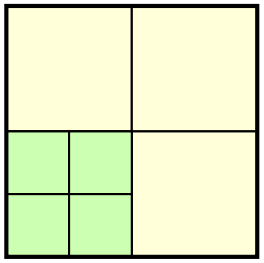
LECTURE 4

Numerical Stability Analysis



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12th School "Mathematical Theory in Fluid Mechanics",
Kacov, May 27 - June 3, 2011

4.0 Content

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4.1 The problem of hydrodynamic stability

4.1.1 Stability of stationary solutions (an example)

Stationary Navier-Stokes system: $u := \{v, p\}$

$$\mathcal{A}(u) := -\nu\Delta v + v \cdot \nabla v + \nabla p = 0, \qquad \nabla \cdot v = 0$$

v velocity, p pressure, ν viscosity ($\rho \equiv 1$),

$$v|_{\Gamma_{\text{rigid}}} = 0, \quad v|_{\Gamma_{\text{in}}} = v^{\text{in}}, \quad \nu\partial_n v - np|_{\Gamma_{\text{out}}} = 0$$

Drag minimization by boundary control

$u \in u^{\text{in}} + V$ state, q "boundary control" (constant at $\Gamma_Q := \Gamma_1 \cup \Gamma_2$)

$$J(u,q) := \frac{2}{\bar{U}^2 D} \int_S n \cdot \sigma(v,p) \cdot e_1 \, \mathrm{ds} \;\rightarrow\; \min$$

S surface of cylinder, D diameter, $\bar{U}$ reference velocity
$\sigma(v,p) = -pI + \nu(\nabla v + \nabla v^T)$ stress,

State equation in variational form:

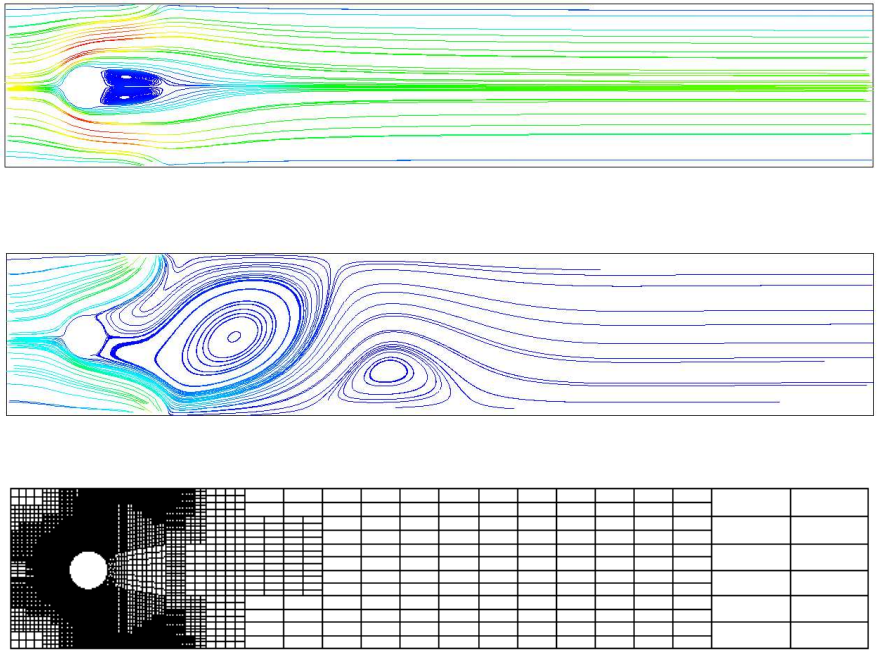
$$a(u)(\varphi) + \mathbf{b(q,\varphi)} = (f, \varphi^v) \qquad \forall \varphi \in V$$

"control form" $b(q,\varphi) := -(q, n \cdot \varphi^v)_{\Gamma_Q}$

Numerical Solution:

Adaptive Galerkin finite element discretization and **Newton iteration**.

Reynolds number $\mathrm{Re} = \bar{\mathrm{U}}^2\mathrm{D}/\nu = 40$ (uncontrolled stationary flow)



Uncontrolled flow (top), controlled flow (middle), adapted mesh (bottom)

Question: Is the stationary "optimal" flow stable?

4.1.2 Linearized stability analysis

Stability of base solution $\hat{u} = \{\hat{v}, \hat{q}\}$ via **linear stability theory**:

a) Spectral approach: non-symmetric eigenvalue problem for $u := \{v, p\} \in V$ and $\lambda \in \mathbb{C}$:

$$\mathcal{A}'(\hat{\mathbf{v}})u := -\nu\Delta v + \hat{\mathbf{v}} \cdot \nabla v + v \cdot \nabla \hat{\mathbf{v}} + \nabla p = \lambda v, \quad \nabla \cdot v = 0$$

$$\mathrm{Re}\,\lambda \geq \mathbf{0} \quad \Rightarrow \quad \hat{\mathbf{u}} \ \textbf{stable} \ (?)$$

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b) Variational approach: growth property of the solution operator $S(t) : \mathbf{J}_0 \to \mathbf{J}_0$ for the linearized perturbation equation

$$(\partial_t v, \varphi) + a'(\hat{v})(v, \varphi) = 0 \quad \forall \varphi \in \mathbf{J}_1, \quad v(0) = v_0.$$

$$\|S(t)\| \approx \mathbf{A} e^{-\mathrm{Re}\lambda\, t} \quad (t \to \infty), \qquad \mathbf{A} \gg \mathbf{1}\,?$$

Variational formulation of stability eigenvalue problem:

$$a'(\hat{\mathbf{u}})(\psi,\varphi) := \nu(\nabla\psi^v,\nabla\varphi^v) + (\hat{\mathbf{v}}\cdot\nabla\psi^v,\varphi^v) + (\psi^v\cdot\nabla\hat{\mathbf{v}},\varphi^v)$$
$$- (\psi^p,\nabla\cdot\varphi^v) + (\varphi^p,\nabla\cdot\psi^v),$$
$$m(\psi,\varphi) := (\psi^v,\varphi^v).$$

Primal and dual eigenvalue problems: $u,\, u^* \in V$:

$$a'(\hat{\mathbf{u}})(u,\varphi) = \lambda\, m(u,\varphi) \qquad \forall\varphi\in V$$
$$a'(\hat{\mathbf{u}})(\varphi,u^*) = \lambda\, m(\varphi,u^*) \qquad \forall\varphi\in V$$

Normalization: $m(u,u) = m(u,u^*) = 1\,.$

If $m(u,u^*) = 0\,,$ the boundary value problem

$$a'(\hat{u})(\tilde{u},\varphi) - \lambda\, m(\tilde{u},\varphi) = m(u,\varphi) \qquad \forall\varphi\in V$$

has a solution $\tilde{u}\in V$ "generalized eigenfunction") $\quad\Rightarrow\quad$ **defect**$(\lambda)\geq \mathbf{1}$

4.1.3 The effect of non-normality and pseudospectra

Generic scenario: There are $m \geq 1$ discrete eigenvalues $\{\lambda_h^{(i)},\ i=1,...,m\}$ which are usually all simple and (for $h \to 0$) approximate some eigenvalue $\lambda \in \Sigma(\mathcal{A})$, such that

(i) λ has geometric multiplicity m and hence **trivial defect**, or
(ii) λ has geometric multiplicity less than m and hence **defect** $\alpha \geq 1$.

