mu(z)\mathcal{S}\omega(z) + \mu(z),$$

or in another words $f_{\bar{z}} = (1 - \mu\mathcal{S})^{-1}\mu$.

Theorem (Iwaniec, Koskela, Martin (2002)) *There is $p_0 > 1$ such that if μ is supported in D and satisfies* $\exp(pK) \in L^p$ *for some $p > p_0$, then the equation*

$$(1 - \mu S)\omega = h$$

has a unique solution $\omega \in L^2(\mathbb{C})$ for every such h that $K^*h \in L^2(\mathbb{C})$, where K^* is a suitable BMO-majorant of K (such a K^* always exists.)

Idea of the proof:

– Construct the BMO majorant using the Coifmann–Rochberg result.
– Assume that ω is a solution. Careful juggling with elementary inequalities yields the estimate

$$(|S\omega|^2 + |\omega|^2) \leq 2K^*(|S\omega|^2 - |\omega|^2) + 4(K^*)^2|h|^2.$$

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Integrate and obtain apriori estimates via observing:
– $|S\omega|^2 - |\omega|^2$ is (essentially) a Jacobian, hence Coifmann, Lions, Meyer and Semmes result shows that its H^1-norm is bounded by constant x L^2-norm of the gradient.
– can make the BMO-norm of K^* small taking p_0 large enough. Apply Fefferman duality, move things to the left hand side.
– finally, approximate μ from below and take the limit to obtain solutions. $\square$

Remarks:

- The above approach has two big advantages:

– the obtained inequality says that $(1 - \mu\mathcal{S})$ is invertible between natural weighted spaces

– it generalizes to $\mathbb{R}^n$, as was done in the original paper.

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- However, to obtain optimal exponents for regularity we return to David's original approach: estimating the Neuman series in the degenerate case.

Part 1. of the proof (of the main result): Reduction to the regularity of principal solutions

Theorem (Iwaniec and Martin (2001)) *Let $Q(t) = \frac{t^2}{\log(e+t)}$ and suppose we are given a homeomorphic solution $f \in W^{1,Q}_{loc}(\Omega)$ to the Beltrami equation*

$$f_{\bar{z}} = \mu(z) f_z, \qquad z \in \Omega,$$

where $|\mu(z)| < 1$ almost everywhere. Then every other solution $h \in W^{1,Q}_{loc}(\Omega)$ takes the form

$$h(z) = \phi(f(z)), \quad z \in \Omega$$

where $\phi : f(\Omega) \to \mathbb{C}$ is holomorphic.

We will not prove this. However, from it follows that it is enough to study the regularity of a principal solution.

$W^{1,Q}_{loc}$ as a natural starting point

Assume that $f \in W^{1,1}_{loc}(\mathbb{C})$ is an orientation preserving homeomorphism whose distortion function $K(z)$ satisfies $e^{K(x)} \in L^p_{loc}$ for some $p > 0$. Apply $ab \le a \log(1+a) + e^b - 1$ to find that

$$\frac{|Df|^2}{\log(e + |Dh|^2)} \le \frac{1}{p} \frac{J}{\log(e+J)} \, pK \lesssim \frac{1}{p} \left(J + e^{pK} - 1 \right)$$

for all $p > 0$. Thus

$$\int_\Omega \frac{|Df|^2}{\log(e+|Df|)} \lesssim \frac{1}{p} \int_\Omega J(z,f) + \frac{1}{p} \int_\Omega [e^{pK(z)} - 1] \, dz$$

for any bounded domain. This shows that f automatically belongs to the Orlicz-Sobolev class $W^{1,Q}_{loc}(\mathbb{C})$.

Part 2. of the proof: The optimal decay of the Neumann series

Recall the Beurling operator $\mathcal{S}$: $\quad \mathcal{S}\varphi(z) := -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\varphi(\tau)}{(z-\tau)^2}\, d\tau$ and that it is an isometry in L^2. Recall also the Neumann series

$$\omega := (1-\mu\mathcal{S})^{-1}\mu = \mu + \mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu\mathcal{S}\mu + \cdots$$

Proposition (Optimal decay in Neumann series) *Suppose* $|\mu(z)| < 1$ *almost everywhere, with* $\mu(z) \equiv 0$ *for* $|z| > 1$. *If the distortion function* $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ *satisfies*

$$e^K \in L^p(\mathbb{D}), \qquad p > 0,$$

then we have for every $0 < \beta < p$,

$$\int_{\mathbb{C}} |(\mu\mathcal{S})^n\mu|^2 \;\leq\; C_0\,(n+1)^{-\beta}, \qquad n \in \mathbb{N}.$$

Proof.

• David's idea: estimate iteratively the terms $(\mu\mathcal{S})^n\mu$ of the Neumann series starting from the Chebychev type estimate (recall that $K(z)+1 = \frac{2}{1-|\mu(z)|}$

$$|\{z\in\mathbb{D} : |\mu(z)| \geq 1-\frac{1}{t}\}| \leq e^{-2pt}\int_{\mathbb{D}} e^{p(K+1)} = C\,e^{-2pt}, \quad t>1$$

and employ the f-isometry property of $\mathcal{S}$.

