Two Conjectures of Astala on Distortion of Sets under Quasiconformal Maps and related Removability Problems

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Size of Sets: Hausdorff Measures and Dimension

Definition

A "gauge function" is a continuous increasing $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$.
Let $E \subset \mathbb{C}$ be compact and let $0 < \delta \leq \infty$. Define

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} h(r_i) : E \subset \cup_{i=1}^{\infty} B(x_i, r_i), 2r_i \leq \delta \right\}$$

If $\delta = \infty$: $\mathcal{H}^h_\infty$ is called Hausdorff content.

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E) \text{ ``Hausdorff Measure''}$$

If $h(t) = t^s$, then $\mathcal{H}^h(E) = \mathcal{H}^s(E)$
$Dim(E) = \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{t > 0 : \mathcal{H}^t(E) = \infty\}$
$\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s_\infty(E) = 0$

An orientation-preserving homeomorphism $\phi : \Omega \to \Omega'$ between planar domains $\Omega, \Omega' \subset \mathbb{C}$ is called K-quasiconformal (K-QC) if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the *distortion inequality*

$$\max_{\alpha} |\partial_\alpha \phi| \leq K \min_{\alpha} |\partial_\alpha \phi| \quad \text{a.e. in } \Omega \,. \tag{1}$$

Actually, $J(z, \phi) \leq |D\phi(z)|^2 \leq K \; J(z, \phi)$

If ϕ is just required to be continuous instead of homeomorphism: ϕ is K-quasiregular (K-QR)

In $\mathbb{C}$, any ϕ K-QR solves the Beltrami equation $\overline{\partial}\phi = \mu \partial \phi$, where $\|\mu\|_\infty \leq \frac{K-1}{K+1} =: k < 1$. (Recall $\overline{\partial} f = \frac{1}{2}(f_x + i f_y)$ and $\partial f = \frac{1}{2}(f_x - i f_y)$.)

Using Calderón-Zygmund theory: there is a unique homeomorphic normalized solution f to Beltrami (e.g. fixing 0, 1, ∞.)

- Generalize conformal maps to $\mathbb{R}^n$
- Geometry: hyperbolic 3-manifolds, Kleinian groups
- Dynamics: quasiconformal removability appears in problems related to MLC, Fatou conjecture (that hyperbolic systems are dense.) It started with $p_i(z) = z^2 + i$: exterior of Julia set is John domain, so its boundary is removable (first non-trivial example, non-quasicircle, -dendrite-): Jones (91).
- PDE: complex gradient $(u_x, -u_y)$ of solutions to p-Laplacian $div(|\nabla u|^{p-2}\nabla u) = 0$ are qr. (Bojarski, Iwaniec, $p > 2$; Manfredi, 88, $1 < p < \infty$)
- PDE: inverse problems: geophysical prospecting, medical imaging (electrical impedance tomography): assume $\Omega \subset \mathbb{R}^n$ is bounded domain, connected complement, and $\sigma, \sigma^{-1} : \Omega \to (0, \infty)$ measurable, bounded.

Why QC maps are important II: inverse problems cont'd

Given boundary values $\phi \in H^{\frac{1}{2}}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the unique solution to the conductivity equation:

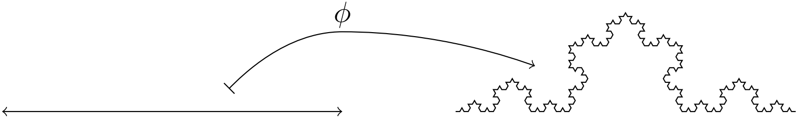
$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega \tag{2}$$

It describes behavior of electric potential in a conductive body.
A.P. Calderón (80): can one recover conductivity σ from boundary measurements? I.e. from Dirichlet to Neumann map
$\Lambda_\sigma : \phi \to \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$
ν : outer normal to $\partial\Omega$; $\sigma \frac{\partial u}{\partial \nu} \in H^{-\frac{1}{2}}(\partial\Omega)$, defined by
$\langle \sigma \frac{\partial u}{\partial \nu}, \psi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \psi \, dm$, where $\psi \in H^1(\Omega)$, $dm =$ Lebesgue.
Sylvester, Uhlmann (Annals 87); Nachman (Annals 96): let $v = \sigma^{\frac{1}{2}} u$. From (2), get $\Delta v - qv = 0$ where $q = \sigma^{-\frac{1}{2}} \Delta \sigma^{\frac{1}{2}}$: need 2 derivatives (smoothness in σ.)
Astala, Päivärinta (Annals 06): use qc maps substitution: positive answer to Calderón's problem in $\mathbb{R}^2$ with $\sigma \in L^\infty$.