The existence of an eigenvalue with $\mathrm{Re}\lambda < 0$ causes dynamic instability of the base flow $\hat{v}$, i.e., arbitrarily small perturbations may grow without bound. This is reflected by the growth property

$$\|\mathbf{S(t)}\| \approx (\mathbf{1} + \mathbf{t}^{\alpha})\mathbf{e}^{-\mathrm{Re}\lambda\,\mathbf{t}} \to \infty \quad (\mathbf{t} \to \infty)$$

of the solution operator $S : \mathbf{J}_0 \to \mathbf{J}_0$ for the linearized perturbation equation.

The challenge is the case $\Sigma(\mathcal{A}) \subset \mathbb{C}_+$ with some critical eigenvalues with $0 < \mathrm{Re}\lambda \ll 1$. For detecting whether case (i) or case (ii) is valid, we may employ the following simple indicator:

Deficiency Test: For any small $h > 0$, let λ_h be one of the approximating eigenvalues of $\mathcal{A}'_h(\hat{v})$ with corresponding right and left eigenvectors v_h and v_h^*, satisfying $\|v_h\| = 1$ and $(v_h, v_h^*) = 1$. Nonzero defect of λ is characterized by the existence of limiting eigenvectors $(v, v^*) = 0$. Hence,

$$\|\mathbf{v_h^*}\| \to \infty \quad (\mathbf{h \to 0}),$$

with a certain rate, can be used as an indicator for $\alpha(\lambda) \geq 1$. In this case the blowup behavior of the solution operator $S(t)$ implies

$$\sup_{\mathbf{t>0}} \|\mathbf{S(t)}\| \geq \frac{\alpha}{|\mathrm{Re}\lambda|}.$$

Consequently, for $0 < \mathrm{Re}\lambda \ll 1$ the amplification constant A in the stability estimate becomes very large.

Pseudospectra

A similar destabilizing effect can also occur if λ is non-deficient. This is related to the concept of "pseudospectrum" of Trefethen et al. (1992).

For $\epsilon \in \mathbb{R}_+$ the ϵ-pseudo-spectrum $\Sigma_\epsilon \subset \mathbb{C}$ of $\mathcal{A}'(\hat{v})$ is defined by

$$\Sigma_\epsilon := \left\{ z \in \mathbb{C} \setminus \Sigma(\mathcal{A}'(\hat{v})),\ \|(\mathcal{A}'(\hat{v}) - z\mathcal{I})^{-1}\| \geq \epsilon^{-1} \right\} \cup \Sigma(\mathcal{A}'(\hat{v})).$$

The following result is related to the "easy part" of the so-called "Kreiss matrix theorem".

Proposition. *Let $z \in \Sigma_\epsilon$ be a regular point of the operator $\mathcal{A}'(\hat{v})$ with $\operatorname{Re} z < 0$. Then, for the solution operator $S(t) : \mathbf{J_0} \to \mathbf{J_0}$ of the linear perturbation equation, there holds*

$$\sup_{\mathbf{t \geq 0}} \|\mathbf{S(t)}\| \ \geq\ |\operatorname{Re} \mathbf{z}|\, \|(\mathcal{A}'(\hat{\mathbf{v}}) - \mathbf{z}\mathcal{I})^{-\mathbf{1}}\| \ \geq\ \frac{|\operatorname{Re} \mathbf{z}|}{\epsilon}.$$

Hence a critical pseudo-spectrum may trigger instability even if all eigenvalues have positive real parts.

4.2 The numerical solution

4.2.1 Adaptive finite element discretization

Discretization by stabilized finite element method (Q_1/Q_1 Stokes element)

Stabilized sesquilinear form (GLS stabilization))

$$a'_\delta(\hat{\mathbf{u}}_\mathbf{h})(u_h,\varphi_h) := a'(\hat{\mathbf{u}}_\mathbf{h})(u_h,\varphi_h) + (\mathcal{A}'(\hat{\mathbf{v}}_\mathbf{h})u_h - \lambda_h \mathcal{M}u_h, \mathcal{S}(\hat{\mathbf{v}}_\mathbf{h})\varphi_h)_\delta$$

Discrete primal and dual eigenvalue problems $u_h,\ u_h^* \in V_h,\ \lambda_h \in \mathbb{C}$:

$$a'_\delta(\hat{\mathbf{u}}_\mathbf{h})(u_h,\varphi_h) = \lambda_h\, m(u_h,\varphi_h) \qquad \forall \varphi_h \in V_h$$
$$a'_\delta(\hat{\mathbf{u}}_\mathbf{h})(\varphi_h,u_h^*) = \lambda_h\, m(\varphi_h,u_h^*) \qquad \forall \varphi_h \in V_h$$

Normalization $\quad m(u_h,u_h) = m(u_h,u_h^*) = 1$

Blow-up criterion:

$$m(u_h^*,u_h^*) \to \infty \quad (h \to 0) \quad \Rightarrow \quad \mathbf{defect}(\lambda) \geq \mathbf{1}$$

Points to be observed:

- Reliable and efficient computation of critical eigenvalues (those with smallest real part):
  a) error control by a posteriori error estimates,
  b) work reduction by "goal-oriented" mesh adaptation.

- Reliability of eigenvalue computation for an only approximately known base solution $\hat{\mathbf{u}}$.

- Effect of the non-normality of the operator $\mathcal{A}'(\hat{\mathbf{u}})$.

- Efficient solution of the discretized (algebraic) eigenvalue problems by Krylov space iteration and multigrid techniques.

- Balancing of discretization and iteration error on the basis of a posteriori error estimates.

Prior results on FE eigenvalue approximation

Abstract eigenvalue problem in Hilbert space $H := H_0^1(\Omega)$

$$a(u, \varphi) = \lambda\,(u, \varphi) \quad \forall \varphi \in H.$$

$a(\cdot, \cdot)$ elliptic sesquilinear form (compact w.r.t. $(\cdot, \cdot) = (\cdot, \cdot)_{L^2}$).

