Composition of q-quasiconformal mappings and functions in Orlicz-Sobolev spaces

Luděk Kleprlík and Stanislav Hencl

Faculty of Mathematics and Physics, Charles University

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?
Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?
Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping
BMO - quasiconformal mappings *(H. M. Reimann)*

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?
Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping
BMO - quasiconformal mappings *(H. M. Reimann)*
fractional Sobolev spaces $\dot{\mathbf{M}}^{\mathbf{s}}_{\mathbf{n/s,q}}$, $s \in (0,1]$ - quasiconformal mappings *(P. Koskela, D. Yang and Y. Zhou)*

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?
Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping
BMO - quasiconformal mappings *(H. M. Reimann)*
fractional Sobolev spaces $\dot{\mathbf{M}}^{\mathbf{s}}_{\mathbf{n/s,q}}$, $s \in (0,1]$ - quasiconformal mappings *(P. Koskela, D. Yang and Y. Zhou)*
Absolutely continuous functions of several variables $\mathbf{AC}^{\mathbf{n}}_{\lambda}$ - quasiconformal mappings *(S. Hencl)*
Exponential Orlicz space $\exp \mathbf{L}(\boldsymbol{\Omega})$ in plane - quasiconformal mappings *(F. Farroni and R. Giova)*.

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?
Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping
BMO - quasiconformal mappings *(H. M. Reimann)*
fractional Sobolev spaces $\dot{\mathbf{M}}^{\mathbf{s}}_{\mathbf{n/s,q}}$, $s \in (0, 1]$ - quasiconformal mappings *(P. Koskela, D. Yang and Y. Zhou)*
Absolutely continuous functions of several variables $\mathbf{AC}^{\mathbf{n}}_{\lambda}$ - quasiconformal mappings *(S. Hencl)*
Exponential Orlicz space $\exp \mathbf{L}(\mathbf{\Omega})$ in plane - quasiconformal mappings *(F. Farroni and R. Giova)*.
$\mathbf{W^{1,q}}$ q-quasiconformal mappings *(V. Gold'shtein L. Gurov and A. Romanov; S. K. Vodop'yanov and A. D. Ukhlov)*.

Introduction

Question: When does the composition $u \circ f$ of function u from Banach function space X and homeomorphism f belong again to X?

Known results: $X = \mathbf{W^{1,n}}$ - f has to be quasiconformal mapping

BMO - quasiconformal mappings *(H. M. Reimann)*

fractional Sobolev spaces $\dot{\mathbf{M}}^{\mathbf{s}}_{\mathbf{n/s,q}}$, $s \in (0,1]$ - quasiconformal mappings *(P. Koskela, D. Yang and Y. Zhou)*

Absolutely continuous functions of several variables $\mathbf{AC}^{\mathbf{n}}_{\lambda}$ - quasiconformal mappings *(S. Hencl)*

Exponential Orlicz space $\exp \mathbf{L}(\boldsymbol{\Omega})$ in plane - quasiconformal mappings *(F. Farroni and R. Giova)*.

$\mathbf{W^{1,q}}$ q-quasiconformal mappings *(V. Gold'shtein L. Gurov and A. Romanov; S. K. Vodop'yanov and A. D. Ukhlov)*.

Definition

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that homeomorphism $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ is a q-quasiconformal mapping if there is a constant $K \geq 1$ such that

$$|Df(x)|^q \leq K|J_f(x)| \text{ for a.e. } x \in \Omega \,.$$

Orlicz Space

A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function if $\Phi(0) = 0$, Φ is increasing and convex.
Denote by $L^\Phi(A)$ the corresponding Orlicz space with Young function Φ on a set A with measure $\mathcal{L}^n$. This space is equipped with the Luxemburg norm

$$||f||_{L^\Phi(A)} = \inf\{\lambda > 0 : \int_A \Phi(|f(x)|/\lambda)\, dx \leq 1\} .$$

Orlicz Space

A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function if $\Phi(0) = 0$, Φ is increasing and convex.
Denote by $L^\Phi(A)$ the corresponding Orlicz space with Young function Φ on a set A with measure $\mathcal{L}^n$. This space is equipped with the Luxemburg norm

$$||f||_{L^\Phi(A)} = \inf\{\lambda > 0 : \int_A \Phi(|f(x)|/\lambda)\, dx \leq 1\} .$$

For $q \geq 1$ and $\alpha \in \mathbb{R}$ we denote by $L^q \log^\alpha L(A)$ the Orlicz space with a Young function such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^q \log^\alpha t} = 1 .$$