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$$|\{z \in \mathbb{D} : |\mu(z)| \geq 1 - \frac{1}{t}\}| \leq e^{-2pt} \int_{\mathbb{D}} e^{p(K+1)} = C\, e^{-2pt}, \quad t > 1$$

- Fix the parameter $0 < \beta < p$, and then for each $n \in \mathbb{N}$ divide the unit disk into the "bad" set

$$B_n = \{z \in \mathbb{D} : |\mu(z)| > 1 - \frac{\beta}{2n+\beta}\}$$

and the "good" one, i.e. the complements

$$G_n = \mathbb{D} \setminus B_n$$

According to Chebyschev $\boxed{|B_n| \le C_1\, e^{-4n\,p/\beta}\,, \qquad n \in \mathbb{N}, .}$

- Define $\qquad \psi_n = \mu\mathcal{S}(\psi_{n-1}), \qquad \psi_0 = \mu$

and the auxiliary terms $\qquad g_n = \chi_{G_n}\, \mu\mathcal{S}(g_{n-1}), \qquad g_0 = \mu,$

- The terms g_n are easy to estimate:

$$\|g_n\|_{L^2}^2 = \int_{G_n} |\mu\, \mathcal{S}(g_{n-1})|^2 \le \left(1 - \frac{\beta}{2n+\beta}\right)^2 \|g_{n-1}\|_{L^2}^2$$

Thus

$$\|g_n\|_{L^2} \le \prod_{j=1}^{n} \left(1 - \frac{\beta}{2j+\beta}\right) \|\mu\|_{L^2} \le C_\beta\, n^{-\beta/2}.$$

- Decompose

$$\psi_n - g_n = \chi_{G_n}\mu\,\mathcal{S}(\psi_{n-1} - g_{n-1}) + \chi_{B_n}\mu\,\mathcal{S}(\psi_{n-1})$$

This gives the norm bounds

$$\|\psi_n - g_n\|_{L^2}^2 \le \left(1 - \frac{\beta}{2n+\beta}\right)^2 \|\psi_{n-1} - g_{n-1}\|_{L^2}^2 + R(n) \qquad (1)$$

where

$$R(n) = \|\chi_{B_n}\mu\,\mathcal{S}(\psi_{n-1})\|_{L^2}^2 = \int_{B_n} |(\mu\mathcal{S})^n\mu|^2$$

- Induction argument finally yields

$$
\begin{aligned}
\|\psi_n - g_n\|_{L^2}^2 &\le \sum_{j=1}^{n} R(j) \prod_{k=j+1}^{n} \left(1 - \frac{\beta}{2k+\beta}\right)^2 \\
&\le \tilde{C}_\beta^2 \, n^{-\beta} \sum_{j=1}^{n} j^\beta \, R(j)
\end{aligned}
$$

$\Longrightarrow$ $\boxed{\text{remains to verify that } R(n) \text{ decays quickly enough!}}$

It is possible to estimate $R(n)$ using known spectral estimates for the operator μS. Let us give another proof that uses more directly the know area distortion bounds for ordinary qc-maps.

- Let $f = f^\lambda$ be the principal solution to

$$f_{\overline{z}}(z) = \lambda\,\mu(z) f_z(z)$$

Then the dependence $\lambda \to f^\lambda$ is holomorphic for $\lambda \in \mathbb{D}$. Namely,

$$f^\lambda_{\overline{z}} = \lambda\mu + \lambda^2 \mu \mathcal{S} \mu + \cdots + \lambda^n (\mu\mathcal{S})^{n-1}\mu + \cdots$$

- Let $E \subset \mathbb{D}$. Then

$$\chi_E\,(\mu\mathcal{S})^n \mu = \frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{1}{\lambda^{n+1}}\,(f^\lambda)_{\overline{z}}\,\chi_E\,d\lambda, \qquad E \subset \mathbb{D},$$

valid for any $0 < \rho < 1$. We are hence to estimate the norms

$$\|(f^\lambda)_{\overline{z}}\,\chi_E\|^2_{L^2} = \int_E |(f^\lambda)_{\overline{z}}|^2 \leq \frac{|\lambda|^2}{1-|\lambda|^2}\int_E J(z, f^\lambda)$$

$$= \frac{|\lambda|^2}{1-|\lambda|^2}\,|f^\lambda(E)|$$

Classical area distortion estimate reads:

$$\boxed{\ |f_\lambda(E)| \leq \pi M |E|^{1/M}, \qquad |\lambda| = \tfrac{M-1}{M+1}, \quad M > 1.}$$

Choose $\rho := |\lambda| =: \frac{M-1}{M+1}$ and combine our previous bound with the above estimates to obtain

$$(**) \qquad \|\chi_E\,(\mu \mathcal{S})^n \mu\|_2 \leq \sqrt{\pi} \left(\frac{M+1}{M-1} \right)^n \frac{M+1}{2}\, |E|^{1/(2M)}$$

The estimate is valid for every $M > 1$ and for any Beltrami coefficient with $|\mu| \leq \chi_{\mathbb{D}}$ almost everywhere.