Galerkin approximation in subspace $H_h \subset H$

$$a(u_h, \varphi_h) = \lambda_h\,(u_h, \varphi_h) \quad \forall \varphi_h \in H_h.$$

a) A priori error analysis (based on operator theory)

Approximation property

$$\|\nabla(u - I_h u)\| \;\leq\; ch\|\nabla^2 u\|.$$

A priori error estimates (Bramble/Osborn 1973):

$$|\lambda_h - \lambda| \;\leq\; c(\lambda)\, h^2, \qquad \|u_h - u\| \;\leq\; c(\lambda)\, h^2.$$

b) A posteriori error analysis (based on "energy technique")

i) Symmetric case (Nystedt 1995, Larson 1996, Verfürth 1996):

$$|\lambda_h - \lambda| \leq c_I \sum_{K \in \mathbb{T}_h} h_K^2 \, \rho_K^2,$$

cell residuals

$$\rho_K = \rho_K(u_h, \lambda_h) := \left(\|\Delta u_h + \lambda_h u_h\|_K^2 + \tfrac{1}{2} \|[\partial_n u_h]\|_{\partial K}^2 \right)^{1/2}.$$

ii) Nonsymmetric case (Heuveline/Ra. 2001):

$$|\lambda_h - \lambda| \leq c_I \sum_{K \in \mathbb{T}_h} h_K^2 \{ \rho_K^2 + \rho_K^{*2} \},$$

primal and dual cell residuals

$$\rho_K = \rho_K(u_h, \lambda_h), \qquad \rho_K^* = \rho_K^*(u_h^*, \lambda_h^*).$$

c) A posteriori error estimation via duality theory

Embedding into the general framework of variational equations:

$$\mathcal{V} := V \times V \times \mathbb{C}, \quad \mathcal{V}_h := V_h \times V_h \times \mathbb{C}$$
$$U := \{\hat{u}, u, \lambda\}, \quad U_h := \{\hat{u}_h, u_h, \lambda_h\}, \quad \Phi = \{\hat{\varphi}, \varphi, \mu\} \in \mathcal{V}$$

Semi-linear form:

$$A(U)(\Phi) := \underbrace{f(\hat{\varphi}) - a_\delta(\hat{u})(\hat{\varphi})}_{\textbf{base solution}} + \underbrace{\lambda\, m(u,\varphi) - a'_\delta(\hat{u})(u,\varphi)}_{\textbf{eigenvalue problem}} + \underbrace{\overline{\mu}\big\{m(u,u) - 1\big\}}_{\textbf{normalization}}$$

Compact variational formulation:

$$A(U)(\Phi) = 0 \qquad \forall \Phi \in \mathcal{V}$$
$$A(U_h)(\Phi_h) = 0 \qquad \forall \Phi_h \in \mathcal{V}_h$$

Error control functional:

$$J(\Phi) := \mu\, m(\varphi,\varphi) \quad \Rightarrow \quad J(U) = \lambda m(u,u) = \lambda.$$

Proposition. *There holds the error representation*

$$\lambda-\lambda_h \;=\; \underbrace{\tfrac{1}{2}\rho(\hat{u}_h)(\hat{u}^*-i_h\hat{u}^*)+\tfrac{1}{2}\rho^*(\hat{u}_h^*)(\hat{u}-i_h\hat{u})}_{\textbf{base solution residuals}}$$

$$+\;\underbrace{\tfrac{1}{2}\rho(\{u_h,\lambda_h\})(u^*-i_hu^*)+\tfrac{1}{2}\rho^*(\{u_h^*,\lambda_h\})(u-i_hu)}_{\textbf{eigenvalue residuals}}\;+\mathcal{R}_h,$$

for arbitrary $i_h\hat{u}^*$, $i_h\hat{u}$, i_hu^*, $i_hu\in V_h$. *The remainder* $\mathcal{R}_h$ *is cubic in the errors* $\hat{e}_h^v:=\hat{v}-\hat{v}_h$, *etc.*

Residuals:

$$\begin{aligned}
\rho(\hat{u}_h)(\cdot) &:= (f,\cdot)-a_\delta(\hat{u}_h)(\cdot)\\
\rho^*(\hat{u}_h^*)(\cdot) &:= -a_\delta''(\hat{u})(\cdot,u_h,u_h^*)-a_\delta'(\hat{u}_h)(\cdot,\hat{u}_h^*)\\
\rho(\{u_h,\lambda_h\})(\cdot) &:= \lambda_h\,m(u_h,\cdot)-a_\delta'(\hat{u}_h)(u_h,\cdot)\\
\rho^*(\{u_h^*,\lambda_h\})(\cdot) &:= \lambda_h\,m(\cdot,u_h^*)-a_\delta'(\hat{u}_h)(\cdot,u_h^*)
\end{aligned}$$

Sketch of proof (by Euler-Lagrange approach).

Lagrangian: $\mathcal{L}(u,z) := J(u) - A(u)(z).$

Stationary points $\{u,z\} \in V \times V$, $\{u_h, z_h\} \in V_h \times V_h$:

$$\mathcal{L}'(u,z)(\varphi,\psi) = \left\{ \begin{array}{c} J'(u)(z) - A'(u)(\varphi,z) \\ -A(u)(\psi) \end{array} \right\} = 0 \quad \forall \{\varphi,\psi\}$$

$$\mathcal{L}'(u_h,z_h)(\varphi_h,\psi_h) = \left\{ \begin{array}{c} J'(u_h)(z_h) - A'(u_h)(\varphi_h,z_h) \\ -A(u_h)(\psi_h) \end{array} \right\} = 0 \quad \forall \{\varphi_h,\psi_h\}$$

"Primal" and "dual" residuals defined on V :

$$\rho(u_h)(\cdot) := -A(u_h)(\cdot)$$
$$\rho^*(z_h)(\cdot) := J'(u_h)(\cdot) - A'(u_h)(\cdot, z_h)$$

Setting $L(x) := \mathcal{L}(u,z)$ for $x := \{u,z\}$, and $\epsilon := x - x_h$, there holds

$$\begin{aligned} J(u) - J(u_h) &= L(x)\underbrace{-A(u)(z)}_{=0} - L(x_h)\underbrace{-A(u_h)(z_h)}_{=0} \\ &= \int_0^1 L'(x_h + s\epsilon;\epsilon)\,\mathrm{ds} \quad \textbf{(trapezoidal rule with remainder)} \\ &= \tfrac{1}{2}\big\{L'(x_h;\epsilon) + \underbrace{L'(x;\epsilon)}_{=0}\big\} + \tfrac{1}{2}\int_0^1 L'''(x_h+s\epsilon)(\epsilon,\epsilon,\epsilon)\,s(s-1)\,ds \end{aligned}$$