- PDE: nonlinear elasticity: Antman, Ball, Ciarlet: maps minimizing energy integrals: they are not always qr, but same governing PDEs.

Infinitesimally, quasiconformal mappings carry circles to ellipses with eccentricity at most K.
Macroscopically: they take discs to "quasidiscs" (one can inscribe and circunscribe circles of radii r and R such that $\frac{R}{r} \leq C(K)$).
I.e. something somewhat round, neither "cigarettes" nor "horseshoes".

These mappings can distort dimension



Theorem (Stoilow Factorization)

Let f be the homeomorphic normalized solution to the Beltrami equation:

$$\overline{\partial}\phi = \mu\partial\phi, \text{ where } \|\mu\|_\infty < 1. \tag{3}$$

Let $g \in W^{1,2}_{loc}(\Omega)$ be another solution to (3) Then there exists a holomorphic $\psi : f(\Omega) \to \mathbb{C}$ such that $g(z) = \psi(f(z))$. Conversely, for any holomorphic ψ in $f(\Omega)$, $\psi \circ f \in W^{1,2}_{loc}$ and solves Beltrami in $f^{-1}(\Omega)$.

Examples of K-quasiconformal f:

- f conformal
- f bilipschitz ($c|z-w| \leq |f(z)-f(w)| \leq C|z-w|$)
- radial stretching: $z \to |z|^{K-1}z$; $z \to |z|^{\frac{1}{K}-1}z$

Theorem (Astala - Area Distortion - Acta '94)

If ϕ is K-quasiconformal, then

$$|\phi(E)| \lesssim |E|^{1/K}.$$

Theorem (Astala - Higher Integrability - Acta '94)

If ϕ is K-quasiconformal, then

$$\|\phi\|_{W^{1,p}_{\mathrm{loc}}} < \infty, \qquad p < \frac{2K}{K-1} (> 2!).$$

Theorem (Astala - H-Dimension Distortion - Acta '94)

For any compact set E with Hausdorff dimension $0 < t < 2$ and any K-quasiconformal mapping ϕ we have

$$\frac{1}{K}\left(\frac{1}{t}-\frac{1}{2}\right) \leq \frac{1}{\dim(\phi E)} - \frac{1}{2} \leq K\left(\frac{1}{t}-\frac{1}{2}\right) \qquad (4)$$

Finally, these bounds are optimal, in that equality may occur in either estimate.

In other words, $\dim(E) \leq t$ implies $\dim(\phi E) \leq t' = \frac{2Kt}{2+(K-1)t}$.
Equation (4) conjectured by Iwaniec, Martin - Acta '93 (n instead of 2.)

Progress on these questions in higher dimensions is much harder.

Main Theorem (Lacey, Sawyer, UT - Acta '10)

If ϕ is a planar K-quasiconformal mapping, $0 \le t \le 2$ and $t' = \frac{2Kt}{2+(K-1)t}$, then we have the implication below for all compact sets $E \subset \mathbb{C}$.

$$\mathcal{H}^t(E) = 0 \quad \Longrightarrow \quad \mathcal{H}^{t'}(\phi E) = 0, \tag{5}$$

IF ϕ IS K-QUASICONFORMAL, ϕ^{-1} IS ALSO K-QUASICONFORMAL

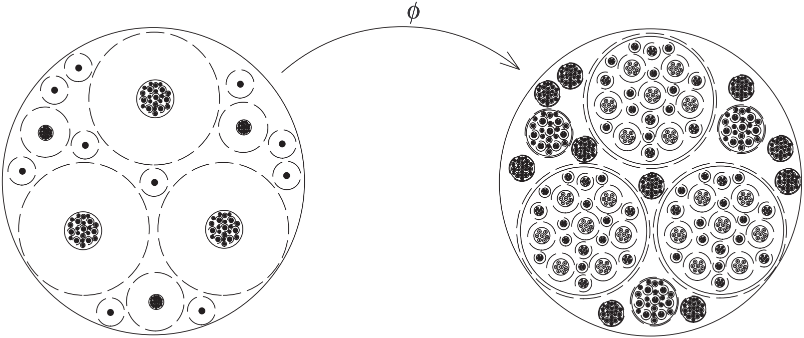
So, it follows that

$$\mathcal{H}^{t'}(\phi E) > 0 \quad \Longrightarrow \quad \mathcal{H}^{t}(E) > 0\,.$$

$\mathcal{H}^t(E) < \infty$ NEED NOT IMPLY $\mathcal{H}^{t'}(\phi E) = 0$. (UT - IMRN '08)

THE CONCLUSION $\mathcal{H}^{t'}(E) = 0$ CANNOT BE STRENGTHENED TO H-D W.R.T. A GAUGE. (UT - IMRN '08)

For any gauge function h satisfying $\lim_{s\to 0} \frac{s^{t'}}{h(s)} = 0$, there exists a compact set E and K-quasiconformal mapping ϕ with $\mathcal{H}^t(E) = 0$ but $\mathcal{H}^h(\phi E) = \infty$.