- We apply the above on the set $E = B_n$ recalling that $|B_n| \leq C_1\, e^{-4n\,p/\beta}$.

We obtain

$$
\begin{aligned}
R(n) &\leq 4M^2 \left(\frac{M+1}{M-1}\right)^{2n} \; |E|^{1/M} \\
&\leq 4C_1 M^2 \left(\frac{M+1}{M-1}\right)^{2n} \; e^{-\frac{4n}{M}\frac{p}{\beta}}
\end{aligned}
$$

Given $\beta < p$ we can choose $M > 1$ so that

$$
\log\left(\frac{M+1}{M-1}\right) - \frac{2}{M}\frac{p}{\beta} < -\delta < 0
$$

for some $\delta > 0$. With this choice $R(n) \leq Ce^{-2\delta n}$. $\square$

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- One may verify by concrete examples that the obtained decay of the Neumann series is optimal, which looks quite bad right now!.

Part 3. of the proof: $W^{1,2}$-solution if $p > 2$.

From here on we are slightly more sketchy!

Lemma 1. (tentative area distortion) *Assume that $e^K \in L^p(\mathbb{D})$ for some $p > 2$. Then the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \quad \textit{almost every } z \in \mathbb{C}$$

admits a principal solution $f \in W^{1,2}_{loc}(\mathbb{C})$. Moreover, this solution satisfies the area distortion estimate

$$f(E) \leq C \log^{2-\beta}(e + |E|) \quad \text{for } E \subset \mathbb{D}.$$

for any $\beta < p$.

Proof.

- Existence: cut in the standard way the Beltrami coefficient $\mu(z)$:

$$\mu_m(z) := \begin{cases} \mu(z) & \text{if } |\mu(z)| \leq 1 - \frac{1}{m} \\ \left(1 - \frac{1}{m}\right) \frac{\mu(z)}{|\mu(z)|} & \text{otherwise.} \end{cases} \tag{2}$$

Since $p > 2$ the Neumann series of ∂f^m – the principal solution corresponding to μ_m – converges uniformly in L^2 with respect to m. Since we are in case $Df \in L^2$, our previous considerations yield the homeomorphic solution function

$$f = \lim f_m \in W^{1,2}_{loc}.$$

- Suppose $2 < \beta < p$ and observe that

$$\begin{aligned} f(E) &= \int_E J_f \leq \int_E |\partial f|^2 = \|\chi_E \partial f\|_2^2 \\ &\leq \left(\sum_{k=0}^{\infty} \|\chi_E (\mathcal{S}\mu)^k\|_2\right)^2 \end{aligned} \tag{3}$$

Our Proposition and its proof give us two different ways to estimate the terms. First,

$$\|\chi_E (\mu\mathcal{S})^n \mu\|_2 \leq C_0\,(n+1)^{-\beta/2}$$

Summing this up gives

$$\sum_{n=m+1}^{\infty} \|\chi_E (\mu\mathcal{S})^n \mu\|_2 + \leq c\frac{C_0}{\beta - 2}\, m^{1-\beta/2}, \qquad m \in \mathbb{N}$$

Secondly, choose $M = 3$ in the estimate (**). Then

$$\|\chi_E(\mu\mathcal{S})^n\mu\|_2 \le c\cdot 2^n\,|E|^{1/6},$$

whence

$$\sum_{n=0}^{m}\|\chi_E(\mu\mathcal{S})^n\mu\|_2 \le c\cdot 2^{m+1}|E|^{1/6}$$

Combining we arrive at

$$\sum_{n=0}^{\infty}\|\chi_E\,\sigma_\mu\|_2 \le \frac{c\,C_0}{\beta-2}\,m^{1-\beta/2} + c'\cdot 2^m|E|^{1/6}$$

Simply optimize over m. $\square$

Part 4. of the proof: Optimal area distorsion via factorization

Lemma 2. (factorization) *Suppose the distortion function* $K = K(z)$ *satisfies* $e^K \in L^p(\mathbb{D})$ *for some* $p > 0$. *Then for any* $M \geq 1$ *the principal solution to* $f_{\bar{z}}(z) = \mu(z)\, f_z(z)$ *admits a factorization*

$$f = g \circ F$$

where both g *and* F *are principal mappings,* g *is* M*-quasiconformal and* F *satisfies*

$$\int_{\mathbb{D}} e^{pMK(z,F)} \leq C_0 < \infty.$$

Proof. Quite standard – just construct the dilatation of F as a suitable multiple (which may vary pointwise) of μ. $\square$

Lemma 3. (area distortion) *Assume that a principal solution satisfies $e^K \in L^p$ for some $p > 0$,*

then for any $0 < \beta < p$ we have

$$|f(E)| \leq C \log^{-\beta}(e + \frac{1}{|E|}), \qquad E \subset \mathbb{D}$$

Proof. Choose $\beta_0 \in (\beta, p)$. Apply the factorization $f = g \circ F$ from Lemma 2 with large enough M to deduce with the use of Lemma 1:

$$|f(E)| = |g \circ F(E)| \leq \pi M |F(E)|^{1/M} \leq \pi C \left(\log(e + \frac{1}{|E|}) \right)^{(2-\beta_0 M)/M}$$

Choosing M large enough one has $\beta < \beta_0 - 2/M = (2 - \beta_0 M)/M$ and the result follows. $\square$

Part 5. (conclusion) of the proof: Optimal integrability of Jacobian and the derivatives.

