Fast Solvers for Incompressible Flow Problems I

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Lecture Schedule

- $$-\nabla^2 \vec{u} + \nabla p = \vec{0}; \quad \nabla \cdot \vec{u} = 0$$

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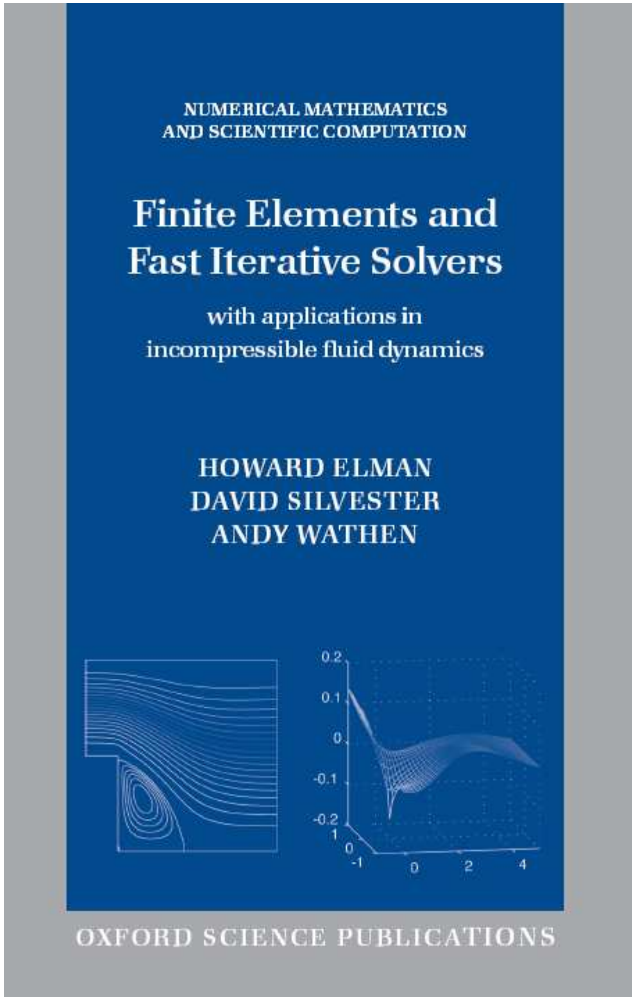
- $$\left.\begin{array}{c} \dfrac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = \vec{j} T; \quad \nabla \cdot \vec{u} = 0 \\ \dfrac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - \nu \nabla^2 T = 0 \end{array}\right\}$$

Reference — lectures I & II



Chapters 5–6 (Stokes) & 7–8 (Steady Navier–Stokes) .

References — lectures III & IV

- David Kay & Philip Gresho & David Griffiths & David Silvester Adaptive time-stepping for incompressible flow; part II: Navier-Stokes equations
SIAM J. Scientific Computing, 32: 111–128, 2010.
- Howard Elman, Milan Mihajlović and David Silvester.
Fast iterative solvers for buoyancy driven flow problems
J. Computational Physics, 230: 3900–3914, 2011.

Lecture I

- $$-\nabla^2 \vec{u} + \nabla p = \vec{0}; \quad \nabla \cdot \vec{u} = 0$$

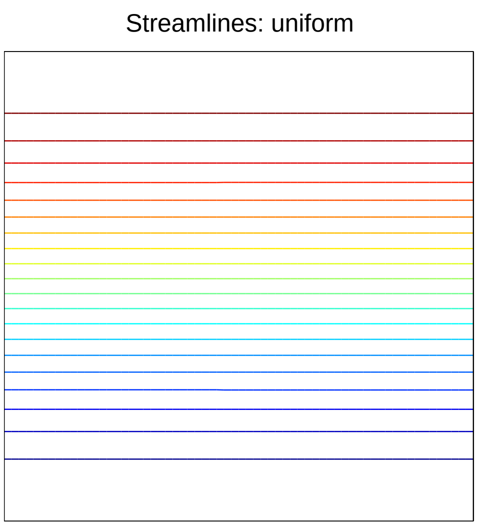
Poiseuille Flow: problem 5.1

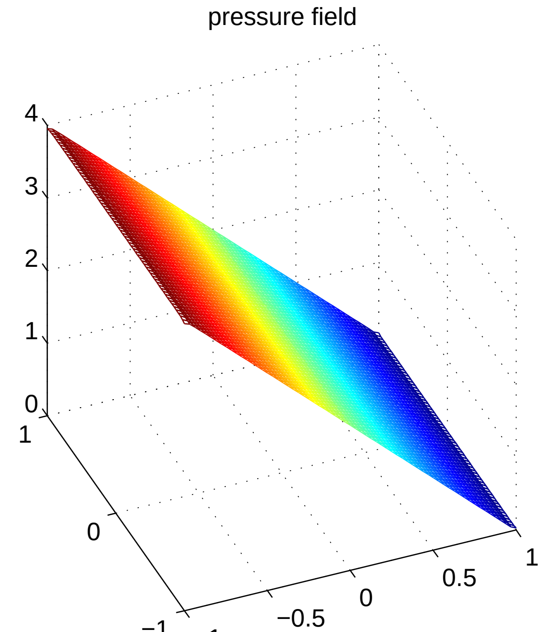
Flow in $[-1,1] \times [-1,1]$ with Dirichlet boundary conditions:

$$\begin{aligned} \vec{u}(x,y) &= \vec{0} \text{ for all } (x,y) \in (-1,1) \times \{-1,1\}, \\ \vec{u}(x,y) &= \left(1-y^2, 0\right) \text{ for all } (x,y) \in \{-1\} \times (-1,1), \end{aligned}$$

and the Neumann condition

$$\left.\begin{aligned} \frac{\partial u_x(x,y)}{\partial x} - p(x,y) &= 0 \\ \frac{\partial u_y(x,y)}{\partial x} &= 0 \end{aligned}\right\} \text{ for all } \quad (x,y) \in \{1\} \times (-1,1)$$