ϕ

Definition

A *K- quasicircle* is the image of the unit circle under a K-quasiconformal homeomorphism of the Riemann sphere $\hat{\mathbb{C}}$. Sometimes quasilines more convenient (images of $\mathbb{R}$ under a quasiconformal homeomorphism of the finite plane $\mathbb{C}$).

- A simply connected planar domain is a quasidisk if and only if it is a non-tangentially accessible domain (i.e. Harnack chain condition and two-sided corckscrew condition) (Jones).
- A simply connected planar domain D is a quasidisk if and only if it is a BMO extension domain (BMO functions relative to D, i.e. usual def. but for cubes $Q \subset D$, are planar BMO functions restricted to D) (Jones)
- A bounded and finitely connected domain D is extension domain for Sobolev spaces if and only if ∂D is a finite union of points and quasicircles (Jones). (For simply connected domains proved by Goldshtein, Latfullin, Vodopyanov.)

- Complex dynamics (Julia sets, limit sets of quasi-Fuchsian groups): rich source of examples of quasicircles with Hausdorff dimension greater than one.
- Dimension estimates of quasicircles important in theory of Kleinian groups.

Astala's Hausdorff dimension distortion theorem implies that the dimension of a $K = \frac{1+k}{1-k}$-quasicircle Γ satisfies

$$1 - k \leq \dim \Gamma \leq 1 + k \ .$$

The lower bound can clearly be improved: Γ is connected and compact, so $\dim \Gamma = 1$!! Might the upper bound be also improvable?

Examples from complex dynamics (Ruelle) and a previous result of Becker-Pommerenke suggest the behavior

$$\dim \Gamma \leq 1 + ck^2 \ .$$

Conjecture (Astala '94)

Let Γ_k be a $K = \frac{1+k}{1-k}$-quasicircle. Then

$$\dim(\Gamma_k) \leq 1 + k^2 \ ,$$

and this bound is sharp.

Theorem (Smirnov)

Let Γ_k be a $K = \frac{1+k}{1-k}$-quasicircle. Then

$$\dim(\Gamma_k) \le 1 + k^2 \ .$$

Theorem (Astala, Rohde, Schramm)

There exists a $K = \frac{1+k}{1-k}$-quasicircle Γ_k with

$$\dim(\Gamma_k) \ge 1 + 0.69k^2 \ .$$

So sharpness is still open (important connections to extremal behaviour of harmonic measure -Prause, Smirnov-)

Theorem (Prause, Tolsa, UT)

If Γ is a K-quasicircle in $\hat{\mathbb{C}}$, then

$$\mathcal{H}^{1+k^2}(\Gamma \cap B(z,r)) \leq C(K) r^{1+k^2} \quad \textit{for all } z \in \mathbb{C} \textit{ and with } k = \frac{K-1}{K+1}.$$

Ideas of proof

- Factorization into "conformal inside" and "conformal outside" parts.
- Conformal inside part: taken care of by Smirnov
- Conformal outside part: use techniques from Lacey-Sawyer-UT (for the proof of first mentioned conjecture of Astala): boundedness of the Beurling transform with respect to weights arising from a special packing condition.
- Extra difficulties: [LSUT] well suited to estimate Hausdorff contents, we need Hausdorff measures. Factorization of quasiconformal maps and packing conditions are much more delicate in our case.