Then, by **Galerkin orthogonality**, $L'(x_h)(\cdot) = 0$ on $V_h \times V_h$,

$$J(u) - J(u_h) = \tfrac{1}{2}L'(x_h)(x - I_h x) + \underbrace{\tfrac{1}{2}\int_0^1 L'''(x_h+s\epsilon)(\epsilon,\epsilon,\epsilon)\,s(s-1)\,ds}_{=:\,\mathcal{R}_h}$$

for arbitrary $I_h x \in V_h \times V_h$. This implies

$$J(u) - J(u_h) = \tfrac{1}{2}\underbrace{\rho(u_h)(z - I_h z)}_{\text{primal}} + \tfrac{1}{2}\underbrace{\rho^*(z_h)(u - I_h u)}_{\text{dual}} + \mathcal{R}_h$$

Application to the perturbed eigenvalue problem

Dual solutions $Z = \{\hat{z}, z, \pi\} \in \mathcal{V}$, $Z_h = \{\hat{z}_h, z_h, \pi_h\} \in \mathcal{V}_h$:

$$A'(U)(\Phi, Z) = J'(U)(\Phi) \qquad \forall \Phi \in \mathcal{V}$$

$$A'(U_h)(\Phi_h, z_h) = J'(U_h)(\Phi_h) \qquad \forall \Phi_h \in \mathcal{V}_h$$

Observation: $z = u^*$, $\pi = \lambda$ solution of corresponding adjoint eigenvalue problem, and $\hat{z} = \hat{u}^*$ determined by

$$a'(\hat{u})(\psi, \hat{u}^*) = -a''(\hat{u})(\psi, u, u^*) \qquad \forall \psi \in V.$$

Remarks:

- The error representation does not require the uniqueness of the solutions. It becomes meaningful through the a priori assumption $u_h \to u$ $(h \to 0)$.
- The cubic remainder $\mathcal{R}_h$ is neglected.
- The dual (adjoint) problem providing the error sensitivities is **linear**.

Determination of residuals

$$\begin{aligned}\rho(u_h)(\psi) &= \lambda_h(u_h,\psi) - a(u_h,\psi)\\ &= \sum_{K\in\mathbb{T}_h}\left\{(\lambda_h u_h - \mathcal{A}u_h,\psi)_K - (\partial_n^{\mathcal{A}}u_h,\psi)_{\partial K}\right\}\\ &= \sum_{K\in\mathbb{T}_h}\left\{(\lambda_h u_h - \mathcal{A}u_h,\psi)_K - \tfrac{1}{2}([\partial_n^{\mathcal{A}}u_h],\psi)_{\partial K}\right\}\end{aligned}$$

primal and dual residuals:

$$|\rho(u_h)(u^*-I_h u^*)| \le \sum_{K\in\mathbb{T}_h}\rho_K\omega_K^*, \qquad |\rho^*(u_h^*)(u-I_h u)| \le \sum_{K\in\mathbb{T}_h}\rho_K^*\omega_K$$

$$\begin{aligned}\rho_K &:= \|\lambda_h u_h - \mathcal{A}u_h\|_K + \tfrac{1}{2}h_K^{-1/2}\|[\partial_n^{\mathcal{A}}u_h]\|_{\partial K}\\ \omega_K^* &:= \max\left\{\|u^*-I_h u^*\|_K, h_K^{1/2}\|u^*-I_h u^*\|_{\partial K}\right\}\end{aligned}$$

and anagously for ρ_K^* and ω_K .

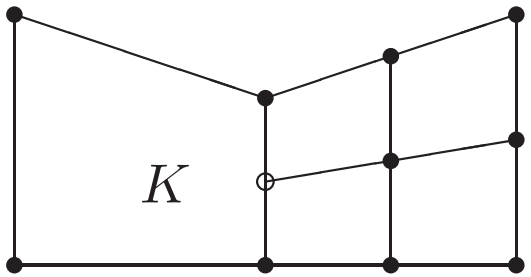
Practical mesh adaptation

a) Evaluation of error estimator

$$\eta_\omega(u_h, z_h) := \tfrac{1}{2}\rho(u_h)(z - I_h z) + \tfrac{1}{2}\rho^*(z_h)(u - I_h u)$$

needs approximations to the exact solutions u and z by **post-processing**:

$$u \approx i_{2h}^{(2)} u_h, \quad z \approx i_{2h}^{(2)} z_h$$



$\text{isopar}\{1, x_1, x_2, x_1 x_2\}$

b) Mesh adaptation by **"error balancing" strategy**: $N = \#\{K \in \mathbb{T}_h\}$

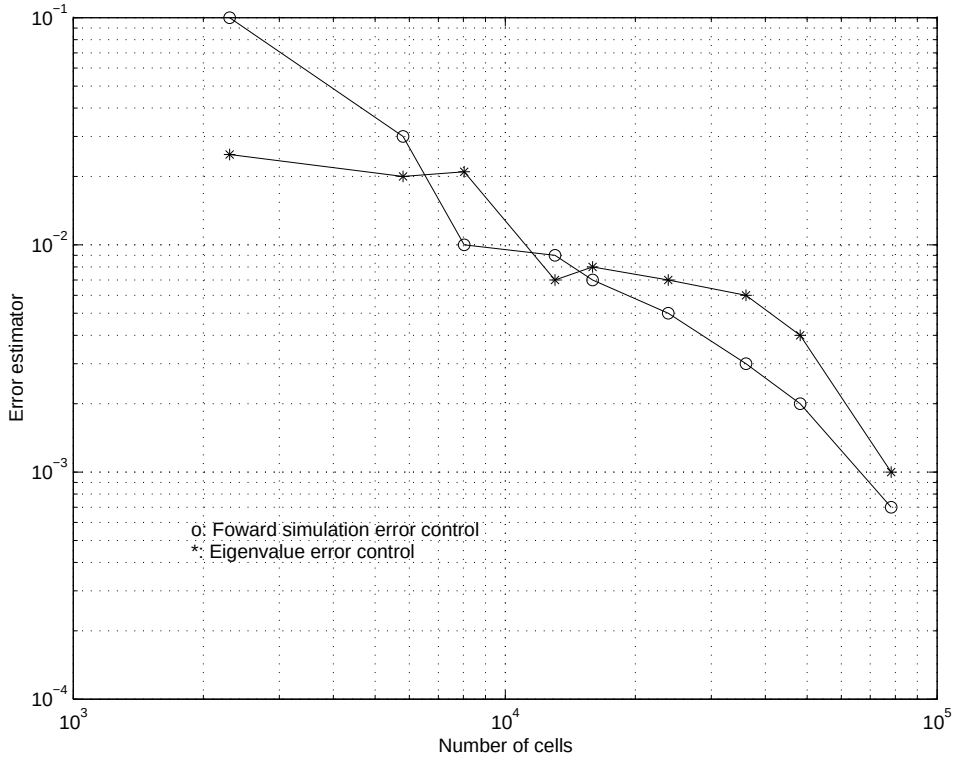
$$\eta_K := \tfrac{1}{2}\rho_K \omega_K^* + \tfrac{1}{2}\rho_K^* \omega_K \approx \frac{TOL}{N} \quad \Rightarrow \quad \eta_\omega(u_h, z_h) \approx TOL$$

