Fast Solvers for Incompressible Flow Problems II

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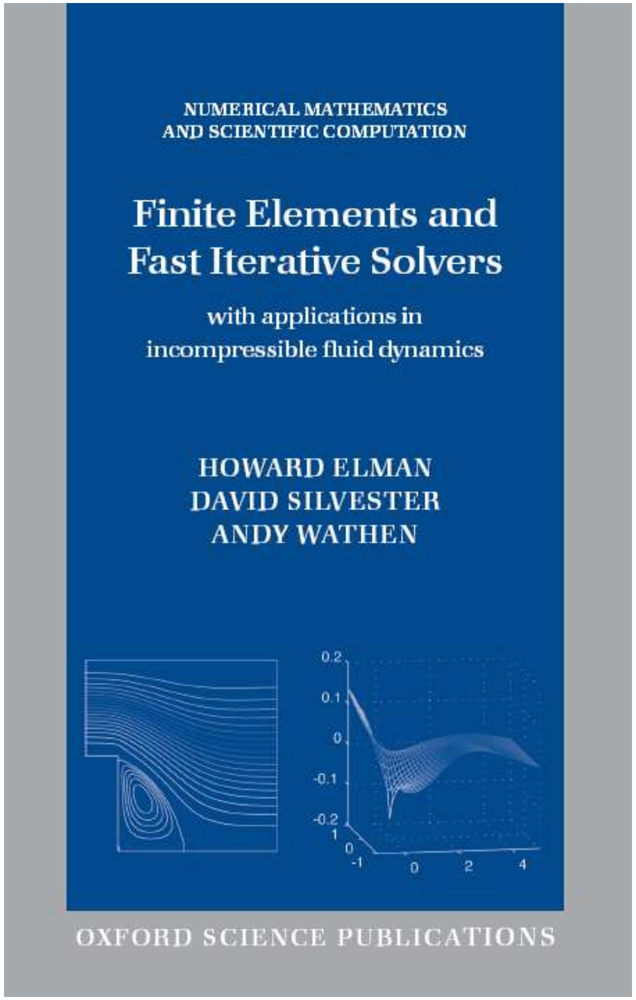
Lecture II

- $$-\nabla^2\,\vec{u} + \nabla p = \vec{0}; \quad \nabla \cdot \vec{u} = 0$$

- $$\vec{u} \cdot \nabla\,\vec{u} - \nu\nabla^2\,\vec{u} + \nabla p = \vec{0}; \quad \nabla \cdot \vec{u} = 0$$

Reference — lectures I & II



Chapters 5–6 (Stokes) & 7–8 (Steady Navier–Stokes) .

Steady-state Navier-Stokes equations

$$\vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = \vec{0} \qquad \text{in } \Omega$$
$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \Omega.$$

Boundary conditions:

$$\vec{u} = \vec{w} \text{ on } \partial\Omega_D, \quad \nu \frac{\partial \vec{u}}{\partial n} - \vec{n}p = \vec{0} \text{ on } \partial\Omega_N.$$

Picard linearization:
Given $\vec{u}^0$, compute $\vec{u}^1$, $\vec{u}^2$, $\ldots$, $\vec{u}^k$ via

$$\vec{u}^k \cdot \nabla \vec{u}^{k+1} - \nu \nabla^2 \vec{u}^{k+1} + \nabla p^{k+1} = 0,$$
$$\nabla \cdot \vec{u}^{k+1} = 0 \qquad \text{in } \Omega$$

together with appropriate boundary conditions.