Relevance of the first conjecture: analytic capacity

- $E \subset \mathbb{C}$ compact
- $\gamma(E) = 0$ if and only if for any $U \supset E$ open, any bounded analytic $f : U \setminus E \to \mathbb{C}$, extends analytically to U.
- Independent of U
- $\mathcal{H}^1(E) = 0$ implies $\gamma(E) = 0$ (Painlevé): hence $dim_H(E) < 1$ implies $\gamma(E) = 0$.
- $dim_H(E) > 1$ implies $\gamma(E) > 0$ (Ahlfors - Duke '47)
- If $0 < \mathcal{H}^1(E) < \infty$ (or sigma-finite): $E = G \cup R \cup N$ (disjoint union), with $\mathcal{H}^1(G) = 0$, $\mathcal{H}^1(R \setminus \cup_{i=1}^{\infty} \Gamma_i) = 0$ (Γ_i Lipschitz or C^1 curves), for any rectifiable Γ, $\mathcal{H}^1(N \cap \Gamma) = 0$ (e.g. Garnett set)
  Then, $\gamma(E) = 0$ if and only if $R = \emptyset$ (David - Rev. Mat. Ib. '98; Tolsa - Acta '03)
- General case: crucial Tb theorem (Nazarov-Treil-Volberg - Acta '03); complete solution (Tolsa - Acta '03).

Relevance of the first conjecture: L^∞-K-removability

- $E \subset \mathbb{C}$ compact is L^∞-K-removable if any $f : \mathbb{C} \setminus E \to \mathbb{C}$ in L^∞, K-quasiregular, can be extended to a global ($\mathbb{C} \to \mathbb{C}$) K-quasiregular map.
  By Stoilow and Liouville's thms, such extension is constant.
- By Stoilow's theorem, the critical dimension is $\frac{2}{K+1}$.

Theorem (Astala, Clop, Mateu, Orobitg, UT - Duke '08)

Let $E \subset \mathbb{C}$ compact and f K-quasiregular

(a) $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$ *(sigma-finite) implies* $\mathcal{H}^1(fE) = 0$ *(resp. sigma-finite)*

(b) $\mathcal{H}^{\frac{2}{K+1}}(E)$ *sigma-finite and* $K > 1$ *implies* E *is* L^∞-K*-removable.*

(c) *There exists* E *with* $\dim_H(E) = \frac{2}{K+1}$ *not* L^∞-K*-removable.*

Relevance of the first conjecture: L^∞-K-removability II

- Proof of easier version of case (B): if $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$, E is L^∞-K-removable. (Conjectured Iwaniec-Martin, Acta'93; proved Astala '04 approx.)
- Let E compact, $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$, and $g : \mathbb{C} \setminus E \to \mathbb{C}$ K-quasiregular, bounded.
- By Stoilow: $g = h \circ f$ with h analytic and bounded, and f K-quasiconformal.
- μ defined a.e. : solve Beltrami : f defined everywhere. The problem is whether h can be extended, i.e. whether $\gamma(fE) = 0$ for any K-quasiconformal f.
- By part (A), $\mathcal{H}^1(fE) = 0$ (quasiconformal distortion of sets: Astala's conjecture)
- By Painlevé, $\gamma(fE) = 0$, so h can be extended analytically (removability results for bounded analytic functions, i.e. analytic capacity).

Relevance of the first conjecture: L^∞-K-removability III

Main Theorem (Tolsa, UT '09)

Let $E \subset \mathbb{C}$ compact and $\varphi : \mathbb{C} \to \mathbb{C}$ K-quasiconformal, $K > 1$. If E is contained in a ball B, then

$$\frac{\dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E)}{\operatorname{diam}(B)^{\frac{2}{K+1}}} \geq c^{-1}\left(\frac{\gamma(\varphi(E))}{\operatorname{diam}(\varphi(B))}\right)^{\frac{2K}{K+1}}.$$

Remarks

- $\dot{C}_{\alpha,p}$ is a Riesz capacity associated to non linear potential.
- Indices $\alpha = \frac{2K}{2K+1}$, $p = \frac{2K+1}{K+1}$ are sharp.
- E non K-removable $\Rightarrow \dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E) > 0 \Rightarrow \mathcal{H}^{\frac{2}{K+1}}(E)$ non σ-finite. (Recover Astala, Clop, Mateu, Orobitg, UT '08 (B))
- Theorem only holds for $K > 1$.

$$\dot{C}_{\alpha,p}(E) = \inf\left\{\|\psi\|^p_{\dot{W}^{\alpha,p}} : \psi \in C_c^\infty(\mathbb{C}),\ \psi \geq 1 \text{ on } E\right\}$$

$$= \sup_{\mu\in\mathcal{M}^+(E)} \left(\frac{\mu(E)}{\|I_\alpha * \mu\|_{p'}}\right)^p;\ I_\alpha(x) = \frac{c_\alpha}{|x|^{n-\alpha}} \text{ in } \mathbb{R}^n.$$

$\mathcal{M}^+(E)$ are the positive Radon measures on E.