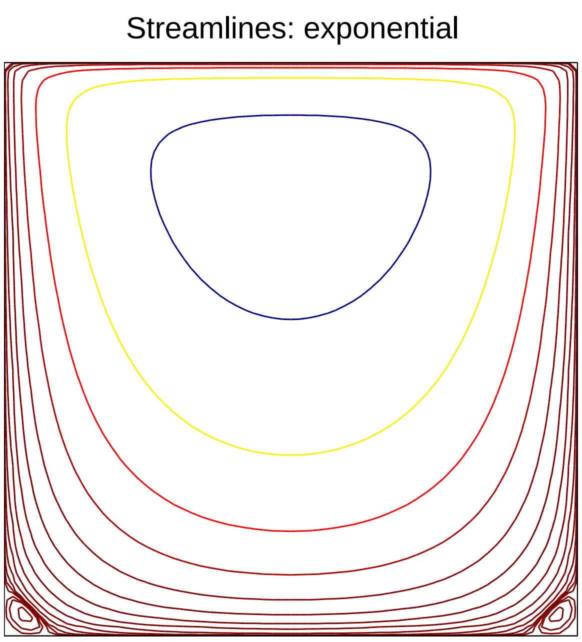


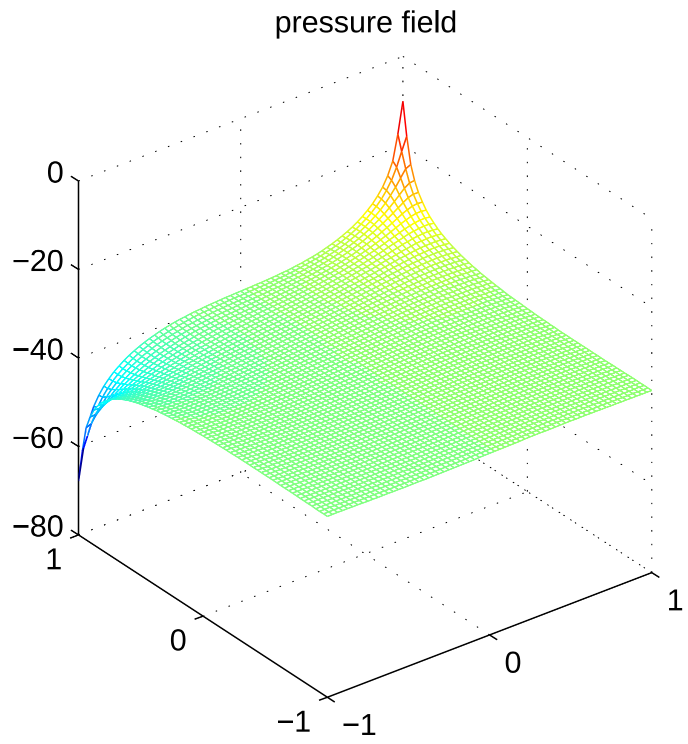
Cavity Flow: problem 5.3

Flow in $[-1,1] \times [-1,1]$ with Dirichlet boundary conditions:

$$\vec{u} = \vec{0} \qquad \text{on} \quad x = -1, 1 \text{ and } y = -1.$$

$$\vec{u} = \left(\left(1 - x^2\right)\left(1 + x^2\right), 0\right)^T \qquad \text{on} \quad y = 1.$$





Stokes flow problem

$$-\nabla^2 \vec{u} + \nabla p = \vec{0} \quad \text{in } \Omega \subset \mathbb{R}^d$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega$$

$$\vec{u} = \vec{g} \quad \text{on } \Gamma_D$$

$$\nabla \vec{u} \cdot \vec{n} - p\,\vec{n} = \vec{0} \quad \text{on } \Gamma_N$$

Stokes flow problem

$$
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\nabla \vec{u} \cdot \vec{n} - p \vec{n} &= \vec{0} \quad \text{on } \Gamma_N
\end{aligned}
$$

Mixed formulation : find $(\vec{u}, p) \in (H_0^1(\Omega))^d \times L^2(\Omega)$ such that

$$
\begin{aligned}
(\nabla \vec{u}, \nabla \vec{v}) - (\nabla \cdot \vec{v}, p) &= f(\vec{v}) \qquad \forall \vec{v} \in (H_0^1(\Omega))^d, \\
-(\nabla \cdot \vec{u}, q) &= g(q) \qquad \forall q \in L^2(\Omega).
\end{aligned}
$$

Generic structure

Find $(\vec{u}, p) \in (H_0^1(\Omega))^d \times L^2(\Omega)$ such that

$$
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-(\nabla \cdot \vec{u}, q) &= g(q) \qquad \forall q \in L^2(\Omega).
\end{aligned}
$$

Abstract formulation : find $(\vec{u}, p) \in V \times Q$ such that

$$
\begin{aligned}
a(\vec{u}, \vec{v}) + b(\vec{v}, p) &= f(\vec{v}) \qquad \forall \vec{v} \in V, \\
b(\vec{u}, q) &= g(q) \qquad \forall q \in Q.
\end{aligned}
\qquad (V)
$$

Where, V and Q represent Hilbert spaces; $a : V \times V \to \mathbb{R}$ is a symmetric bounded bilinear form, $b : V \times Q \to \mathbb{R}$ is a bounded bilinear form and $f : V \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ are linear functionals.

Saddle Point Structure

$$\begin{aligned} a(\vec{u}, \vec{v}) + b(\vec{v}, p) &= f(\vec{v}) \qquad \forall \vec{v} \in V, \\ b(\vec{u}, q) &= g(q) \qquad \forall q \in Q. \end{aligned} \qquad (V)$$

To discover the structure we define dual spaces V^* and Q^* respectively, with a duality pairing $\langle \cdot, \cdot \rangle$.

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To discover the structure we define dual spaces V^* and Q^* respectively, with a duality pairing $\langle \cdot, \cdot \rangle$. Then, if we associate the bilinear forms a and b with operators $\mathcal{A} : V \to V^*$ and $\mathcal{B} : V \to Q^*$ so that

$$\langle \mathcal{A}\vec{u}, \vec{v} \rangle = a(\vec{u},\vec{v}) = \langle \vec{u}, \mathcal{A}\vec{v} \rangle, \qquad \langle \mathcal{B}\vec{u}, q \rangle = b(\vec{u},q) = \langle \vec{u}, \mathcal{B}^* q \rangle\,;$$

we arrive at the infinite-dimensional saddle point system

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \qquad (S)$$

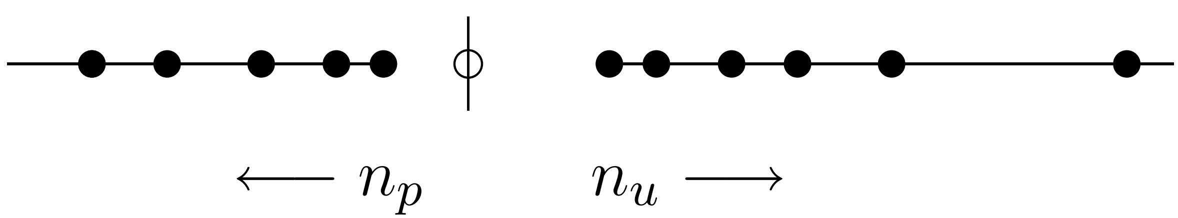
Optimal preconditioning

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \qquad (S)$$

Following Mardal & Winther [2010], a canonical preconditioner is the 2×2 block diagonal matrix operator that maps the dual space $V^* \times Q^*$ back into the original space $V \times Q$:

$$\mathcal{M} = \begin{pmatrix} M_{11}^{-1} & 0 \\ 0 & M_{22}^{-1} \end{pmatrix}.$$

Eigenvalues of the preconditioned operator $\mathcal{M}\mathcal{K}$:



Preconditioning ... Stokes

Mixed formulation : find $(\vec{u}, p) \in V \times Q$ such that

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Spaces : $V := (H_0^1(\Omega))^d$ and $Q = L^2(\Omega)$ so that the dual spaces are $V^* := (H^{-1}(\Omega))^d$ and $Q^* := L^2(\Omega)$ respectively.

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In practice, the velocity approximation needs to be continuous across inter-element edges (e.g. Q_1), whereas the pressure approximation can be discontinuous.

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In practice, the velocity approximation needs to be continuous across inter-element edges (e.g. Q_1), whereas the pressure approximation can be discontinuous.