Steady-state Navier-Stokes equations

$$\vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = 0 \qquad \text{in } \Omega$$
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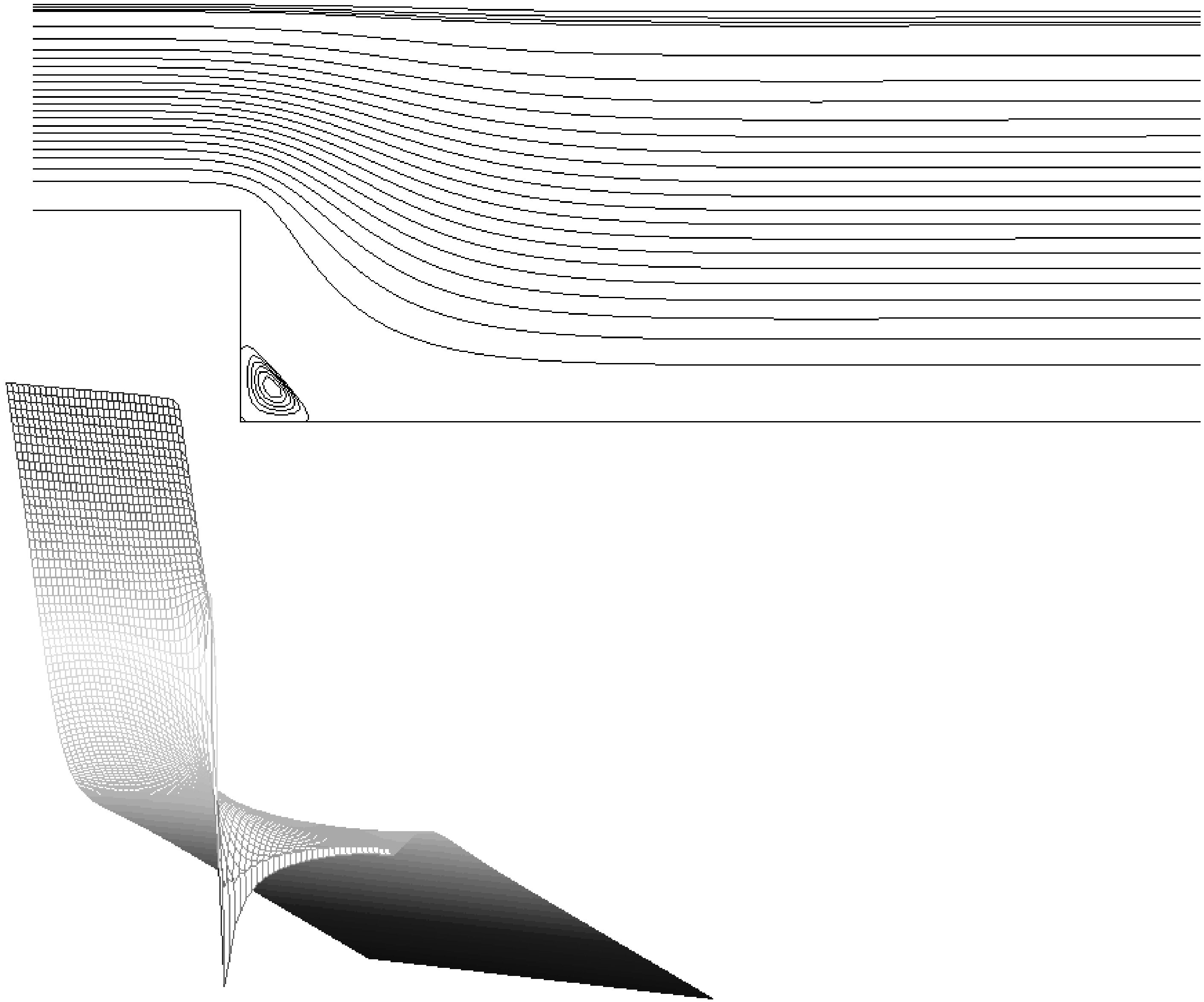
$$\vec{u} = \vec{w} \text{ on } \partial\Omega_D, \quad \nu \frac{\partial \vec{u}}{\partial n} - \vec{n} p = \vec{0} \text{ on } \partial\Omega_N.$$

Newton linearization:
Given $\vec{u}^k$, and residuals R_k and r_k, compute the correction $(\delta \vec{u}^k, \delta p^k)$ via

$$\delta \vec{u}^k \cdot \nabla \vec{u}^k + \vec{u}^k \cdot \nabla \delta \vec{u}^k - \nu \nabla^2 \delta \vec{u}^k + \nabla \delta p^k = R_k,$$
$$\nabla \cdot \delta \vec{u}^k = r_k \qquad \text{in } \Omega$$

together with appropriate boundary conditions.