Wolff:

$$\dot{C}_{\alpha,p}(E) \approx \sup\left\{\mu(E) : \operatorname{supp}(\mu) \subset E;\ \dot{W}^\mu_{\alpha,p}(x) \leq 1\ \forall x \in \mathbb{C}\right\}, \quad (7)$$

where

$$\dot{W}^\mu_{\alpha,p}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{2-\alpha p}}\right)^{p'-1} \frac{dr}{r}.$$

For $\dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E)$,

$$\dot{W}^\mu_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{\frac{2}{K+1}}}\right)^{\frac{K+1}{K}} \frac{dr}{r}.$$

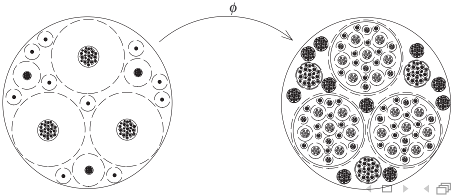
L^∞-K-removability IV: Sharpness

There are sets E with $\mathcal{H}^{\frac{2}{K+1}}(E)$ non σ-finite such that $\dot{C}_{\alpha,p}(E) = 0$ for all α, p with $2 - \alpha\, p = \frac{2}{K+1}$. $\longrightarrow$ New examples of K-removable sets.

Theorem (Tolsa, UT '09)

For all $\beta, q > 0$ such that $2 - \beta q = \frac{2}{K+1}$ and $q' < \frac{K+1}{K}$, there exists $E \subset \mathbb{C}$ and a K-quasiconformal map φ such that $\gamma(\varphi E) > 0$ (and so $\dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E) > 0$), but $\dot{C}_{\beta,q}(E) = 0$.

$$\Rightarrow \qquad \dot{C}_{\alpha,p}(E) = 0 \qquad \text{if } 2 - \alpha p > \frac{2}{K+1}.$$



THEOREM (TOLSA, UT '09)

Let h be a positive function on $(0,\infty)$ such that

$$\varepsilon(r) = \frac{h(r)}{r^{\frac{2}{K+1}}} \to 0 \quad \text{as } r \to 0.$$

Then there is a compact set $E \subset \mathbb{C}$ such that $\mathcal{H}^h(E) = 0$ and a K-quasiconformal map φ such that $\gamma(\varphi(E)) > 0$ (and thus $\dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E) > 0$.)

- E can be taken so that $\mathcal{H}^h(E) = 0$ for all ε which is nondecreasing and

$$\int_0 \left(\frac{h(r)}{r^{\frac{2}{K+1}}}\right)^a \frac{dr}{r} < \infty$$

for some $a > 0$ (E independent of h).

- Instead of L^∞, work with BMO.
- Analytic case: $\gamma_{BMO}(E) = 0$ if and only if $\mathcal{H}^1(E) = 0$ (Král - Wilhelm Pieck Univ. '78; Kaufman - Pacific J. '82)
- K-quasiregular case: if $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$, then E is BMO-K-removable (Astala, Clop, Mateu, Orobitg, UT '08). There exists E with $0 < \mathcal{H}^{\frac{2}{K+1}}(E) < \infty$, not BMO-$K$-removable (UT '08). Hence the L^∞ and BMO K-removability problems are different (answering Question 4.2 in Astala, Clop, Mateu, Orobitg, UT '08)
- Can also work with Lipschitz (or Hölder) (α).
- Analytic case: $\gamma_{Lip(\alpha)}(E) = 0$ if and only if $\mathcal{H}^{1+\alpha}(E) = 0$ (Dolženko - Uspekhi Mat. Nauk. '63)
- K-quasiregular case: let $d_\alpha = \frac{2}{K+1}(1 + \alpha K)$. If $\mathcal{H}^{d_\alpha}(E) = 0$, E is Lip(α)-K-removable (Clop - Ann. Acad. Sci. Fenn. '07). There exists E with $0 < \mathcal{H}^{d_\alpha}(E) < \infty$, not Lip($\alpha$)-$K$-removable (Clop, UT - J. d'Analyse, '09).

Ideas of Proof of the first Conjecture of Astala

Recall statement of Astala's conjecture: (5)

- Can reduce to the case of K-quasiconformal ϕ, with $0 < K < 1 + \epsilon$ very close to one.
- ϕ is a solution to the Beltrami equation $\overline{\partial}\phi = \mu\partial\phi$, where $\|\mu\|_\infty \leq K - 1 \leq \epsilon$.
- We have the Beurling operator which intertwines ∂ and $\overline{\partial}$:

$$\mathrm{B}\, f(z) = \int \frac{f(w)}{(z-w)^2}\, dz \wedge d\overline{z}$$

$$\partial f = 1 + \mathrm{B}(\overline{\partial} f) \qquad (\overline{\partial} f = \mu \partial f; \textit{homeo}, \textit{normalized})$$

- And so, we have a formal expansion

$$\overline{\partial} f = \mu \partial f = \mu + \mu\, \mathrm{B}\, \mu + \mu\, \mathrm{B}(\mu\, \mathrm{B}\, \mu) + \cdots$$

- There is one additional factorization that we will come to.