Canonical Stokes Preconditioner :

$$\mathcal{M} = \begin{pmatrix} (-\nabla^2)^{-1} & 0 \\ 0 & I^{-1} \end{pmatrix}.$$

Discretized approximation

$$\begin{bmatrix} \mathbb{A} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (S_h)$$

That is, given $V_h \subset V$ and $Q_h \subset Q$: find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(u_h, v) + b(v, p_h) &= f(v) \qquad \forall v \in V_h, \\ b(u_h, q) &= g(q) \qquad \forall q \in Q_h. \end{aligned} \qquad (V_h)$$

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Ideal Stokes Preconditioner :

$$M = \begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & \mathbb{I}^{-1} \end{pmatrix}.$$

See Rusten & Winther [1992], Silvester & Wathen [1994].

Optimal Solver

Energy arguments lead to a natural norm for measuring the quality of approximation for functions in the space $V \times Q$,

$$\|(u,p)\|_{V\times Q} = \|u\|_V + \|p\|_Q \,.$$

This will be referred to as the energy norm.

Our goal is to construct an optimal iterative solver for (S)...

Optimal Solver

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Our goal is to construct an optimal iterative solver for (S)...

that is, we would like to construct a sequence of rapidly converging iterates $(u_h^{(1)}, p_h^{(1)}), (u_h^{(2)}, p_h^{(2)}), (u_h^{(3)}, p_h^{(3)}), \ldots$ with the property that the iteration is terminated once the energy norm of the algebraic error $(u_h - u_h^{(m)}, p_h - p_h^{(m)})$ is commensurate with the discretization error:

$$\|u_h - u_h^{(m)}\|_V + \|p_h - p_h^{(m)}\|_Q \sim \|u - u_h^{(m)}\|_V + \|p - p_h^{(m)}\|_Q.$$

Issues

- The most natural iterative solver for a symmetric indefinite system $K\mathbf{x} = \mathbf{b}$ is MINRES. This minimizes the ℓ_2–norm of the mth residual

$$\|\mathbf{r}^{(m)}\| = \|\mathbf{b} - K\mathbf{x}^{(m)}\| = \|K(\mathbf{x} - \mathbf{x}^{(m)})\|$$

  over the Krylov space

$$\mathcal{K}_m(K, \mathbf{b}) = \text{span}\,\{\mathbf{b}, K\mathbf{b}, \ldots, K^{m-1}\mathbf{b}\}.$$

  It does not minimize the energy norm of the error.
- How does one compute an accurate estimate of the discretization error $\|u - u_h^{(m)}\|_V + \|p - p_h^{(m)}\|_Q$?

Rest of the talk

- Well-posedness of (V) and (V_h)

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  - Negative Laplacian preconditioning
- Estimating the inf-sup constant on the fly:
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- A proof-of-concept implementation:
  - EST_MINRES
  - The IFISS 3.1 MATLAB Toolbox

Well posedness ...

Abstract formulation : find $(\vec{u}, p) \in V \times Q$ such that

$$\begin{aligned} (\nabla \vec{u}, \nabla \vec{v}) - (\nabla \cdot \vec{v}, p) &= \vec{f} \qquad \forall \vec{v} \in V, \\ -(\nabla \cdot \vec{u}, q) &= 0 \qquad \forall q \in Q, \end{aligned} \qquad (V)$$

with norms $\|\vec{u}\|_V := (\nabla u, \nabla u)^{1/2}$ and $\|p\|_Q := (p, p)^{1/2}$.

Discrete formulation : find $(\vec{u}_h, p_h) \in V_h \times Q_h$

$$\begin{aligned} a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) &= \vec{f} \qquad \forall \vec{v}_h \in V_h \\ b(\vec{u}_h, q_h) &= 0 \qquad \forall q_h \in Q_h. \end{aligned} \qquad (V_h)$$

... inf-sup stability

Theorem Brezzi [1974]. Given bounded bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, two conditions are sufficient for the existence and uniqueness of solutions to the Stokes problem in its discrete form:

... inf-sup stability

Theorem Brezzi [1974]. Given bounded bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, two conditions are sufficient for the existence and uniqueness of solutions to the Stokes problem in its discrete form:

1. V_h – coercivity: there exists a constant $\alpha\ (=1)$ such that

$$a(\vec{v}_h,\vec{v}_h) \geq \alpha \|\vec{v}_h\|_V^2 \quad \forall \vec{v}_h \in V_h.$$

2. Discrete "inf-sup" condition: there exists a constant $\gamma \geq \gamma_* > 0$ such that

$$\sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq \vec{0}}} \frac{b(\vec{v}_h, q_h)}{\|\vec{v}_h\|_V} \geq \gamma \|q_h\|_Q \quad \forall q_h \in Q_h.$$

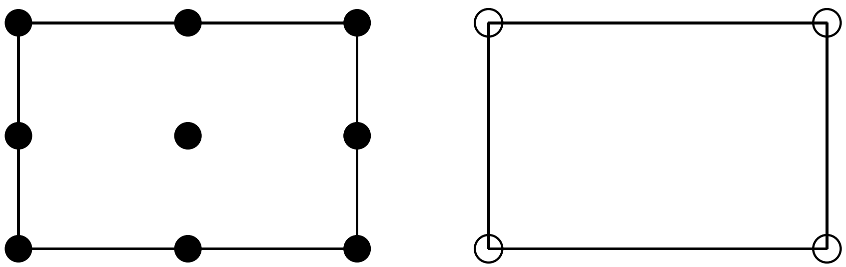
Furthermore, if $(\vec{u}, p)$ is the weak solution of the Stokes problem, and if there exists a constant $\gamma \geq \gamma_* > 0$ such that

$$\sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq \vec{0}}} \frac{b(\vec{v}_h, q_h)}{\|\vec{v}_h\|_V} \geq \gamma \|q_h\|_Q \quad \forall q_h \in Q_h.$$

then there exists a constant $C(\gamma_*) > 0$ such that

$$\|\vec{u} - \vec{u}_h\|_V + \|p - p_h\|_Q \leq C \left\{ \inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right\}.$$

Two different inf-sup stable mixed approximation methods are implemented in IFISS:



Q_2–Q_1 element (also referred to as Taylor-Hood).

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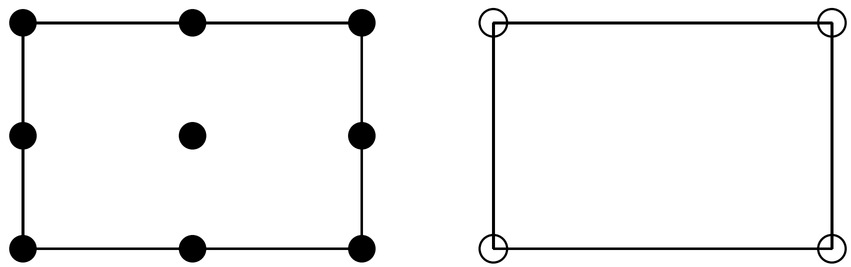