Example: Flow over a Step



Rest of the talk

- A posteriori (energy–) error estimation

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  - Navier-Stokes flow

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  - Pressure Convection-Diffusion preconditioner
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Rest of the talk

- A posteriori (energy–) error estimation
  - Stokes flow
  - Navier-Stokes flow
- Optimally preconditioned GMRES:
  - Pressure Convection-Diffusion preconditioner
  - Least-squares commutator preconditioner
- A proof-of-concept implementation:
  - The IFISS 3.1 MATLAB Toolbox

Stokes recap ...

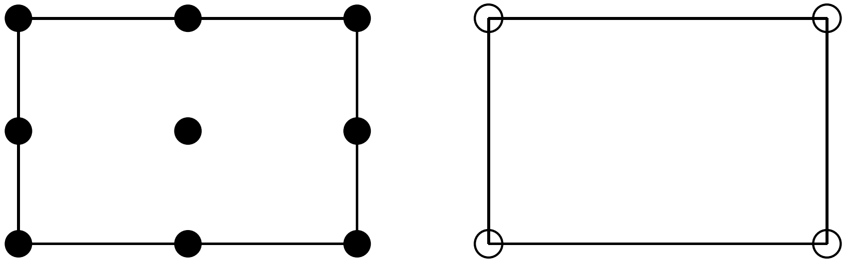
Mixed formulation : find $(\vec{u}, p) \in V \times Q$ such that

$$\begin{aligned} (\nabla \vec{u}, \nabla \vec{v}) + (\nabla \cdot \vec{v}, p) &= f(\vec{v}) \qquad \forall \vec{v} \in V, \\ (\nabla \cdot \vec{u}, q) &= g(q) \qquad \forall q \in Q. \end{aligned} \qquad (V)$$

Spaces : $V := (H_0^1(\Omega))^d$ and $Q = L^2(\Omega)$ so that the dual spaces are $V^* := (H^{-1}(\Omega))^d$ and $Q^* := L^2(\Omega)$ respectively.

In practice, the velocity approximation needs to be continuous across inter-element edges (e.g. Q_1), whereas the pressure approximation can be discontinuous.

Two different inf-sup stable mixed approximation methods are implemented in IFISS:



Q_2–Q_1 element (also referred to as Taylor-Hood).

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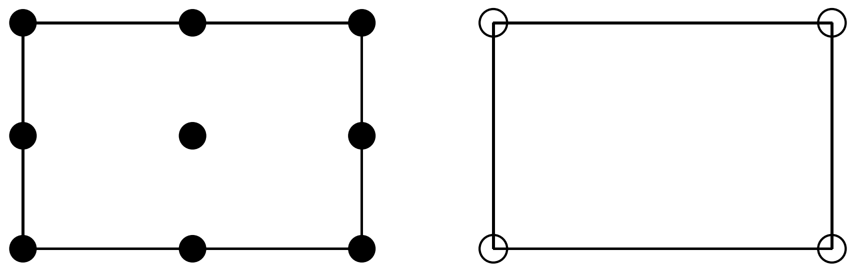

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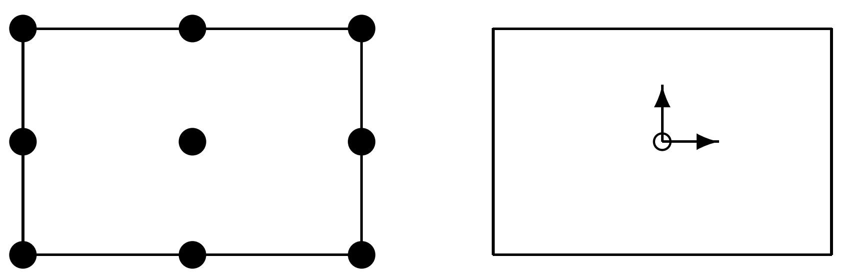
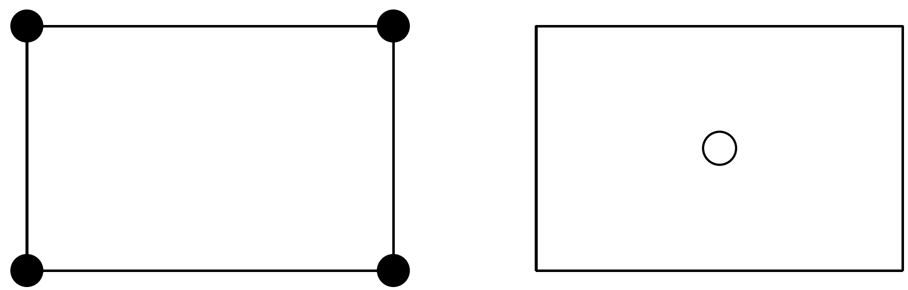