Orlicz Space

A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function if $\Phi(0) = 0$, Φ is increasing and convex.
Denote by $L^\Phi(A)$ the corresponding Orlicz space with Young function Φ on a set A with measure $\mathcal{L}^n$. This space is equipped with the Luxemburg norm

$$||f||_{L^\Phi(A)} = \inf\{\lambda > 0 : \int_A \Phi(|f(x)|/\lambda)\, dx \leq 1\} .$$

For $q \geq 1$ and $\alpha \in \mathbb{R}$ we denote by $L^q \log^\alpha L(A)$ the Orlicz space with a Young function such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^q \log^\alpha t} = 1 .$$

We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u : |Du| \in L^\Phi(A)\}$$

Stability of $WL^q \log^\alpha L$ under q-quasiconformal mappings

Theorem

Let

- $q \geq n$ *and* $\alpha \geq 0$

or

- $1 < q \leq n$ *and* $\alpha \leq 0$

and suppose that $f : \Omega_1 \to \Omega_2$ *is a* q*-quasiconformal mapping.*
Then T_f *maps* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2) \cap C(\Omega_2)$ *into* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ *for* $q > n$
and T_f *maps* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2)$ *into* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ *for* $q \leq n$.

Stability of $WL^q \log^\alpha L$ under q-quasiconformal mappings

Theorem

Let

- $q \geq n$ *and* $\alpha \geq 0$

or

- $1 < q \leq n$ *and* $\alpha \leq 0$

and suppose that $f : \Omega_1 \to \Omega_2$ *is a* q*-quasiconformal mapping.*
Then T_f *maps* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2) \cap C(\Omega_2)$ *into* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ *for* $q > n$
and T_f *maps* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2)$ *into* $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ *for* $q \leq n$.

Theorem

Let $f : \Omega_1 \to \Omega_2$ *be a homeomorphism,* $q \geq 1$ *and let* $\alpha \in \mathbb{R}$*. For* $q \leq n-1$ *we moreover assume that* f *is differentiable a.e. Suppose that* T_f *maps* $WL^q \log^\alpha L(\Omega_2)$ *into* $WL^q \log^\alpha L(\Omega_1)$ *continuously, that is*

$$\|Du \circ f\|_{L^q \log^\alpha L(\Omega_1)} \leq C \|Du\|_{L^q \log^\alpha L(\Omega_2)}$$

for every $u \in WL^q \log^\alpha L(\Omega_2) \cap C(\Omega_2)$*. Then* f *is a* q*-quasiconfomal mapping.*

Proof. Let $u \in WL^q \log^\alpha L_{\mathrm{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\mathrm{loc}}$ and therefore $u \circ f \in W^{1,q}_{\mathrm{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

Proof. Let $u \in WL^q \log^\alpha L_{\mathrm{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\mathrm{loc}}$ and therefore $u \circ f \in W^{1,q}_{\mathrm{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

$$\begin{aligned}\int_A &|Du \circ f|^q \log^\alpha(e + |Du \circ f|) \\ &\leq \int_A |Du(f(x))|^q |Df(x)|^q \log^\alpha(|Du(f(x))||Df(x)|)\, dx.\end{aligned}$$

Proof. Let $u \in WL^q \log^\alpha L_{\mathrm{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\mathrm{loc}}$ and therefore $u \circ f \in W^{1,q}_{\mathrm{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

$$\int_A |Du \circ f|^q \log^\alpha(e + |Du \circ f|)$$
$$\leq \int_A |Du(f(x))|^q |Df(x)|^q \log^\alpha(|Du(f(x))||Df(x)|)\, dx.$$

There exists $p > q$ such that $f \in W^{1,p}(A, \mathbb{R}^n)$.

Proof. Let $u \in WL^q \log^\alpha L_{\text{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\text{loc}}$ and therefore $u \circ f \in W^{1,q}_{\text{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

$$\int_A |Du \circ f|^q \log^\alpha(e + |Du \circ f|)$$
$$\leq \int_A |Du(f(x))|^q |Df(x)|^q \log^\alpha(|Du(f(x))||Df(x)|)\, dx.$$

There exists $p > q$ such that $f \in W^{1,p}(A, \mathbb{R}^n)$. Let us fix $q < s < p$ and we will divide integral into two integrals over sets

$$U = \{x \in A : \ |Du(f(x))| \geq |Df(x)|^{\frac{s-q}{q}}\} \ \text{ and}$$
$$V = \{x \in A : \ |Du(f(x))| \leq |Df(x)|^{\frac{s-q}{q}}\}.$$