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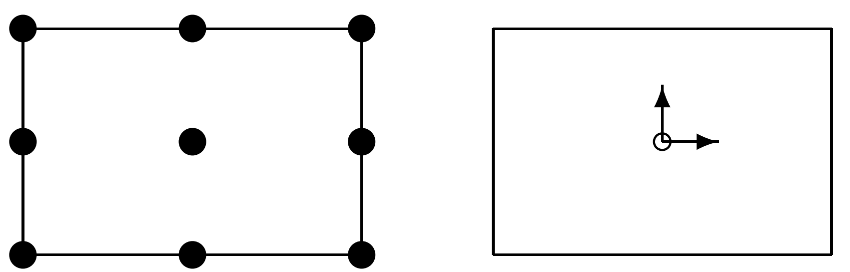
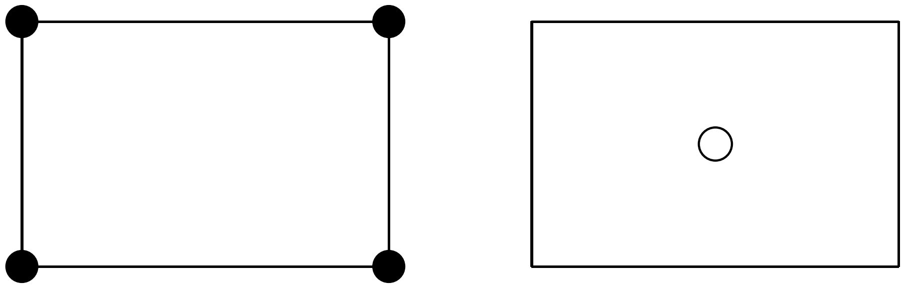


Q_2–P_{-1} element : $\circ$ pressure; $\overset{\uparrow}{\rightarrow}$ pressure derivative

Two unstable low-order mixed approximation methods are implemented in IFISS:



Q_1–P_0 element : • two velocity components; ∘ pressure

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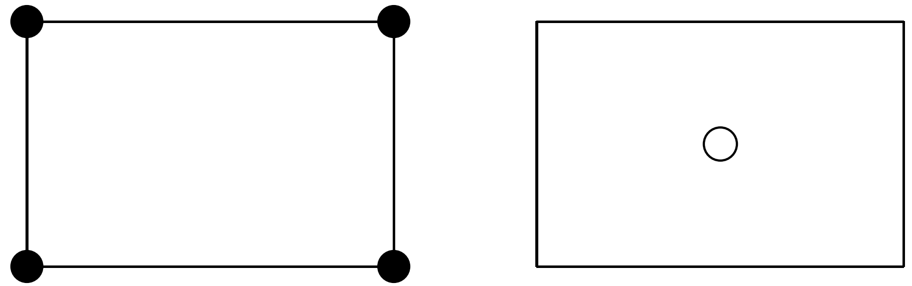

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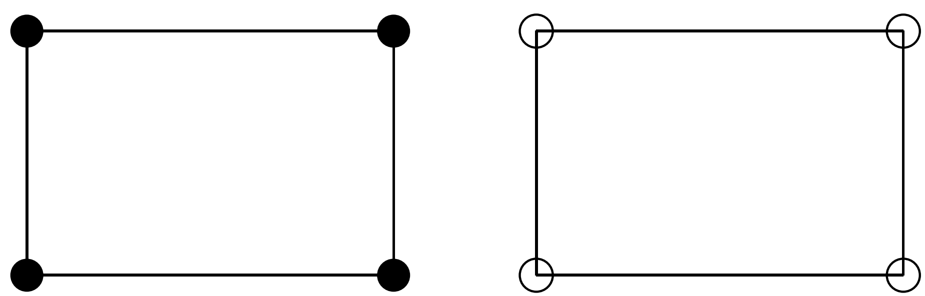

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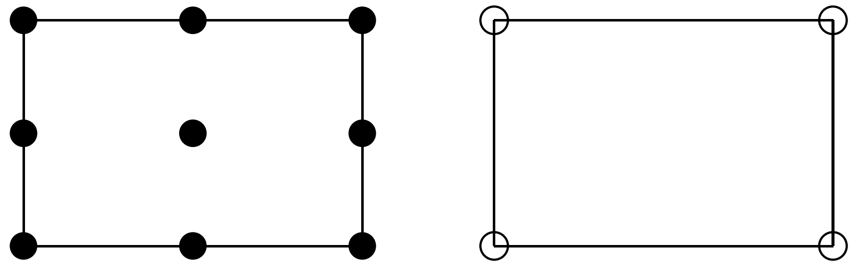
Stokes Preconditioner I



Q_2–Q_1 element (• two velocity components; ∘ pressure).

$$M = \begin{pmatrix} \mathbb{A}_*^{-1} & 0 \\ 0 & \mathbb{I}_*^{-1} \end{pmatrix}.$$

- Negative Laplacian preconditioning ($(-\nabla^2)^{-1}$ operator)

$$\lambda \leq \frac{\mathbf{u}^T \mathbb{A} \mathbf{u}}{\mathbf{u}^T \mathbb{A}_* \mathbf{u}} \leq \Lambda.$$

HSL

HSL_MI20

PACKAGE SPECIFICATION HSL 2007

1 SUMMARY

Given an $n \times n$ sparse matrix $\mathbf{A}$ and an $n-$vector $\mathbf{z}$, `HSL_MI20` computes the vector $\mathbf{x} = \mathbf{Mz}$, where $\mathbf{M}$ is an algebraic multigrid (AMG) v-cycle preconditioner for $\mathbf{A}$. A classical AMG method is used, as described in [1] (see also Section 5 below for a brief description of the algorithm). The matrix $\mathbf{A}$ must have positive diagonal entries and (most of) the off-diagonal entries must be negative (the diagonal should be large compared to the sum of the off-diagonals). During the multigrid coarsening process, positive off-diagonal entries are ignored and, when calculating the interpolation weights, positive off-diagonal entries are added to the diagonal.

Reference

[1] K. Stüben. *An Introduction to Algebraic Multigrid*. In U. Trottenberg, C. Oosterlee, A. Schüller, eds, 'Multigrid', Academic Press, 2001, pp 413-532.

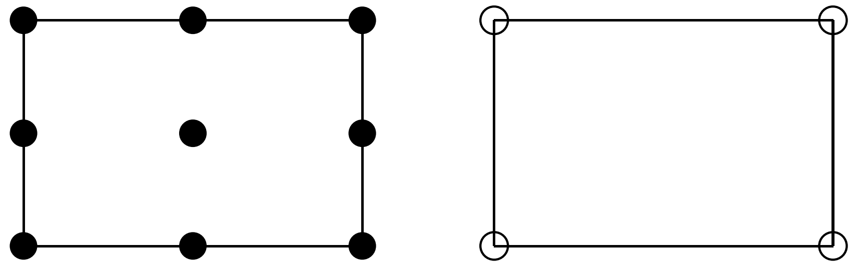
ATTRIBUTES — Version: 1.1.0 **Types:** Real (single, double). **Uses:** `HSL_MA48`, `HSL_MC65`, `HSL_ZD11`, and the `LAPACK` routines `_GETRF` and `_GETRS`. **Date:** September 2006. **Origin:** J. W. Boyle, University of Manchester and J. A. Scott, Rutherford Appleton Laboratory. **Language:** Fortran 95, plus allocatable dummy arguments and allocatable components of derived types. **Remark:** The development of `HSL_MI20` was funded by EPSRC grants EP/C000528/1 and GR/S42170.

$$\lambda \leq \frac{\mathbf{u}^T \mathbb{A} \mathbf{u}}{\mathbf{u}^T \mathbb{A}_* \mathbf{u}} \leq \Lambda.$$

Using black-box AMG
nv is the number of V-cycles performed.

nv	1		2		4	
grid	λ	Λ	λ	Λ	λ	Λ
uniform 8×8	0.864	1.000	0.981	1.000	1.000	1.00
uniform 32×32	0.831	1.000	0.971	1.000	0.999	1.00
stretched 32×32	0.447	1.000	0.694	1.000	0.906	1.00

Stokes Preconditioner II



Q_2–Q_1 element (• two velocity components; ∘ pressure).

$$M = \begin{pmatrix} \mathbb{A}_*^{-1} & 0 \\ 0 & \mathbb{I}_*^{-1} \end{pmatrix}.$$

- Mass matrix preconditioning (I^{-1} operator)

$$\theta \le \frac{\mathbf{p}^T \mathbb{I}\, \mathbf{p}}{\mathbf{p}^T \mathbb{I}_*\, \mathbf{p}} \le \Theta.$$

$$\theta \leq \frac{\mathbf{p}^T \mathbb{I}\, \mathbf{p}}{\mathbf{p}^T \mathbb{I}_* \, \mathbf{p}} \leq \Theta.$$

Wathen & Rees [2009]
Using Chebyshev accelerated Jacobi
`its` is the number of acceleration steps performed.