Q_2–P_{-1} element : $\circ$ pressure; $\overset{\uparrow}{\rightarrow}$ pressure derivative

Two unstable low-order mixed approximation methods are implemented in IFISS:



Q_1–P_0 element : $\bullet$ two velocity components; $\circ$ pressure

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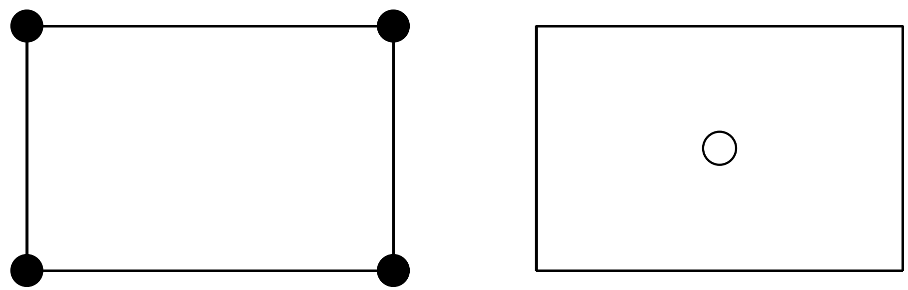
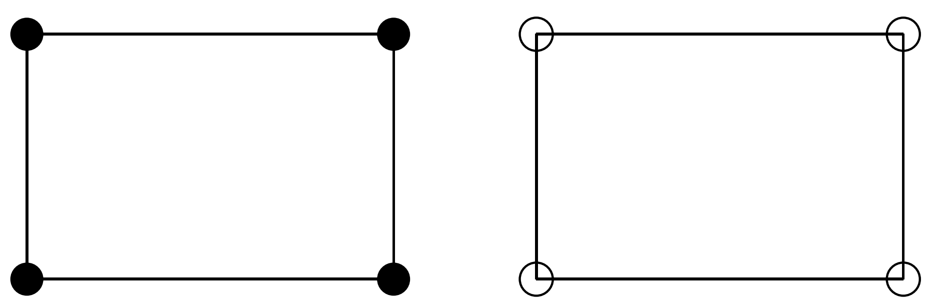


Q_1–P_0 element : • two velocity components; ○ pressure



Q_1–Q_1 element : • two velocity components; ○ pressure

Q_1–P_0 stabilization

(Kechkar & S. (1992))
Given a suitable macroelement (e.g. 2×2) partitioning $\mathcal{M}_h$.
Find $(\vec{u}_h, p_h) \in V_h \times Q_h$ such that:

$$\begin{aligned} a\,(\vec{u}_h, \vec{v}_h) + b\,(\vec{v}_h, p_h) &= \vec{f} \quad \forall \vec{v}_h \in V_h, \\ b\,(\vec{u}_h, q_h) - \beta^* c\,(p_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned}$$

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Where $\beta^* = 1/4$ and

$$c\left(p_h, q_h\right) = \sum_{M \in \mathcal{M}_h} |M| \sum_{E \in \mathcal{E}(M)} \langle [\![p_h]\!], [\![q_h]\!] \rangle_{\bar{E}} \qquad \forall q_h \in Q_h,$$

where $|M|$ is the mean element area within the macroelement and $\langle p, q \rangle_{\bar{E}} = \frac{1}{|E|} \int_E pq$.