4.2.2 Case of inexact base solution

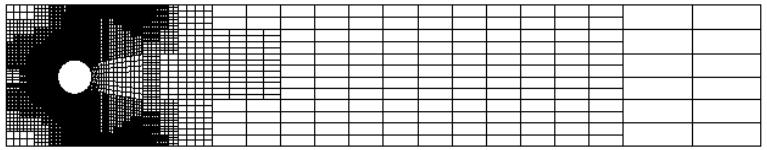
Error estimator and **balancing criterion**:

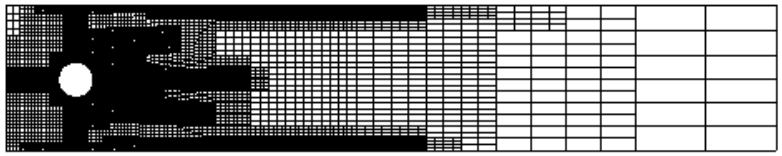
$$|\lambda - \lambda_h| \approx \sum_{K \in \mathbb{T}_h} \{\hat{\eta}_K + \eta_K^\lambda\}, \qquad \sum_{K \in \mathbb{T}_h} \hat{\eta}_K \leq \sum_{K \in \mathbb{T}_h} \eta_K^\lambda \ !$$

Balance of discretization errors for base solution and eigenvalue problem

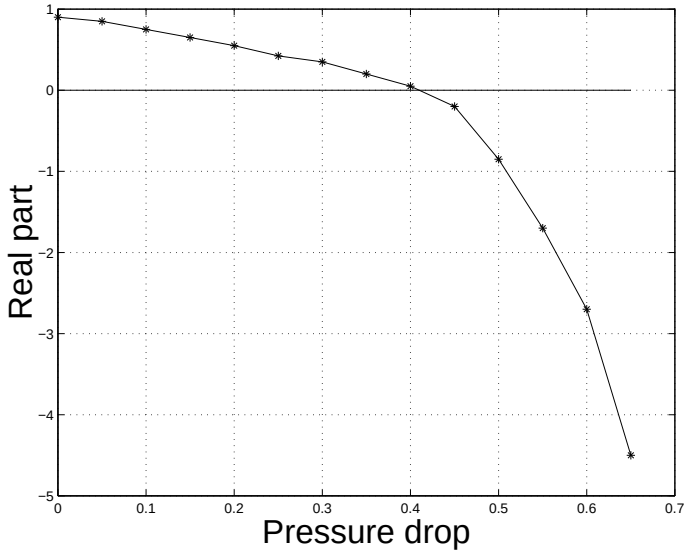


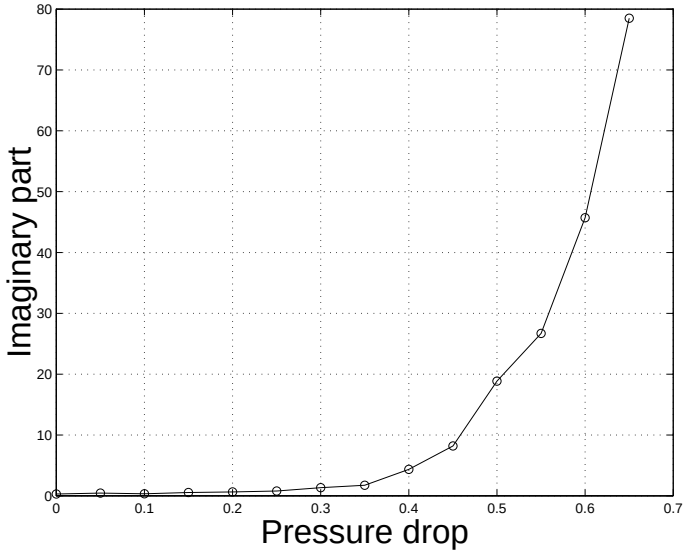
Meshes obtained by the error estimators for the drag minimization (left) and the eigenvalue computation (right)





Real and imaginary parts of the critical eigenvalue as function of the control pressure; $q_{\text{opt}} = 0.5$ ($\Rightarrow$ 'optimal' state **unstable**)





4.2.3 Control of algebraic iteration (Arnoldi method)

Algebraic eigenvalue problem in $\mathbb{R}^N$ (linearized Navier-Stokes operator)

$$\mathbf{A}u = \lambda \mathbf{M}u$$

with "stiffness" matrix and "mass" matrix

$$\mathbf{A} := \begin{bmatrix} A & B \\ B^T & C_\delta \end{bmatrix}, \qquad \mathbf{M} := \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$$

Here, $u = \{v, p\}$ stands for the vector of velocity and pressure nodal values.

Consider the inverse eigenvalue problem

$$\mathbf{T}u := \mathbf{A}^{-1}\mathbf{M}u = \lambda^{-1}u =: \mu u,$$

since the "smallest" eigenvalues are to be computed.

Algebraic solution process

- Arnoldi method using Krylov spaces $K_k = \text{span}\{u_1, \mathbf{T}u_1, \ldots, \mathbf{T}^{k-1}u_1\}$
- Computation of $v_j := \mathbf{T}^j u_1$, $j = 1, \ldots, k-1$, by geometric MG on sequences of locally adapted meshes (most cost critical part)
- Orthonormalization by mod. Gram-Schmidt or Householder algorithm
- Reduction to k-dimensional Hessenberg matrix $\mathbf{H}_k \in \mathbb{R}^{k\times k}$ $(k^2 \ll N)$:

$$\mathbf{H}_k = \mathbf{V}_k^T \mathbf{A}^{-1}\mathbf{M}\mathbf{V}_k, \qquad \mathbf{V}_k := (v_1, ..., v_k) \in \mathbb{R}^{n\times k}.$$

- Approximate eigenvalues $\lambda_{h,k}$ are obtained by the k-dimensional eigenvalue problem

$$\mathbf{H}_k \tilde{v}_{h,k} = \lambda_{h,k}^{-1} \tilde{v}_{h,k},$$

  which can be solved by a QR-method in $\mathcal{O}(k^2)$ operations.
- Corresponding eigenvectors are obtained by a modified QR decomposition of the singular system and by setting

$$v_{h,k} := \mathbf{V}_k \tilde{v}_{h,k}.$$

Question: Stopping criterion for the Arnoldi iteration at the level of the discretization error?