`its`	5		10		20	
grid	θ	Θ	θ	Θ	θ	Θ
uniform 16×16	0.883	1.234	0.986	1.003	1.000	1.00
uniform 64×64	0.883	1.234	0.986	1.003	1.000	1.00
stretched 64×64	0.883	1.234	0.986	1.003	1.000	1.00

Back to the Issues ...

- The most natural iterative solver for a symmetric indefinite system $K\mathbf{x} = \mathbf{b}$ is MINRES. This minimizes the ℓ_2–norm of the mth residual

$$\|\mathbf{r}^{(m)}\| = \|\mathbf{b} - K\mathbf{x}^{(m)}\| = \|K(\mathbf{x} - \mathbf{x}^{(m)})\|$$

over the Krylov space

$$\mathcal{K}_m(K, \mathbf{b}) = \text{span}\,\{\mathbf{b}, K\mathbf{b}, \ldots, K^{m-1}\mathbf{b}\}.$$

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- We want to compute constants c and C such that

$$c\,\|\mathbf{e}^{(m)}\|_E \leq \|\mathbf{r}^{(m)}\|_M \leq C\,\|\mathbf{e}^{(m)}\|_E,$$

where $\mathbf{e}^{(m)} = \mathbf{x} - \mathbf{x}^{(m)}$, $\mathbf{r}^{(m)} = K\mathbf{e}^{(m)}$, and $M = E^{-1}$ with E the block diagonal matrix representing the norms associated with the underlying space $V \times Q$.

... Stokes flow case

$$c\,\|\mathbf{e}^{(m)}\|_E \leq \|\mathbf{r}^{(m)}\|_M \leq C\,\|\mathbf{e}^{(m)}\|_E$$

$$K = \begin{bmatrix} \mathbb{A} & B^T \\ B & 0 \end{bmatrix},\; E = \begin{bmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{I} \end{bmatrix},\; M^{-1} = \begin{bmatrix} \mathbb{A}_* & 0 \\ 0 & \mathbb{I}_* \end{bmatrix}.$$

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Inf-Sup stability :

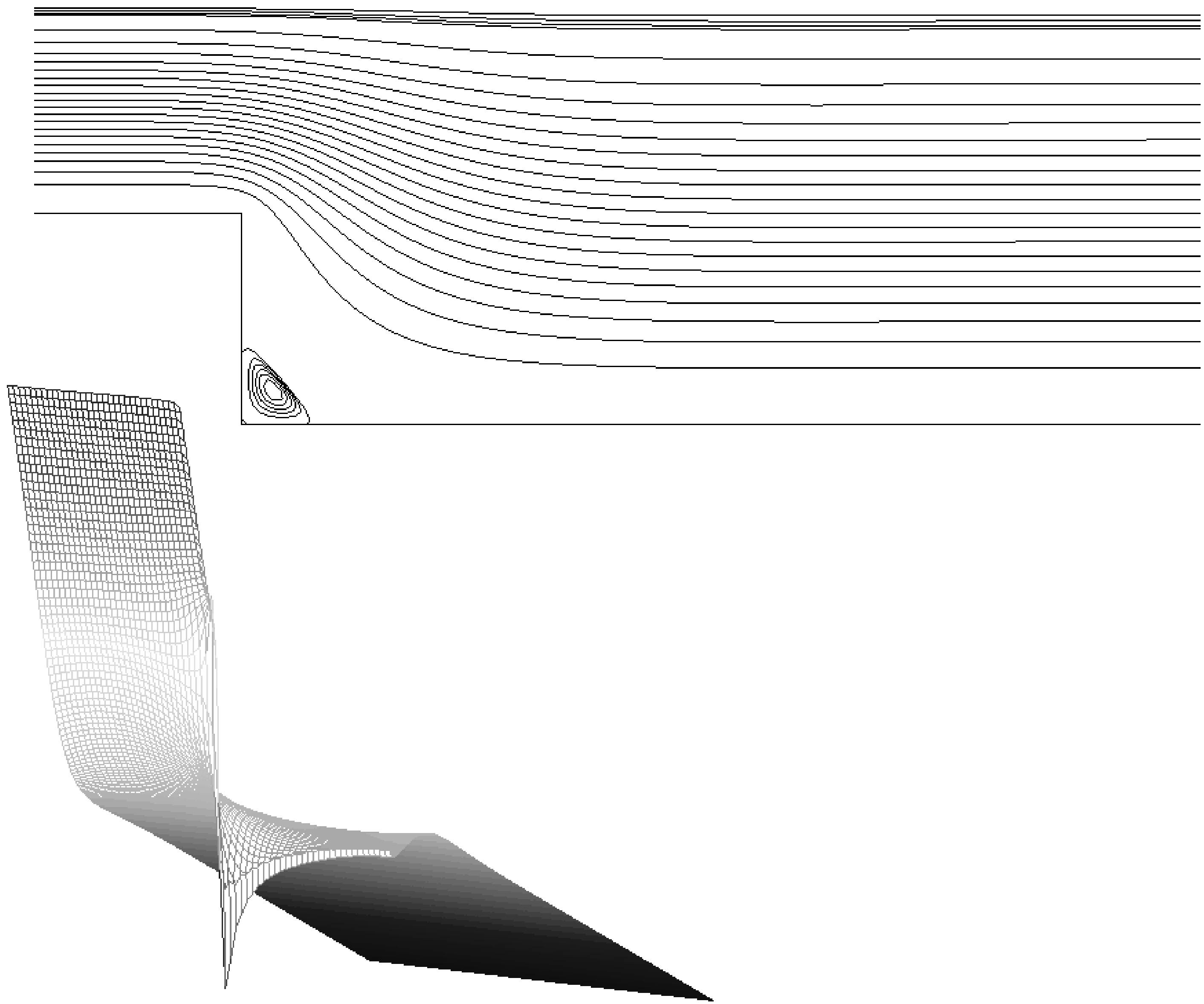
$$\gamma^2 \le \frac{\mathbf{q}^T B \mathbb{A}^{-1} B^T \mathbf{q}}{\mathbf{q}^T\, \mathbb{I}\, \mathbf{q}} \le \Gamma^2 \le d$$

Eigenvalue bounds : (from Silvester & Wathen [1994])