Q_1–Q_1 stabilization

(Dohrmann & Bochev (2004))
Find $(\vec{u}_h, p_h) \in V_h \times Q_h$ such that:

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Find $(\vec{u}_h, p_h) \in V_h \times Q_h$ such that:

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a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) &= \vec{f} \quad \forall \vec{v}_h \in V_h, \\
b(\vec{u}_h, q_h) - c(p_h, q_h) &= 0 \quad \forall q_h \in Q_h.
\end{aligned}
$$

Where

$$
c(p_h, q_h) = (p_h - \Pi_0 p_h, q_h - \Pi_0 q_h),
$$

and

$$
\Pi_0 p_h|_T = \frac{1}{|T|} \int_T p_h \, d\Omega \quad \forall T \in \mathcal{T}_h.
$$

Stress Jump

In general $\mathbf{P}_0$ approximation is discontinuous and $\mathbf{Q}_1$ approximation has a discontinuous normal derivative across the edges.

Consequently it is convenient to define the stress jump across edge E adjoining elements T and S:

$$[\![\nabla \vec{u}_h - p_h \vec{\mathbf{I}}\,]\!] := ((\nabla \vec{u}_h - p_h \vec{\mathbf{I}}\,)|_T - (\nabla \vec{u}_h - \; p_h \vec{\mathbf{I}}\,)|_S)\, \vec{n}_{E,T}.$$

(If a C^0 pressure approximation is used the jump in $p_h \vec{\mathbf{I}}$ is zero.)

Stokes Error Estimation

Define the interior edge stress jump $\vec{R}_E := \frac{1}{2}[\![\nabla\,\vec{u}_h - p_h\vec{I}]\!]$ and the element PDE residuals

$$\vec{R}_T := \{\nabla^2\,\vec{u}_h - \nabla p_h\}|_T \quad \text{and} \quad R_T := \{\nabla\cdot\,\vec{u}_h\}|_T.$$

This gives the error characterization:
$\vec{e} := \vec{u} - \vec{u}_h \in V$ and $\epsilon := p - p_h \in Q$ satisfies

$$\sum_{T\in\mathcal{T}_h}(\nabla\,\vec{e},\nabla\,\vec{v})_T - \sum_{T\in\mathcal{T}_h}(\epsilon,\nabla\cdot\,\vec{v})_T$$

$$= \sum_{T\in\mathcal{T}_h}\left\{(\,\vec{R}_T,\,\vec{v})_T - \sum_{E\in\mathcal{E}(T)}\langle\,\vec{R}_E,\,\vec{v}\rangle_E\right\} \quad \forall\,\vec{v}\in V$$

$$\sum_{T\in\mathcal{T}_h}(\nabla\cdot\,\vec{e},q) = \sum_{T\in\mathcal{T}_h}(R_T,q) \quad \forall q\in Q.$$

Q_1–Q_1 approximation

$\vec{R}_E$ is piecewise linear and

$$\vec{R}_T = \{\nabla^2 \vec{u}_h - \nabla p_h\}|_T \subset \left(\boldsymbol{P}_0(T)\right)^d.$$

$$R_T = \{\nabla \cdot \vec{u}_h\}|_T \subset \boldsymbol{P}_1(T).$$

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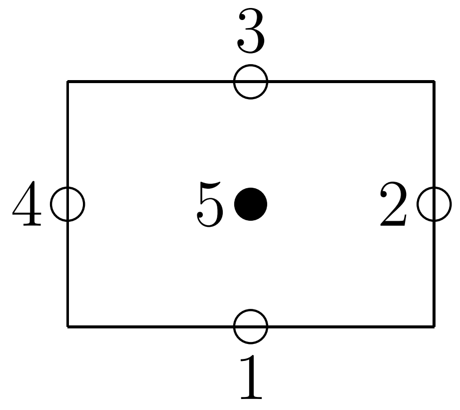
Also, we recall the high order correction space:

$$\mathcal{Q}_T = Q_T \oplus B_T$$



The local (vector-) Poisson problem is to compute $\vec{e}_T \in \vec{\mathcal{Q}}_T$ and $\epsilon_T \in \boldsymbol{P}_1(T)$ such that

$$(\nabla \vec{e}_T, \nabla \vec{v})_T = (\vec{R}_T, \vec{v})_T - \sum_{E \in \mathcal{E}(T)} \langle \vec{R}_E, \vec{v} \rangle_E \qquad \forall \vec{v} \in \vec{\mathcal{Q}}_T$$

$$(\epsilon_T, q)_T = (\nabla \cdot \vec{u}_h, q)_T \qquad \forall q \in \boldsymbol{P}_1(T).$$

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Note that $R_T \subset \boldsymbol{P}_1(T)$ implies that $\epsilon_T = \nabla \cdot \vec{u}_h$.