Just for simplicity of notations, assume that $p > 1$. We then must first prove that $J(z, f) \log(e + J(z, f)) \in L^1_{loc}(\mathbb{C})$.

- Replace $J(z,f)$ by its nondecreasing radial rearrangement J^*.

- Set $A_n := \{2^{-n} \le |z| < 2^{1-n}\}$ and denote
$j_n := 2^{-2n} J^*(2^{-n})$ for $n \ge 1$.

Apply area distortion on set $\{|z| \le 2^{-\ell}\}$:

$$\begin{aligned} r_\ell &:= \sum_{n=\ell}^{\infty} j_n = \sum_{n=\ell}^{\infty} J^*(2^{-n}) 2^{-2n} \lesssim \sum_{n=\ell}^{\infty} J^*(2^{-n}) m(A_{n+1}) \\ &\le \int_{\{|z| \le 2^{-\ell}\}} J^* \le c\ell^{-\beta} \quad \text{with } \beta > 1. \end{aligned}$$

to obtain

$$
\begin{aligned}
\int_{\{|z|\le 1\}} J(z,f)\log(e+J(z,f)) &= \int_{\{|z|\le 1\}} J^*\log(e+J^*) \\
\le \quad & \sum_{n=1}^{\infty} J^*(2^{-n})\log(e+J^*(2^{-n}))m(A_n) \\
& \lesssim \sum_{n=1}^{\infty} j_n\log(e+2^{2n}j_n) \lesssim \sum_{n=1}^{\infty} nj_n \\
= \quad & \sum_{n=1}^{\infty} n(r_n - r_{n+1}) \lesssim \sum_{n=1}^{\infty} r_n < \infty,
\end{aligned}
$$

The second claim $|Df|^2 \in L^1_{loc}(\Omega)$ follows from what we just proved by applying the inequality

$$
xy \le C_p(x\log(e+x) + e^{p\,y} \qquad \text{for all } x,y>0
$$

Simply choose $x = J$ and $y = K$. $\square$

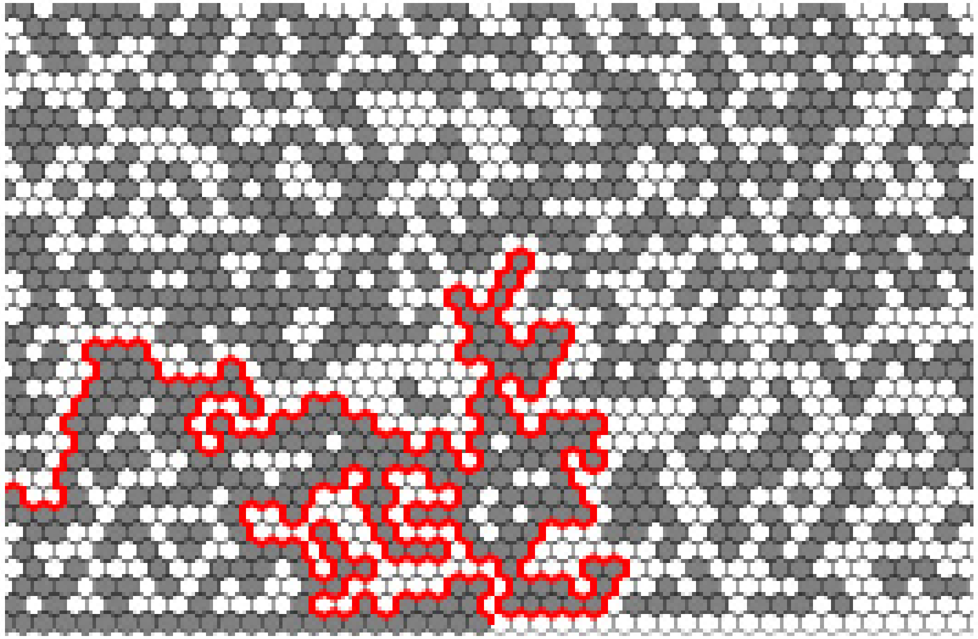
Part II: Lehtos method and random weldings

'Conformally invariant' Random Curves in the plane.

2d Statistical Mechanics:

- E.g. boundaries between phases
- One often gets curves joining boundary points
- Critical temperatures...