Proof. Let $u \in WL^q \log^\alpha L_{\mathrm{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\mathrm{loc}}$ and therefore $u \circ f \in W^{1,q}_{\mathrm{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

$$\int_A |Du \circ f|^q \log^\alpha(e + |Du \circ f|)$$
$$\leq \int_A |Du(f(x))|^q |Df(x)|^q \log^\alpha(|Du(f(x))||Df(x)|)\, dx.$$

There exists $p > q$ such that $f \in W^{1,p}(A, \mathbb{R}^n)$. Let us fix $q < s < p$ and we will divide integral into two integrals over sets

$$U = \{x \in A : \ |Du(f(x))| \geq |Df(x)|^{\frac{s-q}{q}}\} \ \text{ and}$$
$$V = \{x \in A : \ |Du(f(x))| \leq |Df(x)|^{\frac{s-q}{q}}\}.$$

$$\int_U |Du(f(x))|^q |Df(x)|^q \log^\alpha\big(e + |Du(f(x))||Df(x)|\big)\, dx + \int_V \ldots dx$$
$$\leq \int_U |Du(f(x))|^q K|J_f(x)| \log^\alpha\big(e + |Du(f(x))|^{\frac{s}{s-q}}\big)\, dx +$$
$$+ \int_V |Df(x)|^s \log^\alpha\big(e + |Df(x)|^{\frac{s}{q}}\big)\, dx$$

Proof. Let $u \in WL^q \log^\alpha L_{\text{loc}} \cap C$ and $A \subset\subset \Omega_1$. It follows that $u \in W^{1,q}_{\text{loc}}$ and therefore $u \circ f \in W^{1,q}_{\text{loc}}$ and $D(u \circ f) = \big((Du) \circ f\big) \cdot Df$.

$$\int_A |Du \circ f|^q \log^\alpha(e + |Du \circ f|)$$
$$\leq \int_A |Du(f(x))|^q |Df(x)|^q \log^\alpha(|Du(f(x))||Df(x)|)\, dx.$$

There exists $p > q$ such that $f \in W^{1,p}(A, \mathbb{R}^n)$. Let us fix $q < s < p$ and we will divide integral into two integrals over sets

$$U = \{x \in A : \ |Du(f(x))| \geq |Df(x)|^{\frac{s-q}{q}}\} \ \text{ and}$$
$$V = \{x \in A : \ |Du(f(x))| \leq |Df(x)|^{\frac{s-q}{q}}\}.$$

$$\int_U |Du(f(x))|^q |Df(x)|^q \log^\alpha\big(e + |Du(f(x))||Df(x)|\big)\, dx + \int_V \dots dx$$
$$\leq \int_U |Du(f(x))|^q K |J_f(x)| \log^\alpha\big(e + |Du(f(x))|^{\frac{s}{s-q}}\big)\, dx +$$
$$+ \int_V |Df(x)|^s \log^\alpha\big(e + |Df(x)|^{\frac{s}{q}}\big)\, dx$$
$$\leq CK \int_{f(A)} \big(1 + |Du(y)|^q \log^\alpha(e + |Du(y)|)\big)\, dy + C \int_A (|Df(x)|^p + 1)\, dx.$$

Theorem

Let $f : \Omega_1 \to \Omega_2$ *be a homeomorphism,* $q \geq 1$ *and let* $\alpha \in \mathbb{R}$*. For* $q \leq n-1$ *we moreover assume that* f *is differentiable a.e. Suppose that* T_f *maps* $WL^q \log^\alpha L(\Omega_2)$ *into* $WL^q \log^\alpha L(\Omega_1)$ *continuously, that is*

$$\|Du \circ f\|_{L^q \log^\alpha L(\Omega_1)} \leq C\|Du\|_{L^q \log^\alpha L(\Omega_2)}$$

for every $u \in WL^q \log^\alpha L(\Omega_2) \cap C(\Omega_2)$*. Then* f *is a q-quasiconfomal mapping.*

Theorem

Let $f:\Omega_1\to\Omega_2$ be a homeomorphism, $q\geq 1$ and let $\alpha\in\mathbb{R}$. For $q\leq n-1$ we moreover assume that f is differentiable a.e. Suppose that T_f maps $WL^q\log^\alpha L(\Omega_2)$ into $WL^q\log^\alpha L(\Omega_1)$ continuously, that is

$$\|Du\circ f\|_{L^q\log^\alpha L(\Omega_1)}\leq C\|Du\|_{L^q\log^\alpha L(\Omega_2)}$$

for every $u\in WL^q\log^\alpha L(\Omega_2)\cap C(\Omega_2)$. Then f is a q-quasiconfomal mapping.