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Note that $R_T \subset \boldsymbol{P}_1(T)$ implies that $\epsilon_T = \nabla \cdot \vec{u}_h$.

The local estimator is given by $\eta_T^2 = \|\nabla \vec{e}_T\|_T^2 + \|\epsilon_T\|_T^2$, and the global error estimator is

$$\eta := (\sum_{T \in \mathcal{T}_h} \eta_T^2)^{1/2} \approx \|\nabla(\vec{u} - \vec{u}_h)\| + \|p - p_h\|$$

Theory

(Ainsworth & Oden (1997), Kay & S. (1999))
Assuming a shape regular subdivision, the local problem estimator is reliable in the case of either the unstabilized or the stabilized formulation:

$$\|\nabla(\vec{u} - \vec{u}_h)\| + \|p - p_h)\| \leq \frac{C}{\gamma_*} \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}$$

where γ_* is the inf-sup constant associated with the continuous problem.

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where γ_* is the inf-sup constant associated with the continuous problem.
The local problem estimator is also efficient :

$$\eta_T \leq C \left(\|\nabla(\vec{u} - \vec{u}_h)\|_{\omega_T} + \|p - p_h)\|_{\omega_T} \right)$$

where C is independent of γ_*.

Example: Poiseuille flow

Grid	$\|\nabla(\vec{u} - \vec{u}_h)\|$	$\|\nabla \cdot \vec{u}_h\|$	η
4×4	6.112×10^{-1}	3.691×10^{-2}	5.600×10^{-1}
8×8	2.962×10^{-1}	6.656×10^{-3}	2.912×10^{-1}
16×16	1.460×10^{-1}	1.196×10^{-3}	1.458×10^{-1}
32×32	7.255×10^{-2}	2.129×10^{-4}	7.267×10^{-2}

Stabilized Q_1–Q_1 approximation

Error estimators for Q_2–Q_1 and Q_2–P_{-1} approximation are include in IFISS 3.1. They are discussed in

- Qifeng Liao & David Silvester. A simple yet effective a posteriori estimator for classical mixed approximation of Stokes equations. MIMS Eprint 2009.75.

What have we achieved?

- A simple a posteriori energy estimator for inf-sup stable (and stabilized) mixed approximation.
- Efficiency index is very close to unity.

- Computational interlude . . .

NS Error Estimation I

Define the interior edge stress jump $\vec{R}_E := \frac{1}{2}[\![\nu \nabla \vec{u}_h - p_h \vec{I}]\!]$ the element PDE residuals

$$\vec{R}_T := \{-\vec{u}_h \cdot \nabla \vec{u}_h + \nu \nabla^2 \vec{u}_h - \nabla p_h\}|_T \quad \text{and} \quad R_T := \{\nabla \cdot \vec{u}_h\}|_T .$$

and the difference operator

$$\begin{aligned} D(\vec{u}_h, \vec{e}, \vec{v}) &:= c(\vec{e} + \vec{u}_h, \vec{e} + \vec{u}_h, \vec{v}) - c(\vec{u}_h, \vec{u}_h, \vec{v}) \\ &= c(\vec{u}, \vec{u}, \vec{v}) - c(\vec{u}_h, \vec{u}_h, \vec{v}) \\ &= (\vec{u} \cdot \nabla \vec{u}, \vec{v}) - (\vec{u}_h \cdot \nabla \vec{u}_h, \vec{v}) \,. \end{aligned}$$

NS Error Estimation II

This gives the error characterization: $\vec{e} := \vec{u} - \vec{u}_h \in V$ and $\epsilon := p - p_h \in Q$ satisfies

$$\sum_{T \in \mathcal{T}_h} (\nabla \vec{e}, \nabla \vec{v})_T - \sum_{T \in \mathcal{T}_h} (\epsilon, \nabla \cdot \vec{v})_T + D(\vec{u}_h, \vec{e}, \vec{v})$$

$$= \sum_{T \in \mathcal{T}_h} \left\{ (\vec{R}_T, \vec{v})_T - \sum_{E \in \mathcal{E}(T)} \langle \vec{R}_E, \vec{v} \rangle_E \right\} \quad \forall \vec{v} \in V$$

$$\sum_{T \in \mathcal{T}_h} (\nabla \cdot \vec{e}, q) = \sum_{T \in \mathcal{T}_h} (R_T, q) \quad \forall q \in Q.$$