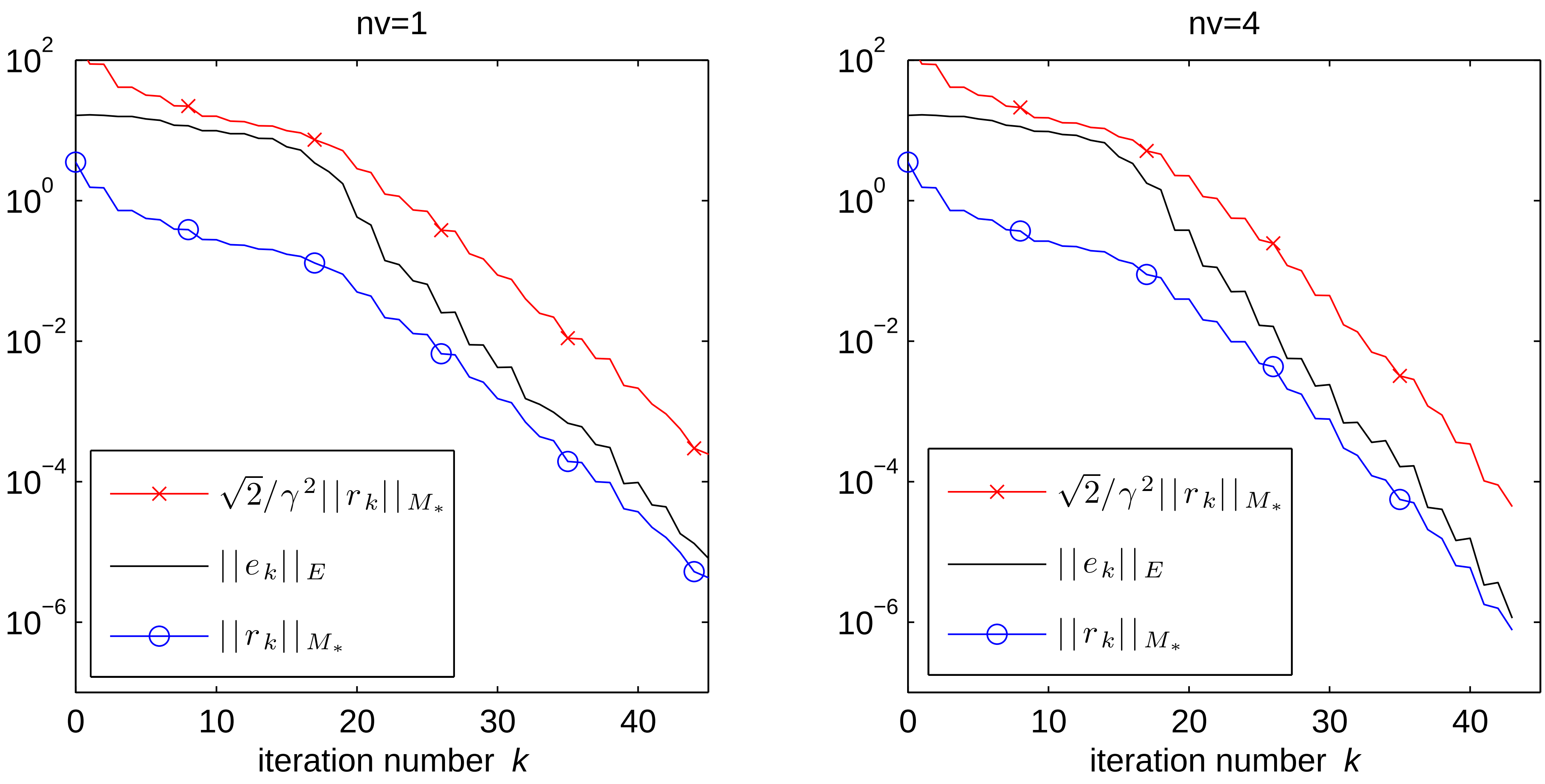
$$c^2 = \gamma^2 \left(1 + {}^1\!/_2\,\gamma^2 - \sqrt{1 + {}^1\!/_4\,\gamma^4}\right) \sim {}^1\!/_2\gamma^4; \qquad C^2 = \max\left\{2 + \Gamma^2, 2\Gamma^2\right\}$$

Stopping heuristic : $\|\mathbf{e}^{(m)}\|_E \le \frac{\sqrt{2}}{\gamma^2}\,\|\mathbf{r}^{(m)}\|_M.$

Flow over a Step: problem 5.2



Step flow: precomputed value: $\gamma^2 \approx \mathbf{0.0247}$



Dynamic inf-sup constant ...

Can we estimate the value of γ^2 on-the-fly?

Dynamic inf-sup constant ...

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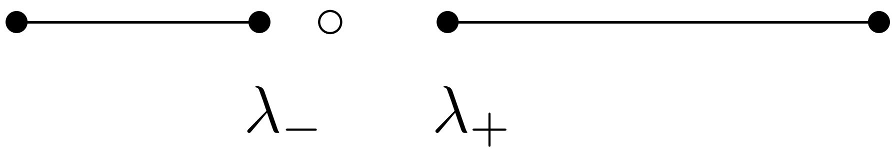
Maybe yes:



Inverting the eigenvalue bounds on the largest negative value λ_- and the smallest positive eigenvalue λ_+ of the matrix MK:

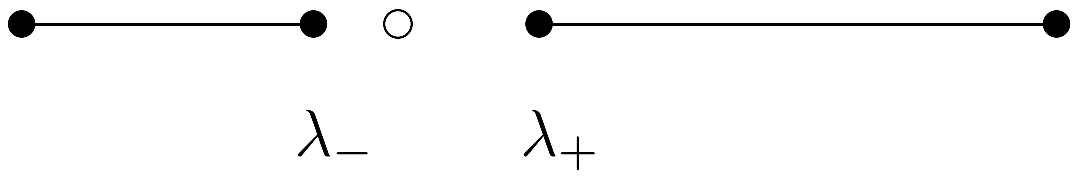
$$\lambda_- \leq 1/2\left(\delta - \sqrt{\delta^2 + 4\delta\gamma^2}\right) \quad \text{and} \quad \delta < \lambda_+,$$

leads to the computable estimate

$$\gamma_k^2 = \left(\lambda_-^2 - \lambda_-\lambda_+\right)/\lambda_+.$$

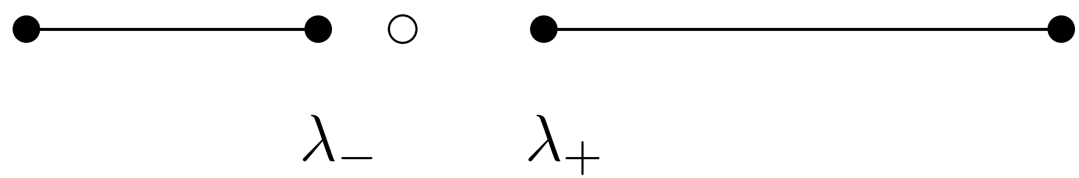
All we need to do is to estimate λ_- and λ_+ using the harmonic Ritz values... EST_MINRES.

Harmonic Ritz values ...



Eigenvalues of the preconditioned matrix $\hat{K} = MK$.

Harmonic Ritz values ...