Percolation;   Brownian frontier;   etc. ....



Pictures: Oded Schramm

Scaling limit: SLE_κ

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- Curves ($\kappa \leq 4$) growing in fictious time are contructed (somewhat indirectly) using an explicit equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - B(\kappa\, t)}, \qquad g_0(z) = z.$$

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- Statistics of the full curve less explicit.

Scaling limit: SLE_κ

- Curves growing in fictious time are contructed using an explicit equation:

Proposal of Peter Jones:

construct natural random Jordan curves by describing the *statistics of welding homeomorphisms on the circle*.

Conformal welding:

Closed Jordan curves in $\widehat{\mathbb{C}} \longleftrightarrow$ Homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$.

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Jordan curve $\Gamma \subset \widehat{\mathbb{C}}$ splits $\widehat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$.

Take Riemann mappings:

$$f_+ : \mathbb{D} \to \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \to \Omega_-$$

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$$f_+ : \mathbb{D} \to \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \to \Omega_-$$

f_- and f_+ extend continuously to $\mathbb{T} = \partial\mathbb{D} = \partial\mathbb{D}_\infty \qquad \Rightarrow$

$$\text{get homeo}: \quad \phi = \left(f_+\right)^{-1} \circ f_- : \mathbb{T} \to \mathbb{T}$$

The Welding problem: invert this !

Given homeo $\phi : \mathbb{T} \to \mathbb{T}$, find Γ and Riemann maps $f_\pm$ so that

$$\boxed{\phi = \left(f_+\right)^{-1} \circ f_- : \mathbb{T} \to \mathbb{T}}$$

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Problem: Not possible for every homeomorphism ϕ !

Welding by QuasiConformal homeomorphisms:

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Suppose first that $\phi : \mathbb{T} \to \mathbb{T}$ is a restriction

$$\phi = f|_{\mathbb{T}},$$

where $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *quasiconformal* homeo:

$$\partial_{\overline{z}} f = \mu(z) \partial_z f \qquad \text{with } |\mu(z)| \le k < 1 \text{ a.e.}$$

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Then: $f_- := F|_{\mathbb{D}_\infty} : \mathbb{D}_\infty \to \Omega_-$ is conformal and $\Gamma = F(\mathbb{T}) = f_-(\mathbb{T})$ is a Jordan curve.

Beltrami equation: Now have in $\mathbb{D}$ two solutions

$$\partial_{\overline{z}} f = \mu(z) \partial_z f, \quad \partial_{\overline{z}} F = \mu(z) \partial_z F$$

Uniqueness of solutions in the uniformly elliptic case $\|\mu\|_\infty < 1$:

$$\Longrightarrow \qquad F(z) = f_+ \circ f(z), \quad z \in \mathbb{D},$$

where

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$f_\pm$ solve welding: Since $F|_{\mathbb{D}_\infty} := f_-$ and $f|_{\mathbb{T}} = \phi$ then

$$\phi(z) = f|_{\mathbb{T}}(z) = f_+^{-1} \circ f_-(z), \qquad z \in \mathbb{T}.$$

When does the reduction to Beltrami equations work?

When is $\phi : \mathbb{T} \to \mathbb{T}$ a restriction $\phi = f|_{\mathbb{T}}$ of a qc homeo $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$?

In the uniformly elliptic case $\|\mu\|_\infty < 1$, this happens $\Leftrightarrow$
ϕ is quasisymmetric:

$$\frac{|\phi(s+t) - \phi(s)|}{|\phi(s-t) - \phi(s)|} \leq K < \infty, \qquad s, t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

These have QC extensions with $|\mu| \leq m(K) < 1$.

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These have QC extensions with $|\mu| \leq m(K) < 1$.

In the random setting: Our ϕ will not be quasisymmetric and our Beltrami will not be uniformly elliptic !

Random homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$?

Random homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$

We take

$$\phi(t) = \frac{\int_0^t e^{\beta X(s)}\, ds}{\int_0^1 e^{\beta X(s)}\, ds}$$

where

- $X = X(t)$ is a *Gaussian random field*, the restriction of 2D Gaussian free field on the unit circle,
- $0 \leq \beta < \beta_0$, where β_0 is a 'critical value'.

Recall: Gaussian random variables are determined by their expectation (take zero) and covariance.

Gaussian free field (GFF), restricted to $\mathbb{T}$:

- $X(\zeta)$ is a *Gaussian random field* with covariance

$$\mathbb{E}X(\zeta)X(\xi) = \log \frac{1}{|\zeta - \xi|}, \quad \zeta, \xi \in \mathbb{T}.$$

(Conformally invariant modulo constants !)

- X is $\mathcal{D}'(\mathbb{T})$ -valued random field: need some care!

Existence:

Set

$$X = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \Big(A_n \cos(2\pi n t) + B_n \sin(2\pi n t) \Big), \qquad t \in [0,1),$$

where

$$A_n \sim N(0,1) \sim B_n \quad (n \geq 1)$$

are independent standard Gaussians.