The local (vector-) Poisson problem is to compute $\vec{e}_T \in \vec{\mathcal{Q}}_T$ such that

$$\nu(\nabla \vec{e}_T, \nabla \vec{v})_T = (\vec{R}_T, \vec{v})_T - \sum_{E \in \mathcal{E}(T)} \langle \vec{R}_E, \vec{v} \rangle_E \quad \forall \vec{v} \in \vec{\mathcal{Q}}_T$$

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$$\eta := (\sum_{T \in \mathcal{T}_h} \eta_T^2)^{1/2} \approx \|\nabla(\vec{u} - \vec{u}_h)\| + \|p - p_h\|$$

Theory

(Verfürth (1996))
Given that the Navier-Stokes problem is guaranteed to have a unique solution, and assuming a shape regular subdivision, the local problem estimator is reliable in the case of either the unstabilized or the stabilized formulation:

$$\|\nabla(\vec{u} - \vec{u}_h)\| + \|p - p_h\| \leq C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}$$

where C is independent of ν

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where C is independent of ν
The efficiency of the local problem estimator is an open problem:

$$\eta_T \overset{?}{\leq} C \left(\|\nabla(\vec{u} - \vec{u}_h)\|_{\omega_T} + \|p - p_h\|_{\omega_T} \right).$$

Example: Driven cavity flow

Grid	$\|\nabla \cdot \vec{u}_h\|$	η
16×16	5.288×10^{-2}	4.116×10^{0}
32×32	1.997×10^{-2}	2.595×10^{0}
64×64	5.970×10^{-3}	1.370×10^{0}
128×128	1.593×10^{-3}	7.030×10^{-1}

Stabilized Q_1–P_0 approximation

Back to fast solvers ...

Introducing the basis sets

$$V_h = \text{span}\{\vec{\phi}_i\}_{i=1}^{n_u}, \quad \text{Velocity basis functions;}$$
$$Q_h = \text{span}\{\psi_j\}_{j=1}^{n_p}, \quad \text{Pressure basis functions.}$$

gives the discretized Oseen system:

$$\begin{pmatrix} N + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix},$$

with associated matrices

$$N_{ij} = (\vec{w}_h \cdot \nabla \vec{\phi}_i, \vec{\phi}_j), \quad \text{convection}$$
$$A_{ij} = (\nabla \vec{\phi}_i, \nabla \vec{\phi}_j), \quad \text{diffusion}$$
$$B_{ij} = -(\nabla \cdot \vec{\phi}_j, \psi_i), \quad \text{divergence} \; .$$

Block triangular preconditioning

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

A perfect preconditioner is given by

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

Here $F = N + \nu A$ and $S = BF^{-1}B^T$.

Block triangular preconditioning

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Here $F = N + \nu A$ and $S = BF^{-1}B^T$. Note that

$$\underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} \underbrace{\begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}}_{\mathcal{P}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} N + \nu A & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}$$

With discrete matrices

$$N_{ij} = (\vec{w}_h \cdot \nabla \vec{\phi}_i, \vec{\phi}_j), \quad \text{convection}$$

$$A_{ij} = (\nabla \vec{\phi}_i, \nabla \vec{\phi}_j), \quad \text{diffusion}$$

$$B_{ij} = -(\nabla \cdot \vec{\phi}_j, \psi_i), \quad \text{divergence}$$

For an efficient block triangular preconditioner $\mathcal{P}$ we require a sparse approximation to the “exact” Schur complement

$$S^{-1} = (B(N + \nu A)^{-1} B^T)^{-1} =: (BF^{-1}B^T)^{-1}$$

Two possible constructions ...