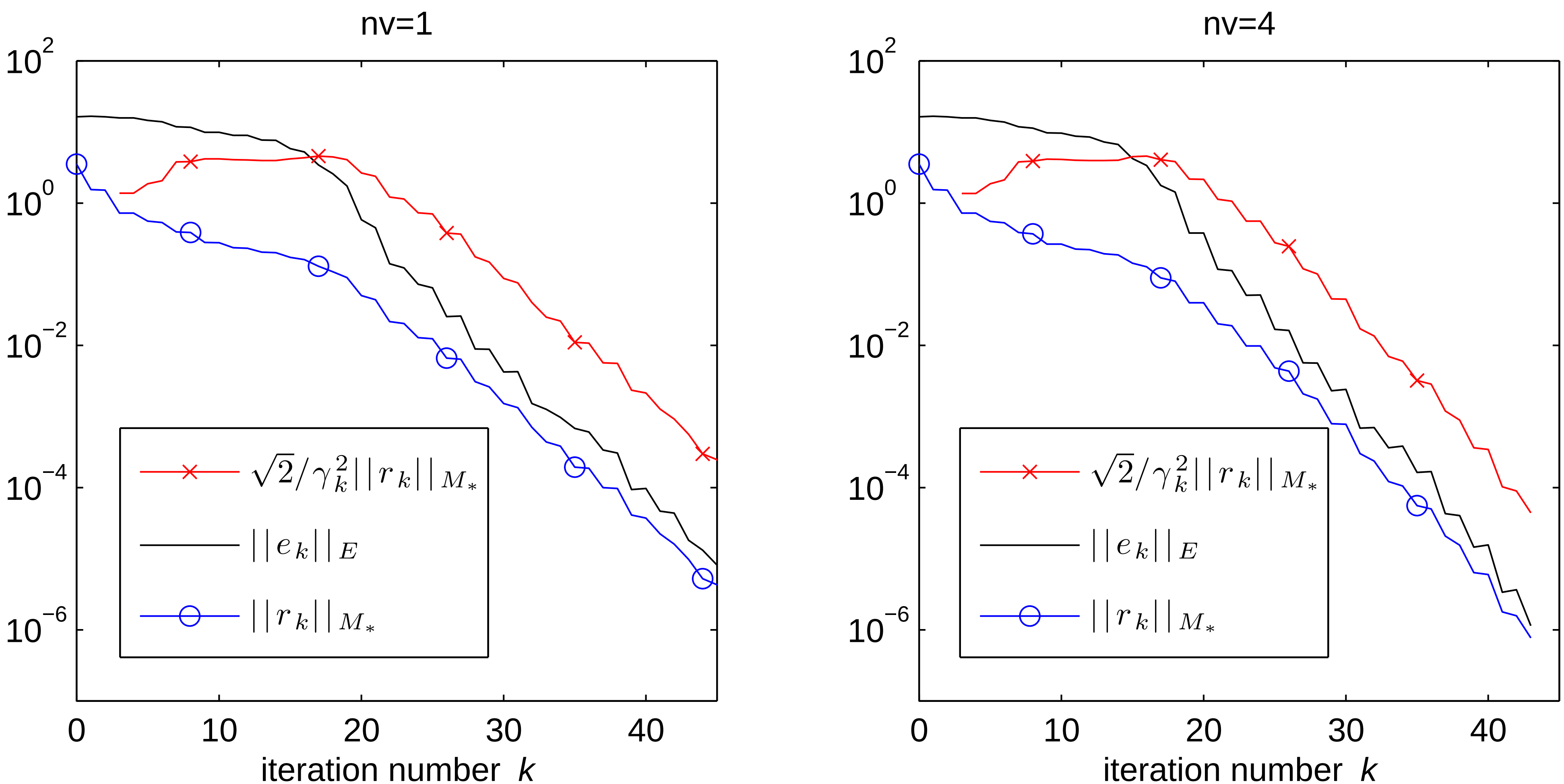
Eigenvalues of the preconditioned matrix $\hat{K} = MK$.
At step m of MINRES the Harmonic Ritz values $\theta_1, \ldots, \theta_m$ are the roots of the residual polynomial ϕ_m, defined as $\mathbf{r}^{(m)} = \phi_m(\hat{K})\hat{\mathbf{b}}$, with $\phi_m(\theta) = \frac{1}{\hat{\phi}_m(0)}\hat{\phi}(\theta)$.

They are also the eigenvalues of the following problem:

$$\underline{T}_m^T \, \underline{T}_m \mathbf{u} = \theta \, T_m \mathbf{u},$$

where T_m is the tridiagonal Lanczos matrix, and $\underline{T}_m$ is the row augmented counterpart.
See Morgan [1991], Freund [1992] and Paige et al. [1995].

Step flow: γ^2 estimated at each iteration



Back to the Issues ...

- How does one compute an accurate estimate of the discretization error $\| \vec{u} - \vec{u}_h^{(m)} \|_V + \|p - p_h^{(m)}\|_Q$?

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- That is, given a candidate solution $(\vec{u}_h, p_h) \in V_h \times Q_h$ (not necessarily the Galerkin solution), we want to compute an estimate η which is equivalent to the exact error in the sense that

$$c\,\eta \leq \| \vec{u} - \vec{u}_h \|_V + \|p - p_h\|_Q \leq C\,\eta,$$

with $C/c \sim O(1)$.

- Qifeng Liao & David Silvester.
A simple yet effective a posteriori estimator for classical mixed approximation of Stokes equations
Appl. Numer. Math., 2011.

$$c\,\eta^{(m)} \leq \left\| \nabla(\vec{u} - \vec{u}_h^{(m)}) \right\| + \left\| p - p_h^{(m)} \right\| \leq C\,\eta^{(m)}$$

with $C/c \sim O(1)$.