Geometric representation of our free field in terms of white noise:

(Bacry, Muzy):

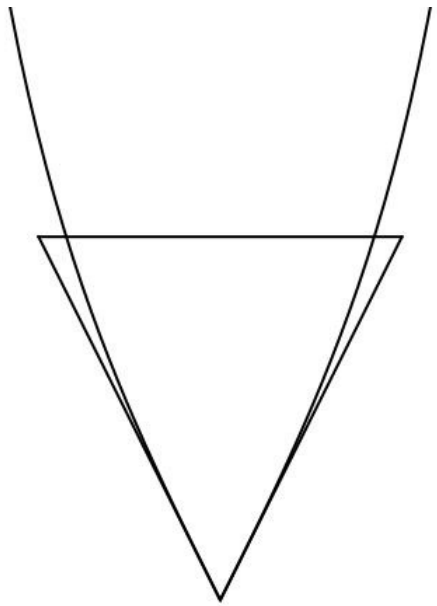
$$X(s) = \int_{H+s} W(dxdy)$$

Here W is the (periodic) white noise in the upper half plane $\mathbb{H}$ (with respect to the hyperbolic measure in $\mathbb{H}$) and H is the domain

$$H := \{(x, y) \in \mathbb{H} \; : \; -1/2 < x < 1/2, \;\; y > \frac{2}{\pi} \tan(|\pi x|)\}.$$

The field obtained by replacing H with the triangular domain V gives a good approximation of the GFF:

$$V := \{(x, y) \in \mathbb{H} \; : \; -1/4 < x < 1/4, \;\; 2|x| < y < 1/2\}.$$



Integration domain H for GFF and its approximation V.

Random homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$: We take

$$\phi(t) = \int_0^t e^{\beta X(s)} ds \Big/ \int_0^1 e^{\beta X(s)} ds, \qquad \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

where $0 \le \beta < \sqrt{2} =: \beta_0$ and X is the restriction of GFF on $\mathbb{T}$.

- X is $\mathcal{D}'(\mathbb{T})$-valued random field: not clear how to define!
- Regularize $e^{\beta \widetilde{X}_\epsilon(s)} := e^{\beta X_\epsilon(s)} / \mathbb{E} e^{\beta X_\epsilon(s)}$ so that you obtain martingale in decreasing ε.
- Almost surely $e^{\beta \widetilde{X}_\epsilon(s)} ds$ converges weakly to a random Borel measure $\tau(ds) \equiv: e^{\beta X(s)} ds$ on $\mathbb{T}$

Properties of the random measure $\tau(ds) =: e^{\beta X(s)} ds \asymp \phi'$ (case $\beta < \sqrt{2}$):

- τ has no atoms

- $\mathbb{E}\,\tau(I)^p < \infty, \quad$ for $-\infty < p < 2/\beta^2$ and all intervals $I \subset \mathbb{T}$

Hence by Hölder, the distortion satisfies

$$\frac{|\phi(s+t) - \phi(s)|}{|\phi(s-t) - \phi(s)|} = \frac{\tau([s, s+t])}{\tau([s-t, s])} \in L^p(d\omega), \quad p < 2/\beta^2.$$

Main Theorem. (Astala-Jones-Kupiainen-S) (i) *Let* $\phi = \phi_\beta$ *be the random homeomorphism*

$(*)$ $\qquad \phi_\beta(s) = \tau([0,s])/\tau([0,1]) \qquad$ *(here* $\tau(ds) = e^{\beta X(s)} ds$, $\beta < \sqrt{2}$*).*

Then a.s. in ω*, the random homeo* ϕ_β *admits a conformal welding*

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(ii) *Dependence on* β *is continuous 'pathwise'.*

(iii) *The above statements hold true if* ϕ_β *is replaced by*

$$\psi = \phi_\beta \circ (\widetilde{\phi}_{\widetilde{\beta}})^{-1},$$

where $\beta, \widetilde{\beta} < \sqrt{2}$ *and* ϕ_β *and* $\widetilde{\phi}_{\widetilde{\beta}}$ *are independent copies of (*).*

Outline of the proof.

1.Extension of ϕ to $f : \mathbb{C} \to \mathbb{C}$ by a Beurling-Ahlfors-type extension

$\implies$ bound for $\mu = \bar{\partial} f / \partial f$ in terms of the measure τ.

2. Existence for Beltrami equation by the Lehto method to control moduli of annuli

3. The crucial ingredient for step 2: specialized large deviation estimates for Lehto integrals that control moduli of annuli

4. Uniqueness of welding: theorem of Jones-Smirnov on removability of Hölder curves

Control of the distortion of the Beurling-Ahlfors extension of ϕ:

- Let $\mathcal{D}_n =$ dyadic intervals on $\partial\mathbb{D}$, $n \in \mathbb{N}$, $\mathcal{D} = \cup \mathcal{D}_n$

- Tile $\mathbb{D}$ by Whitney cubes C_I , $I \in \mathcal{D}$, size $\sim 2^{-n}$ distance 2^{-n} to $\partial\mathbb{D}$

Distortion bound for F. For all $z \in C_I$, $I \in \mathcal{D}_n$

$$K(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \leq C \sum_{J,J'} \frac{\tau(J)}{\tau(J')}$$

Here $J, J' \in \mathcal{D}_{n+5}$ contained in I and its dyadic neighbors.