Approximating E: Packing Condition

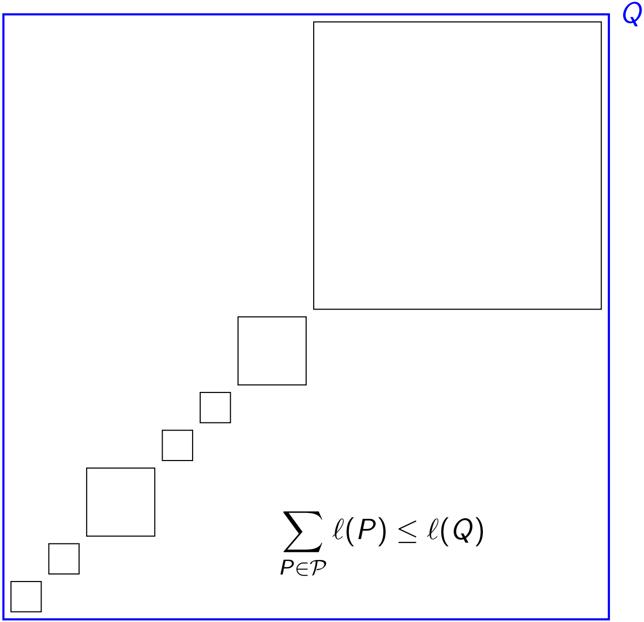
Definition

Let $\mathcal{P}$ be a finite collection of disjoint dyadic cubes in the plane. Let $0 < t < 2$. We denote the t-Carleson packing norm of $\mathcal{P}$ as follows:

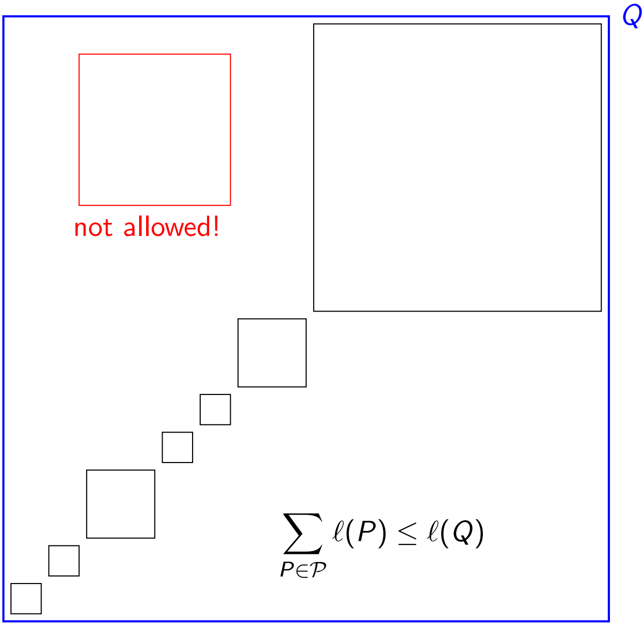
$$\|\mathcal{P}\|_{t\text{-pack}} = \sup_{Q} \left[\ell(Q)^{-t} \sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} \ell(P)^{t} \right]^{1/t},$$

where the supremum is taken over all dyadic cubes Q.


Q



$$\sum_{P \in \mathcal{P}} \ell(P) \leq \ell(Q)$$


Q
not allowed!
$\sum_{P \in \mathcal{P}} \ell(P) \leq \ell(Q)$

LEMMA (PACKING LEMMA)

Suppose E is a compact subset of $(0,1)^2 \subset \mathbb{C}$, $0 < t < 2$, and $\varepsilon > 0$. Then there is a positive constant C and a finite collection of closed dyadic cubes $\mathcal{P} = \{P_i\}_{i=1}^N$ such that

(A) $4P_i \cap 4P_j = \emptyset$ *for* $i \neq j$.

(B) $E \subset \bigcup_{i=1}^N 12P_i$.

(C) $\|\mathcal{P}\|_{t\text{-pack}} \leq 1$.