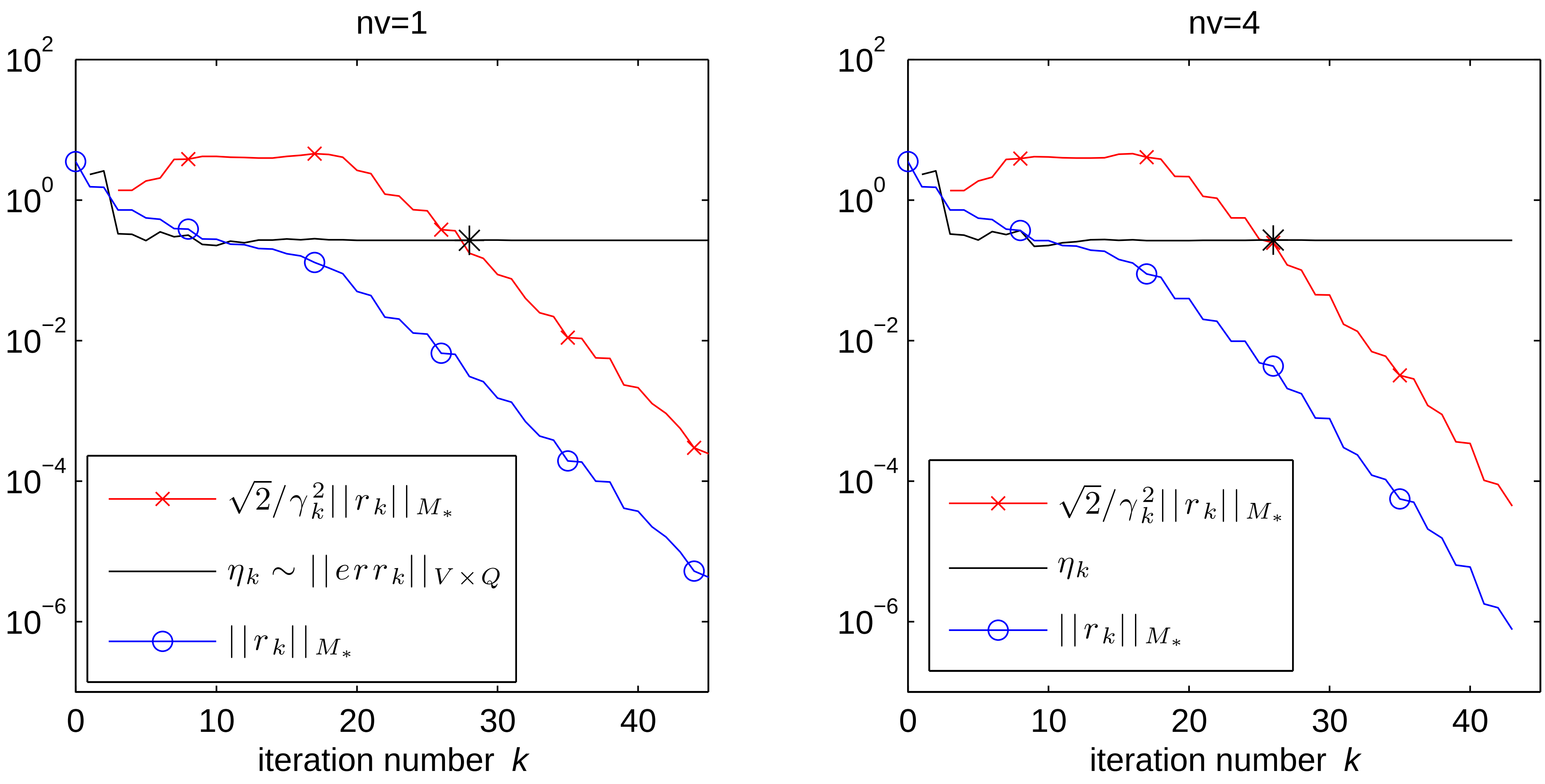
$$c\,\eta^{(m)} \le \left\|\nabla(\vec{u} - \vec{u}_h^{(m)})\right\| + \left\|p - p_h^{(m)}\right\| \le C\,\eta^{(m)}$$

with $C/c \sim O(1)$.

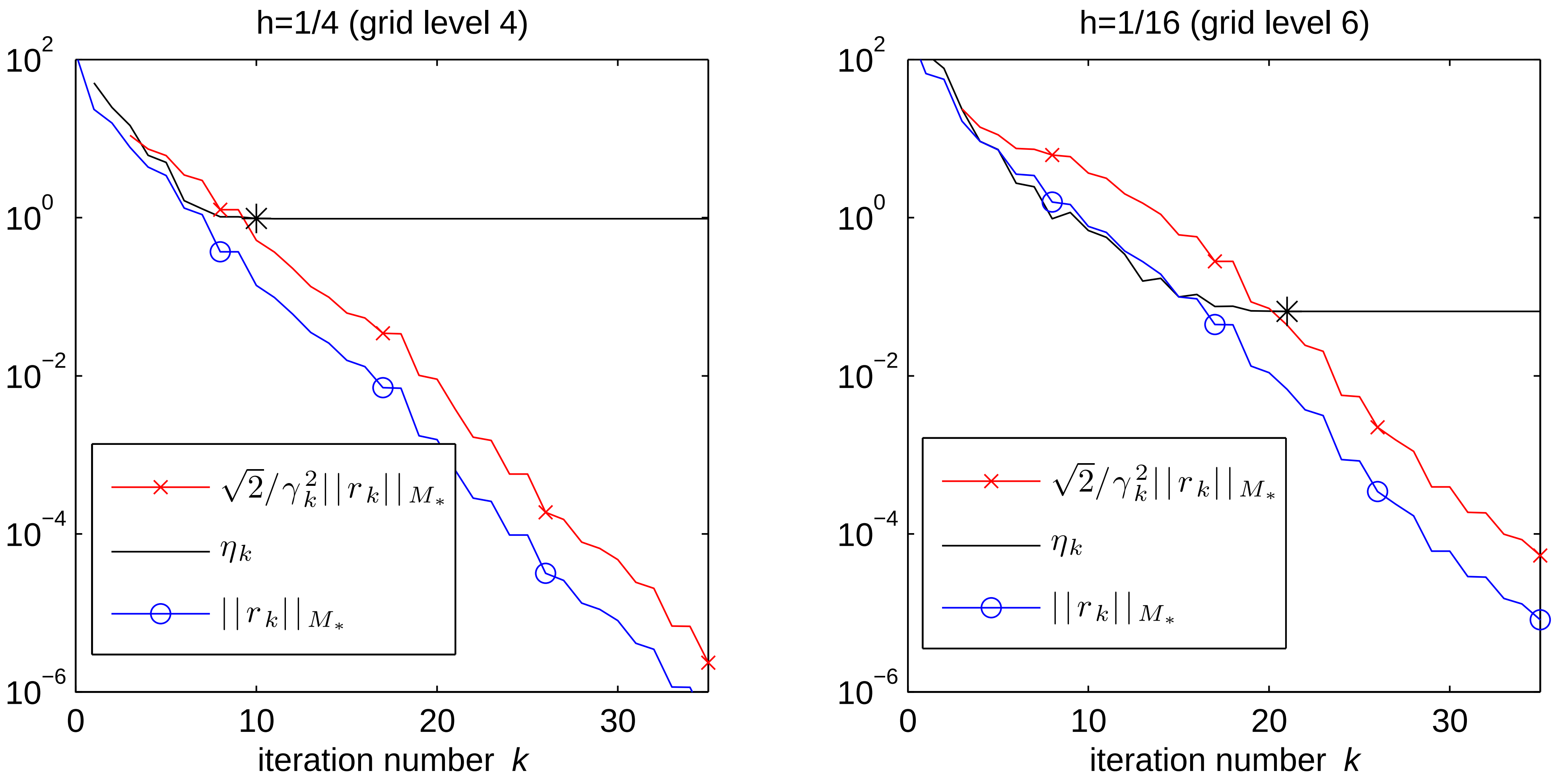
Refined stopping heuristic :

$$\|\mathbf{e}^{(m)}\|_E \le \frac{\sqrt{2}}{\gamma_m^2}\,\|\mathbf{r}^{(m)}\|_M \le \eta^{(m)}$$

Step flow: refined stopping heuristic



Square cavity flow: refined stopping heuristic



Square cavity flow: refined stopping heuristic iteration counts k_* vs spatial accuracy resolution

grid	k_*	η	$\|\nabla \cdot \vec{u}_h\|$
uniform 8×8	10	9.71×10^{-1}	2.97×10^{-2}
uniform 16×16	17	2.54×10^{-1}	3.66×10^{-3}
uniform 32×32	21	6.51×10^{-2}	4.56×10^{-4}
uniform 64×64	24	1.64×10^{-2}	5.69×10^{-5}
		$O(h^2)$	$O(h^3)$

What have we achieved?

- Efficient linear algebra: convergence rate is independent of h.
- Black-box implementation: No parameters have to be estimated a priori!

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- Efficient linear algebra: convergence rate is independent of h.
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Further Reading ...

- David Silvester & Valeria Simoncini.
EST_MINRES: An optimal iterative solver for symmetric indefinite systems stemming from mixed approximation
ACM Trans. Math. Softw., vol. 37 no. 4, 2010.