Extension of the a posteriori error representation for approximations not satisfying Galerkin orthogonality (for "exact" linearization)

Proposition. *Let* $\{u, \lambda\} \in V \times \mathbb{C}$ *be a primal and* $\{u^*, \lambda^*\} \in V \times \mathbb{C}$ *the corresponding dual eigenpair. Then, for any approximations* $\{\tilde{u}_h, \tilde{\lambda}_h\}$, $\{\tilde{u}_h^*, \tilde{\lambda}_h^*\} \in V_h \times \mathbb{C}$, *with* $\|\tilde{u}_h\| = (\tilde{u}_h, \tilde{u}_h^*) = 1$, *there holds*

$$\lambda - \tilde{\lambda}_h = \underbrace{\tfrac{1}{2}\rho(\{\tilde{u}_h, \tilde{\lambda}_h\})(u^* - \tilde{u}_h^*) + \tfrac{1}{2}\rho^*(\{\tilde{u}_h^*, \tilde{\lambda}_h^*\})(u - \tilde{u}_h)}_{\textbf{eigenvalue residuals}}$$

$$+ \underbrace{\rho(\{\tilde{u}_h, \tilde{\lambda}_h)(\tilde{u}_h^*)}_{\textbf{iteration residual}} + \underbrace{\tfrac{1}{2}(\lambda - \tilde{\lambda}_h) m(u - \tilde{u}_h, u^* - \tilde{u}_h^*)}_{=: \sigma_k \textbf{ remainder}}$$

Separation of discretization and iteration errors

We denote the k-th iterate of the eigenvalue computation by $u_{h,k}$ and the k-th iterate of the dual problem by $u^*_{h,k}$.
Then, $u_{h,k} \to u_h, u^*_{h,k} \to u^*_h$ $(k \to \infty)$, and hence

$$\rho(u_{h,k})(u^* - u^*_{h,k}) \to \rho(u_h)(u^* - u^*_h) \qquad (k \to \infty),$$
$$\rho^*(u_{h,k}, u^*_{h,k})(u - u_{h,k}) \to \rho^*(u_h, u^*_h)(u - u_h) \qquad (k \to \infty).$$

For reasonable iteration number k the differences may be rather large and therefore, using

$$\eta^{\mathrm{it}}_k := \rho(u_{h,k})(u^*_{h,k})$$

as an indicator for the iteration error, requires a careful monitoring of these differences. The resulting error estimator for the full eigenvalue error reads

$$\lambda - \lambda_{\mathbf{h,k}} \approx \frac{\mathbf{1}}{\mathbf{1} - \sigma_{\mathbf{h,k}}} \left\{ \eta_{\mathbf{h,k}} + \eta^*_{\mathbf{h,k}} + \eta^{\mathrm{it}}_{\mathbf{k}} \right\}$$

provided that $|\sigma_{h,k}| \ll 1$.

4.2.4 Computational use of pseudospectra

The next proposition relates the size of the resolvent norm $\|(\mathcal{A}'(\hat{v}) - z\mathcal{I})^{-1}\|$ to easily computable quantities in terms of the eigenvalues and eigenmodes of the operator $\mathcal{A}'(\hat{v})$.

Proposition. *Let $\lambda \in \mathbb{C}$ be a non-deficient eigenvalue of the operator $\mathcal{A}'(\hat{v})$ with corresponding primal and dual eigenvectors $v, v^* \in \mathbf{J}_1$ normalized by $\|v\| = (v, v^*) = 1$. Then, there exists a continuous function*

$$\omega(\epsilon) \to 1 \quad (\epsilon \to 0),$$

such that, for $\lambda_\epsilon := \lambda - \epsilon\omega(\epsilon)\|v^*\|$, there holds

$$\|(\mathcal{A}'(\hat{v}) - \lambda_\epsilon\mathcal{I})^{-1}\| \geq \frac{1}{\epsilon},$$

i.e., λ_ϵ belongs to the ϵ-pseudospectrum of $\mathcal{A}'(\hat{v})$.

Proof. (i) Let $b(\cdot,\cdot)$ be a continuous bilinear form on $\mathbf{J}_0$, such that

$$\sup_{\psi,\varphi\in\mathbf{J_1}} \frac{|b(\psi,\varphi)|}{\|\psi\|\,\|\varphi\|} \leq 1.$$

We consider the perturbed eigenvalue problem, for $\epsilon \in \mathbb{R}_+$,

$$a'(\hat{v}; v_\epsilon, \varphi) + \epsilon b(v_\epsilon, \varphi) = \lambda_\epsilon\,(v_\epsilon, \varphi) \quad \forall \varphi \in \mathbf{J}_1.$$

Since this is a regular perturbation and λ non-deficient, there exist corresponding eigenvalues $\lambda_\epsilon \in \mathbb{C}$ and associated eigenfunctions $v_\epsilon \in \mathbf{J_1}$, $\|\mathbf{v}_\epsilon\| = \mathbf{1}$, such that $|\lambda_\epsilon - \lambda| = \mathcal{O}(\epsilon)$, $\|v_\epsilon - v\| = \mathcal{O}(\epsilon)$. Furthermore, from the relation

$$a'(\hat{v}; v_\epsilon, \varphi) - \lambda_\epsilon(v_\epsilon, \varphi) = -\epsilon b(v_\epsilon, \varphi), \quad \varphi \in \mathbf{J_1},$$

we conclude that

$$\sup_{\varphi\in\mathbf{J_1}} \frac{|a'(\hat{v}; v_\epsilon, \varphi) - \lambda_\epsilon(v_\epsilon, \varphi)|}{\|\varphi\|} \leq |\epsilon| \sup_{\varphi\in\mathbf{J_1}} \frac{|b(v_\epsilon, \varphi)|}{\|\varphi\|} \leq \epsilon\,\|v_\epsilon\|,$$

and from this, if λ_ϵ is not an eigenvalue of $\mathcal{A}'(\hat{v})$,

$$\|(\mathcal{A}'(\hat{v}) - \lambda_\epsilon \mathcal{I})^{-1}\|^{-1} = \inf_{\psi \in \mathbf{J_1}} \sup_{\varphi \in \mathbf{J_1}} \frac{|a'(\hat{v}; \psi, \varphi) - \lambda_\epsilon(\psi, \varphi)|}{\|\psi\| \, \|\varphi\|} \leq \epsilon,$$

which implies the asserted estimate. Next, we analyse the dependence of the eigenvalue λ_ϵ on ϵ in more detail. Subtracting the equation for v form that for v_ϵ, we obtain

$$a'(\hat{v}; v_\epsilon - v, \varphi) + \epsilon b(v_\epsilon, \varphi) = (\lambda_\epsilon - \lambda)(v_\epsilon, \varphi) + \lambda(v_\epsilon - v, \varphi).$$