(D) $\sum_{i=1}^N \ell(P_i)^t \leq C\,(\mathcal{H}^t_\infty(E) + \varepsilon)$.

$$d\alpha = \sum_{P\in\mathcal{P}} \frac{\mathbf{1}_P}{\ell(P)^{2-t}}\,dx\,.$$

Remember this!

Theorem

For any $0 < t < 2$*, and any collection* $\mathcal{P}$ *with* $\|\mathcal{P}\|_{t\text{-pack}} < \infty$*, we have*

$$\|\mathrm{B}(f\chi_{\cup_{P\in\mathcal{P}}P})\|_{L^p(\alpha)} \lesssim \|f\|_{L^p(\alpha)}\,, \qquad 1 < p < \infty\,.$$

Heuristics of the Proof:

1. The measures α are not A_p weights, so the inequality above is certainly not trivial.
2. Indeed, one may need to use the general theory of two-weight inequalities, which is a rather thin literature.
3. A direct proof is the hardest route. Slightly easier is the weak-type approach. Easier still is the restricted weak-type approach.
4. The combinatorial properties of t-packing permit a simple self-contained proof.

Combining the Elements of the Proof

1. Given $\epsilon > 0$, approximate E by the Packing Lemma, approximate by the cubes $\mathcal{P}$.
2. Factorize $\varphi = \varphi_{\text{in}} \circ \varphi_{\text{out}}$, where $\varphi_{\text{in/out}}$ is conformal in/out—side $\varphi_{\text{out}}\left(\bigcup P_j\right) / \bigcup P_j$.
3. The inside piece has improved endpoint integrability for Sobolev spaces, so it is easier to handle (Astala - Nesi.)
4. So it remains to control the outside piece, and here the weighted inequality is decisive.

Recall $d\alpha = dx \sum_{P\in\mathcal{P}} \mathbf{1}_P/\ell(P)^{2-t}$, and $\sum_{P\in\mathcal{P}} \ell(P)^t < \epsilon$.

$$\begin{aligned}\left(\sum_{P\in\mathcal{P}} \mathrm{diam}(f(P))^t\right)^{\frac{2}{t}} &\lesssim \int_E J(z,f)\,\alpha(dz\wedge d\bar{z}) \\ &= \int_E (|f_z|^2 - |f_{\bar{z}}|^2)\,\alpha(dz\wedge d\bar{z}) \quad (f_z = 1 + B(f_{\bar{z}})) \\ &\leq 2\int_E (1 + |B(f_{\bar{z}})|^2 + |f_{\bar{z}}|^2)\,\alpha(dz\wedge d\bar{z})\end{aligned}$$

$$\int_E 1\alpha(dz\wedge d\bar{z}) = \sum_{P\in\mathcal{P}} \ell(P)^t < \epsilon \qquad \text{by construction.}$$

$$\int_E |B(f_{\bar{z}})|^2\alpha(dz\wedge d\bar{z}) \lesssim \int_E |(f_{\bar{z}})|^2\alpha(dz\wedge d\bar{z}) \qquad \text{by wtd ineq}$$

Recall: $\overline{\partial} f = \mu \partial f = \mu + \mu \, \mathsf{B} \, \mu + \mu \, \mathsf{B}(\mu \, \mathsf{B} \, \mu) + \cdots$, so

$$\left[\int_E |f_{\overline{z}}|^2 \alpha(dz \wedge d\overline{z}) \right]^{1/2} \leq \sum_{n=0}^{\infty} \|\mu \, \mathsf{B} \, \mu \cdots \mathsf{B} \, \mu\|_{L^2(\alpha)}$$

$$\leq \sum_{n=0}^{\infty} \left[\|\mu\|_\infty \|\mathsf{B}\|_{L^2(\alpha) \to L^2(\alpha)} \right]^n \alpha(\mathbb{C})^{1/2}$$

$$\lesssim \epsilon^{1/2}$$

Ideas of proof of removability result: Analytic capacity and non linear potentials

Theorem (Tolsa - Indiana '02)

$$\gamma(E) \approx \sup\big\{\mu(E) : \operatorname{supp}(\mu) \subset E; M\mu(x) + c_\mu(x) \leq 1 \; \forall x \in \mathbb{C}\big\},$$

where $M\mu(x) = \sup_{r>0} \dfrac{\mu(B(x,r))}{r}$, $c_\mu(x)^2 = \displaystyle\iint \frac{1}{R(x,y,z)^2}\, d\mu(y) d\mu(z)$.