For Proof: we use a Beurling-Ahlfors-type extension of ϕ to $\mathbb{D}$ with controlled distortion:

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Linear and local bound needed to cover all $\beta < \sqrt{2}$.

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But, now μ is not bounded away from 1.

In Brief: Need to solve the welding problem, for homeo's

$$\phi : \mathbb{T} \to \mathbb{T}$$

of unbounded distortion.

$\rightarrow$ Degenerate elliptic systems/ Beltrami equations.

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$\to$ Uniqueness not obvious.

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of unbounded distortion.

$\to$ Degenerate elliptic systems/ Beltrami equations.

$\to$ Existence or uniqueness of homeomorphic solutions not obvious.

Especially: Our μ very strongly non-degenerate, e.g., almost surely $K \notin L^p(\mathbb{D})$ for $p \geq 2 + \varepsilon(\beta)$, where

$$K(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

$\Rightarrow$ we are far outside the realm of mappings of exponentially intergable distortion !

Intermission: the Lehto method

Lehto integral. When solving degenerate Beltrami equations, Lehto had the idea to control images of annuli under f with the distortion K in terms of the Lehto integral:

$$L_K(w, r, R) = L(w, r, R) := \int_r^R \frac{1}{\int_0^{2\pi} K\left(w + \rho e^{i\theta}\right) d\theta} \frac{d\rho}{\rho},$$

over the annulus $A = A(w, r, R)$, with center w, radii $0 < r < R$.

- Consider open topological annulus, with $\mathbb{C} \setminus A = E \cup F$, ($E$ and F are disjoint, connected and closed, E bounded).

- Γ_A stands for the closed curves that lie in A and have winding number 1 with respect to E. One defines (Mod is the standard 2-modulus)

$$\mathrm{Mod}(A) := \mathrm{Mod}(\Gamma_A).$$

- Observe: $\quad \mathrm{Mod}(A(w, r, R)) = (2\pi)^{-1} \log(R/r)$.

Lemma 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be an orientation preserving map of finite distortion, and $K = K_f$. Then one has for every circular annulus $A(w, r, R)$*

$$\mathrm{Mod}(f(A(w, r, R))) \geq L_K(w, r, R)$$

Proof. May assume $w = 0$. Denote $A' = F(A(w,r,R))$ and $S_r = \{|z| = r\}$. Let ρ be any test function for the modulus of A'. Especially one has for any $t \in (r,R)$

$$\int_{f(S_t)} \rho|ds'| \geq 1 \quad \Longrightarrow \quad 1 \leq \int_{S_t} \rho \circ f|Df||ds| = \int_{S_t} \rho \circ f(J_f K)^{1/2}|ds|.$$

By Cauchy-Schwarz

$$\left(\int_{S_t} K|ds|\right)^{-1} \leq \int_{S_t} \rho^2 \circ f J_f|ds|$$

and sfter integration with respect to $t \in (r,R)$

$$L_K(0,r,R) \leq \int_A \rho^2(f)J_f = \int_{A'} \rho^2. \qquad \square$$

For a topological annulus A, let $d(A)$ stand for the inner and $D(A)$ for the outer diameter of $F(A)$.

Corollary. *Let $f:\mathbb{C}\to\mathbb{C}$ be an orientation preserving map of finite distortion, and $K = K_f$. Then one has for every circular annulus $A(w,r,R)$*

$$d(A) \le c\, D(A)\, e^{-c' L_K(w,r,R)}.$$

Proof. One simply uses (iterates suitably) the classical fact that for a topological annulus one has

$$\mathrm{Mod}(A) \le \Phi\left(\frac{\mathrm{dist}(E,F)}{\mathrm{diam}(E)}\right),$$

where $\Phi : (0,\infty) \to (0,\infty)$ is an increasing homeo. $\square$

We need to recall our old Lemma (just slightly reformulated):

Old Lemma:) *Assume that the dilatation μ has compact support and satisfies $\frac{1+|\mu|}{1-|\mu|} = K \in L^1_{loc}$. Assume additionally that we know that approximating functions f_n (corresponding to principal solutions with $\mu_n = \mu(1 - 1/n)$) are equicontinuous. Then a subsequence of (f_n) converges to a principal solution of the Beltrami equation with dilatation μ.*

Theorem a' la Lehto. *Assume that the dilatation μ has compact support and satisfies $\frac{1+|\mu|}{1-|\mu|} = K \in L^1_{loc}$. Then there exists a principal solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ assuming that*

$$L_K(w,r,1) \to \infty \qquad \text{for every } \; w \in \mathbb{D} \quad \text{as } \; r \to 0.$$