Taking $\varphi = v^*$ yields

$$a'(\hat{v}; v_\epsilon - v, v^*) + \epsilon b(v_\epsilon, v^*) = (\lambda_\epsilon - \lambda)(v_\epsilon, v^*) + \lambda(v_\epsilon - v, v^*)$$

and, using the equation satisfied by v^*,

$$\epsilon b(v_\epsilon, v^*) = (\lambda_\epsilon - \lambda)(v_\epsilon, v^*).$$

This yields $\lambda_\epsilon = \lambda + \epsilon\omega(\epsilon) b(v, v^*)$, where, by $v_\epsilon \to v$ and $(v, v^*) = 1$,

$$\omega(\epsilon) := \frac{b(v_\epsilon, v^*)}{(v_\epsilon, v^*) b(v, v^*)} \to 1 \quad (\epsilon \to 0).$$

(ii) It remains to construct an appropriate perturbation form $b(\cdot,\cdot)$. For technical convenience, we consider the renormalized dual eigenfunction $\tilde{v}^* := v^*\|v^*\|^{-1}$, satisfying $\|\tilde{v}^*\| = 1$. With the function $w := (v - \tilde{v}^*)\|v - \tilde{v}^*\|^{-1}$, we set

$$Tv := v - 2\mathrm{Re}(v, w)w, \qquad b(v, \varphi) := -(Tv, \varphi).$$

The operator $T : \mathbf{J}_0 \to \mathbf{J}_0$ acts like a Householder transformation mapping v into $\tilde{v}^*$. In fact, observing $\|v\| = \|\tilde{v}^*\| = 1$, there holds

$$\begin{aligned} Tv &= v - \frac{2\mathrm{Re}(v, v - \tilde{v}^*)}{\|v - \tilde{v}^*\|^2}(v - \tilde{v}^*) \\ &= \frac{(2 - 2\mathrm{Re}(v, \tilde{v}^*))v - 2\mathrm{Re}(v, v - \tilde{v}^*)(v - \tilde{v}^*)}{2 - 2\mathrm{Re}(v, \tilde{v}^*)} \\ &= \frac{2v - 2\mathrm{Re}(v, \tilde{v}^*)v - 2v + 2\mathrm{Re}(v, \tilde{v}^*)v + (2 - 2\mathrm{Re}(v, \tilde{v}^*))\tilde{v}^*}{2 - 2\mathrm{Re}(v, \tilde{v}^*)} = \tilde{v}^*. \end{aligned}$$

This implies that

$$b(v, v^*) = -(Tv, v^*) = -(\tilde{v}^*, v^*) = -\|v^*\|.$$

Further, observing $\|w\| = 1$ and

$$\|Tv\|^2 = \|v\|^2 - 2\mathrm{Re}(v,w)(v,w) - 2\mathrm{Re}(v,w)(w,v) + 4\mathrm{Re}(v,w)^2\|w\|^2 = \|v\|^2,$$

we have

$$\sup_{v,\varphi\in\mathbf{J_1}} \frac{|b(v,\varphi)|}{\|v\|\,\|\varphi\|} \leq \sup_{v,\varphi\in\mathbf{J_1}} \frac{\|Tv\|\,\|\varphi\|}{\|v\|\,\|\varphi\|} = 1.$$

Hence, for this particular choice of the form $b(\cdot,\cdot)$, we have

$$\lambda_\epsilon = \lambda - \epsilon\omega(\epsilon)\|v^*\|, \qquad \lim_{\epsilon\to 0}\omega(\epsilon) = 1.$$

This concludes the proof. Q.E.D.

Combining the two propositions, we obtain the following result. Under the assumption that $\mathrm{Re}\,\lambda_\epsilon = \mathrm{Re}\,\lambda - \epsilon \mathrm{Re}\,\omega(\epsilon)\|v^*\| < 0$, there holds

$$\sup_{\mathbf{t}\geq\mathbf{0}} \|\mathbf{S(t)}\| \;\geq\; \frac{|\mathrm{Re}\,\lambda_\epsilon|}{|\epsilon|}.$$

Therefore, for $\epsilon := \mathrm{Re}\lambda > 0$, we have

$$\mathrm{Re}\,\lambda_\epsilon = \mathrm{Re}\,\lambda - \mathrm{Re}\lambda \mathrm{Re}\,\omega(\mathrm{Re}\,\lambda)\|v^*\| = \mathrm{Re}\,\lambda\big(1 - \mathrm{Re}\,\omega(\mathrm{Re}\,\lambda)\|v^*\|\big) < 0,$$

for $\|v^*\|$ sufficiently large. Consequently,

$$\sup_{\mathbf{t}\geq\mathbf{0}} \|\mathbf{S(t)}\| \;\geq\; \frac{|\mathrm{Re}\,\lambda_\epsilon|}{\mathrm{Re}\lambda} \;=\; \big|\mathbf{1} - \mathrm{Re}\,\omega(\mathrm{Re}\,\lambda)\|\mathbf{v}^*\|\big|$$

Conclusion. For small $\mathrm{Re}\,\lambda_h > 0$, a large value

$$\|\mathbf{v}_\mathbf{h}^*\| \gg \mathbf{1}$$

indicates a large growth constant A and consequently possible nonlinear instability.

4.3 Computation of pseudospectra

Remark. Pseudo-spectra are interesting only for non-normal matrices (closed operators in Banach spaces) since the pseudo-spectra of normal operators are the unions of ϵ-circles around the eigenvalues.

Equivalent formulations for the pseudo-spectrum Σ_ϵ:

$$\Sigma_\epsilon = \left\{ z \in \mathbb{C},\ \exists \mathbf{E} \in \mathbb{C}^{n\times n}, \|\mathbf{E}\| \le \epsilon : z \in \Sigma(\mathbf{A}+\mathbf{E}) \right\}$$

$$\Sigma_\epsilon = \left\{ z \in \mathbb{C},\ \Sigma_{\min}(z\mathbf{I}-\mathbf{A}) \le \epsilon \right\}$$

$$\Sigma_{\min}(\mathbf{B}) := \min\left\{ |\lambda|^{1/2},\ \lambda \in \Sigma(\mathbf{B}\mathbf{B}^*) \right\}$$

Algorithmic approaches (not feasible for PDEs):

- Compute norm $\|(z\mathbf{I}-\mathbf{A})^{-1}\|$ for sufficiently many $z \in \mathbb{C}$.
- Apply inverse iteration with shift to $z\mathbf{I}-\mathbf{A}$ for sufficiently many $z \in \mathbb{C}$.
- "Poor-man's pseudo-spectrum": Compute $\Sigma(\mathbf{A}+\mathbf{E})$ for randomly chosen $\mathbf{E}$ with $\|\mathbf{E}\| = \epsilon$.