Recall:

$$\dot{W}^{\mu}_{2/3,3/2}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r}\right)^2 \frac{dr}{r} \approx \sum_{k\in\mathbb{Z}} \left(\frac{\mu(B(x,2^k))}{2^k}\right)^2 .$$

We have (recall Wolff: (7))

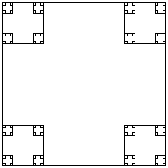
$$M\mu(x)^2 + c_\mu(x)^2 \lesssim \dot{W}^{\mu}_{2/3,3/2}(x) \qquad \Rightarrow \qquad \gamma(E) \gtrsim \dot{C}_{2/3,3/2}(E).$$

$\gamma \gtrsim \dot{C}_{2/3,3/2}$.

But it is false that $\gamma \approx \dot{C}_{2/3,3/2}$.
Example: if L is a segment, then

$$\gamma(L) = \frac{\mathcal{H}^1(L)}{4}, \qquad \dot{C}_{2/3,3/2}(L) = 0.$$

But for "typical Cantor sets", $\gamma(F) \approx \dot{C}_{2/3,3/2}(F)$.



Distortion of $\dot{C}_{2/3,3/2}$:

Theorem (Tolsa, UT '09)

Let $E \subset \mathbb{C}$ compact and $\varphi : \mathbb{C} \to \mathbb{C}$ K-quasiconformal, $K > 1$. If E is contained in a ball B, then

$$\frac{\dot{C}_{\frac{2K}{2K+1},\frac{2K+1}{K+1}}(E)}{\operatorname{diam}(B)^{\frac{2}{K+1}}} \geq c^{-1}\left(\frac{\dot{C}_{2/3,3/2}(\varphi(E))}{\operatorname{diam}(\varphi(B))}\right)^{\frac{2K}{K+1}}.$$

- The theorem also holds for other capacities $\dot{C}_{\alpha,p}$.

$\mathcal{D}$ = dyadic lattice.
$\mathcal{T} \subset \mathcal{D}$ is a tree if:

- There exists $Q_0 \in \mathcal{T}$ which contains all other $Q \in \mathcal{T}$.
- If $Q, R \in \mathcal{T}$, with $Q \subset R$, then $\mathcal{T}$ contains also the intermediate squares

$$P \quad \text{such that} \quad Q \subset P \subset R.$$

Q_0 = Top square of $\mathcal{T}$.
Leaves of $\mathcal{T}$ = stopping squares.

A corona type decomposition is a partition of $\mathcal{D}$ (or $\mathcal{D}(R_0)$) into trees.

We set $\theta(Q) := \dfrac{\mu(Q)}{\ell(Q)}$,

$$c^2(\mu) := \iiint \frac{1}{R(x,y,z)^2}\, d\mu(x)d\mu(y)d\mu(z).$$

LEMMA (CORONA DECOMPOSITION [TOLSA - ANNALS OF MATH. '05])

Let μ with linear growth and $\mathrm{supp}(\mu) \subset Q_0$.
There exists a corona type decomposition of $\mathcal{D}(Q_0)$ such that

$$\sum_{Q \in \mathrm{Top}} \theta(Q)^2 \, \mu(Q) \leq C \left(\mu(\mathbb{C}) + c^2(\mu)\right),$$

where, for each tree $\mathcal{T}$ with top square $Q \in \mathrm{Top}$ there exists a chord arc curve $\Gamma_{\mathcal{T}}$ such that

- *If $P \in \mathcal{T}$, then $8P \cap \Gamma_{\mathcal{T}} \neq \varnothing$.*
- *If $P \in \mathcal{T}$, then $\theta(P) \approx \theta(Q)$.*

Some ideas for the proof of the Main Theorem

Take μ supported on E such that $\mu(E) \approx \gamma(E)$, with linear growth and $c^2(\mu) \leq \mu(E)$.

By Tchebytchev, there exists $F \subset E$, $\mu(F) \geq \mu(E)/2$, such that

$$\sum_{Q \in \mathrm{Top}: x \in Q} \theta(Q)^2 \lesssim 1 \quad \text{for } x \in F.$$

If Top $= \mathcal{D}$, then

$$W^{\mu}_{2/3,3/2}(x) \approx \sum_{Q \in \mathrm{Top}: x \in Q} \theta(Q)^2 \lesssim 1 \quad \text{for } x \in F..$$

In the general case we combine ideas from distortion of $\dot{C}_{2/3,3/2}$ with improved distortion of estimates of subset of chord arc curves.

Recall [ACM+]: If $E \subset \Gamma$, Γ chord arc curve, then

$$\dim_H(\varphi(E)) > \frac{2}{K+1}.$$

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