Schur complement approximation – I

Introducing associated pressure matrices

$$
\begin{aligned}
A_p &\sim (\nabla\psi_i, \nabla\psi_j), && \text{diffusion} \\
N_p &\sim (\,\vec{w}_h \cdot \nabla\psi_i, \psi_j), && \text{convection} \\
F_p &= \nu A_p + N_p, && \text{convection-diffusion}
\end{aligned}
$$

gives the “pressure convection-diffusion preconditioner”:

$$
(BF^{-1}B^T)^{-1} \approx Q^{-1} F_p \underbrace{A_p^{-1}}_{\text{AMG}}
$$

- David Kay & Daniel Loghin (& Andy Wathen)
  A Green’s function preconditioner for the steady-state Navier-Stokes equations
  Report NA–99/06, Oxford University Computing Lab.
  SIAM J. Sci. Comput, 24, 2002.

Schur complement approximation – II

Introducing the diagonal of the velocity mass matrix

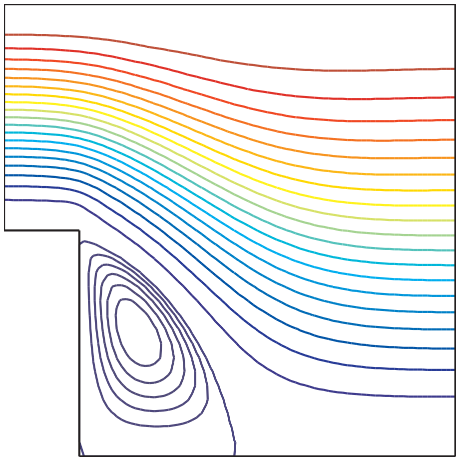
$$M_* \sim M_{ij} = (\vec{\phi}_i, \vec{\phi}_j),$$

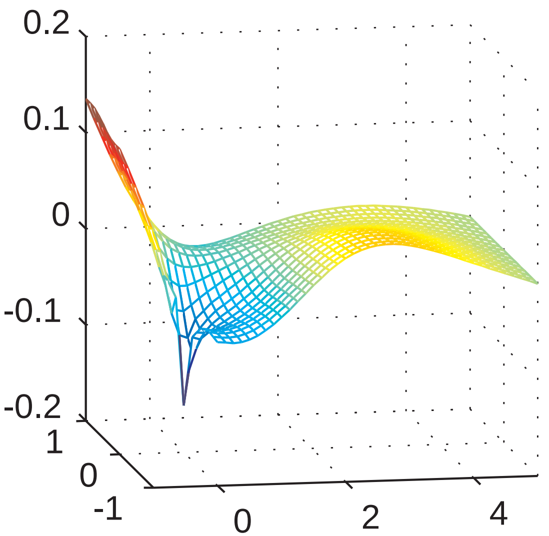
gives the "least-squares commutator preconditioner":

$$(BF^{-1}B^T)^{-1} \approx (\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}(BM_*^{-1}FM_*^{-1}B^T)(\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}$$

- Howard Elman (& Ray Tuminaro et al.)
  Preconditioning for the steady-state Navier-Stokes equations with low viscosity,
  SIAM J. Sci. Comput, 20, 1999.
  Block preconditioners based on approximate commutators,
  SIAM J. Sci. Comput, 27, 2006.

IFISS computational results





Final step of Oseen iteration : $R = 2/\nu$

GMRES iterations using Q_2–Q_1 (Q_1–P_0) — tol $= 10^{-6}$

Exact Least Squares Commutator

$1/h$	$R = 10$	$R = 100$	$R = 200$
5	15 (15)	17 (16)	
6	19 (21)	21 (22)	29 (32)
7	23 (31)	29 (32)	29 (30)

Summary | Navier-Stokes preconditioning

- Grid independent convergence rate for preconditioned GMRES for inf-sup stable (and stabilized) mixed approximation.
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Further Reading ...

- Howard Elman (& Silvester et al.)
Least squares preconditioners for stabilized discretizations of the Navier-Stokes equations,
SIAM J. Sci. Comput, 30, 2007.
- Howard Elman & Ray Tuminaro
Boundary conditions in approximate commutator preconditioners for the Navier-Stokes equations,
Electronic Transactions on Numerical Analysis, 35:257–280, 2009.