Moreover, if there is the uniform bound

$$L_K(w,r,1) \geq c \log(1/r) - c' \qquad \text{for every } \; w \in \mathbb{D} \; \text{ and } \; r \in (0,1),$$

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<u>Proof.</u> Take any two points $z, w \in 2\mathbb{D}$. Simply observe that as $K_{f_n} \leq K$, the Corollary yields uniformly in n

$$|f_n(z) - f(w)| \leq c\, D(f_n(w, |z-w|, 1))\, e^{-c' L_K(w,|z-w|,1)}.$$

By Koebe etc. the outer diameters D are uniformly bounded. $\square$

Back to random weldings

Uniqueness for welding follows from Hölder continuity:

Suppose $f_\pm$ and $\tilde{f}_\pm$ are two solutions, mapping $\mathbb{D}, \mathbb{D}_\infty$ onto $\Omega_\pm$ and $\widetilde{\Omega}_\pm$. Claim:

$$\tilde{f}_\pm = \Psi \circ f_\pm, \qquad \Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ Möbius.}$$

Proof: Define

$$\Psi(z) := \begin{cases} \tilde{f}_+ \circ \left(f_+\right)^{-1}(z) & \text{if } z \in \Omega_+ \\ \tilde{f}_- \circ (f_-)^{-1}(z) & \text{if } z \in \Omega_- \end{cases}$$

Then Ψ is continuous on $\widehat{\mathbb{C}}$ and conformal outside $\Gamma = \partial\Omega_\pm$.

Theorem (Jones-Smirnov): Hölder curves are conformally removable i.e. Ψ extends conformally to $\widehat{\mathbb{C}}$. Hence Ψ is Möbius.

Needed technical estimates:

Key probabilistic estimate: $L(w, r, 1) \geq -c \log r$ with high probability. More precisely: for small enough δ

$$\text{Prob}(L(w, \rho^n, 1) < n\delta) \leq \rho^{-(1+\varepsilon(\beta))n} \qquad (1)$$

where $\rho < 1$.

Assume the key estimate holds true:

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Outline for the rest: it suffices to consider behaviour at points $w \in \partial\mathbb{D} = \mathbb{T}$.

- For any $n \geq 1$ pick a grid $\mathcal{A}_n$ on $\mathbb{T}$ with spacing $\rho^{n(1+\varepsilon')}$. Get points w_i, $i = 1, \ldots \rho^{-n}$.

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- Borell-Cantelli lemma gives a.s. finite $n(\omega)$ such that

$$L(w_i, \rho^n, 1) > n\delta \quad \text{for all } w_i \in \mathcal{A}_n \text{ if } n > n(\omega)$$

$\Longrightarrow$ a.s. Hölder continuity.

To prove the key estimate (1) one must to study

Large Deviations for Lehto integrals.

Recall $L(w,\rho^n,1) = \int_{\rho^n}^{1} \frac{1}{\int_0^{2\pi} K\left(w + re^{i\theta}\right) d\theta} \frac{dr}{r}$

Reduce to sum of weakly correlated random variables:

- Suffices to take $w \in \partial\mathbb{D} = \mathbb{T}$, say $w = 1$.

- Take $\rho < 1/2$ and let $L_k = L(1, \rho^k, 2\rho^k)$. Then

$$L(w,\rho^n,1) \geq \sum_{k=1}^{n} L_k$$

Prove: $\quad \text{Prob}(\sum_{k=1}^{n} L_k < n\delta) < e^{-g(\delta)n}$,

where $\quad L_k = L(1, \rho^k, 2\rho^k)$.

Prove: $\quad$ Prob$(\sum_{k=1}^{n} L_k < n\delta) < e^{-g(\delta)n}$,

Point: $\quad$ • $\quad L_k \geq 0$ are identically distributed and $P(L_k = 0) = 0$.

BUT: they are correlated

$\Longrightarrow$ one needs to estimate the 'damage' caused by the correlation!

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- The correlation structure is quite complicated.

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(We skip the proof, that is the most technical part of the ppaper.)

Relation to SLE:

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• [Sheffield 2010, November, manuscript] states that SLE_κ is indeed obtained from 'welding' of two weighted 'quantum wedges'! Locally this corresponds to welding of $(\phi_\beta)^{-1} \circ \widetilde{\phi}_\beta$, again with $\kappa = 2\beta^2$.

Brownian motion in the space of self-homeos of $\mathbb{T}$

was defined by Malliavin. In this approach

• Weldings for random homeomorphisms generated by the (formal) stochastic flow on $\mathbb{T}$:

$$d\theta_t = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^3 - k}} \Big(\cos(k\theta_t) dX_{2k}(t) + \sin(k\theta_t) dX_{2k+1}(t) \Big).$$

Here X_k:s are independent Brownian motions.

• [Airault, Malliavin,Thalmaier 2004] starts to consider welding problems for corresponding homeos. There is a recent paper [Airault, Malliavin,Thalmaier 2010] containing a new approach..

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- THANKS !