Sketch of proof. Fix r_0 such that for all $r\in(0,r_0)$ we have

$$|f(B(x_0,2r))|\approx|J_f||B(x_0,2r)|.$$

We construct a function such that

(1) support $\psi\subset f(B(x_0,2r))$ (2) ψ is Lipschitz with constant 1
(3) $\psi\in WL^\Phi(f(\Omega))$ (4) ψ is differentiable almost everywhere
(5) $\psi(x)=\pm x_j+const$ in all components of the set $f(B(x_0,r))\setminus E$.

The continuity of T_f give us

$$\|Df_j\|_{L^\Phi(B(x_0,r))} = \|D(\psi\circ f)\|_{L^\Phi} \leq C\|D\psi\|_{L^\Phi(f(B(x_0,2r)))} \leq C\|1\|_{L^\Phi(f(B(x_0,2r)))}.$$

The continuity of T_f give us

$$\|Df_j\|_{L^\Phi(B(x_0,r))} = \|D(\psi\circ f)\|_{L^\Phi} \leq C\|D\psi\|_{L^\Phi(f(B(x_0,2r)))} \leq C\|1\|_{L^\Phi(f(B(x_0,2r)))}.$$

$$\begin{aligned}
\liminf_{r\to 0_+} \frac{\|Df_j\|_{L^\Phi(B(x_0,r))}}{\|1\|_{L^\Phi(B(x_0,r))}} &\leq C \liminf_{r\to 0_+} \frac{\|1\|_{L^\Phi(f(B(x_0,2r)))}}{\|1\|_{L^\Phi(B(x_0,r))}} \\
&\leq C \liminf_{r\to 0_+} \frac{\Phi^{-1}\left(\frac{1}{|B(x_0,r)|}\right)}{\Phi^{-1}\left(\frac{1}{|f(B(x_0,2r))|}\right)} \\
&\lesssim C \liminf_{r\to 0_+} \frac{\Phi^{-1}\left(\frac{1}{|B(x_0,r)|}\right)}{\Phi^{-1}\left(\frac{1}{|J_f||B(x_0,r)|}\right)} \leq C|J_f|^{\frac{1}{q}} .
\end{aligned}$$

The continuity of T_f give us

$$\|Df_j\|_{L^\Phi(B(x_0,r))} = \|D(\psi\circ f)\|_{L^\Phi} \leq C\|D\psi\|_{L^\Phi(f(B(x_0,2r)))} \leq C\|1\|_{L^\Phi(f(B(x_0,2r)))}.$$

$$\begin{aligned}
\liminf_{r\to 0_+} \frac{\|Df_j\|_{L^\Phi(B(x_0,r))}}{\|1\|_{L^\Phi(B(x_0,r))}} &\leq C \liminf_{r\to 0_+} \frac{\|1\|_{L^\Phi(f(B(x_0,2r)))}}{\|1\|_{L^\Phi(B(x_0,r))}} \\
&\leq C \liminf_{r\to 0_+} \frac{\Phi^{-1}\left(\frac{1}{|B(x_0,r)|}\right)}{\Phi^{-1}\left(\frac{1}{|f(B(x_0,2r))|}\right)} \\
&\lesssim C \liminf_{r\to 0_+} \frac{\Phi^{-1}\left(\frac{1}{|B(x_0,r)|}\right)}{\Phi^{-1}\left(\frac{1}{|J_f||B(x_0,r)|}\right)} \leq C|J_f|^{\frac{1}{q}} \ .
\end{aligned}$$

The Lebesgue density theorem for Orlicz functions now gives us

$$|Df_j(x_0)| \leq C(|J_f|)^{\frac{1}{q}}.$$

Lebesgue density theorem for Orlicz functions

Theorem

Suppose that Φ is a Young function and let $f \in L^{\Phi}(\Omega)$ be nonnegative. Then

$$\liminf_{r \to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \geq f(x) \text{ for almost every } x \in \Omega\ .$$

If we moreover assume that our Φ satisfies

$$\Phi(ab) \leq C\Phi(a)\Phi(b) \text{ for every } a, b \geq 0,$$

then

$$\lim_{r \to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = f(x) \text{ for almost every } x \in \Omega\ .$$

Thanks for your attention!