Conceptional problems in the context of PDEs:

- Definition of pseudo-spectrum may be norm-dependent.
- What is an appropriate norm for stability analysis?
- Natural extension to closed operators $\mathbf{A}$ in Banach space is $\Sigma_\epsilon := \left\{ z \in \mathbb{C},\ \Sigma_{\min}(z\mathbf{I} - \mathbf{A}) \leq \epsilon \right\}$.
- Application to compact (inverse) operators $T := \mathbf{A}^{-1}$.

Convergence of pseudo-spectra under FE discretization:

$$\|T - T_h\|_{\mathcal{L}(L^2)} = \mathcal{O}(h^2), \qquad |\lambda - \lambda_h| = \mathcal{O}(h^2)$$

$$\sigma_{\epsilon - c_1 h^2}(T_h) \subset \sigma_\epsilon(T) \subset \sigma_{\epsilon + c_2 h^2}(T_h)$$

Computation of pseudo-spectra via Arnoldi algorithm ($A_h \approx T_h$):

$$H_k = V_k^T A_h V_k, \qquad \lim_{k \to N} \sigma_\epsilon(H_k) = \sigma_\epsilon(H_N) = \sigma_\epsilon(A_h)$$

Choose k sufficiently large such that all eigenvalues of A_h in the relevant subset $D \subset \mathbb{C}$ are sufficiently well approximated by eigenvalues of H_k.

Parameter choices:

- Mesh width $h \sim 2^{-6} - 2^{-8}$ in the discretization of the eigenvalue problem
- Dimension of Krylov space $k = \dim(H_k) \sim 100 - 200$
- Number of grid points $M \sim 10^2 - 100^2$ in covering the subset $D = [a_r, b_r] \times [a_i, b_i] \subset \mathbb{C}$

Compute pseudo-spectrum of H_k in the subset $D \subset \mathbb{C}$:

$$z \in D \setminus \Sigma(T): \qquad \epsilon(z,T) := \|(zI - T)^{-1}\| = \min\left\{\epsilon > 0,\ z \in \Sigma_\epsilon(T)\right\}$$

$$T_h : V_h \to V_h,\ N(h) := \dim(T_h): \qquad |\epsilon(z,T) - \epsilon(z,T_h)| = \mathcal{O}(h^2)$$

$$H_{h,k} \approx A_h = S_h^{-1} M_h: \qquad |\epsilon(z,T) - \epsilon(z,H_{h,k})| \to 0 \quad (h \to 0, k \to N(h))$$

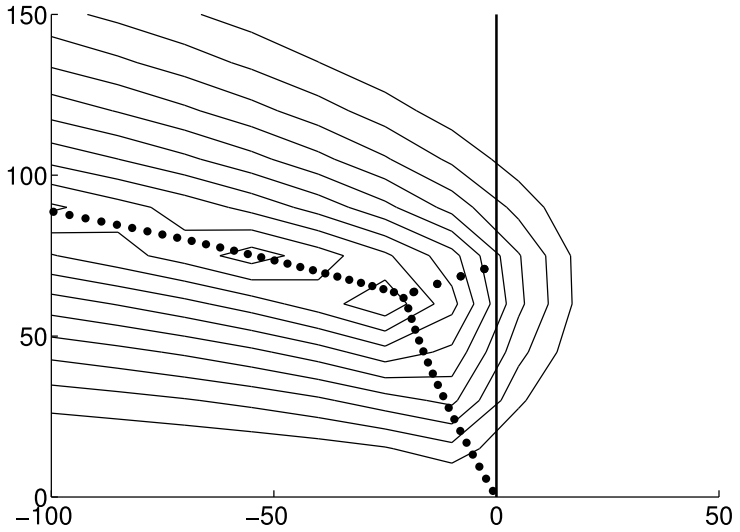
Computation of the singular value decomposition of the Hessenberg matrix $H_{h,k}$ yields approximations to $\Sigma_\epsilon(T)$.

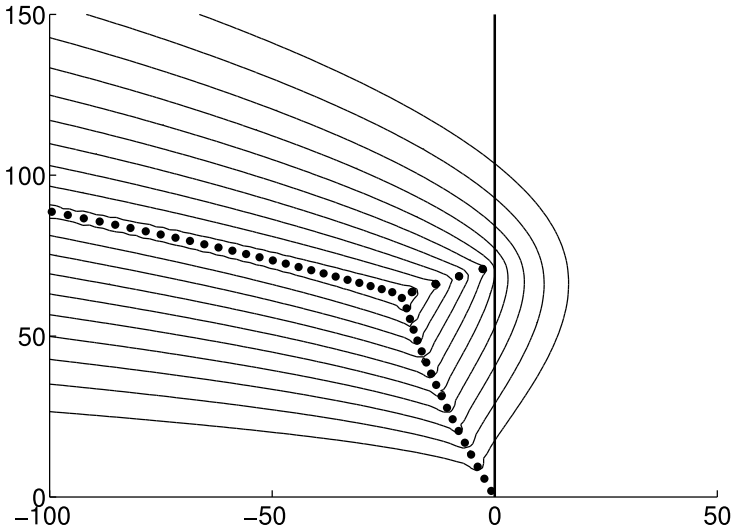
4.4 Computational results

4.4.1 Computation of pseudospectra: accuracy tests

a) Rastering of complex plane

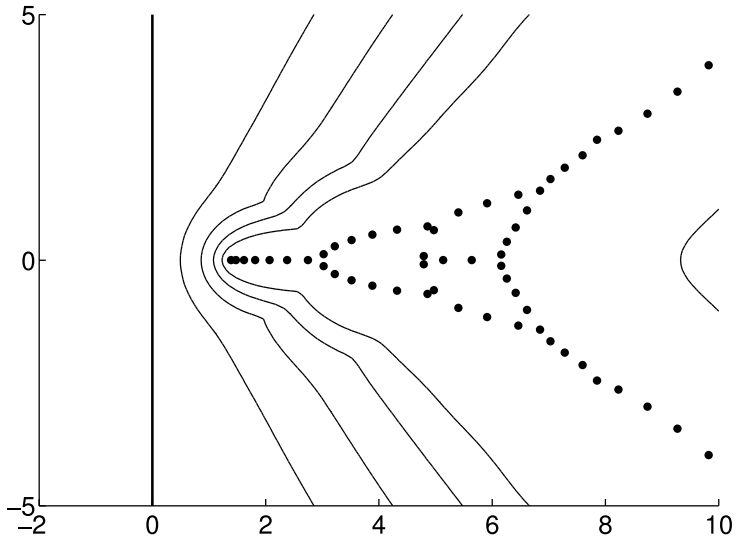
Eigenvalues and pseudospectra computed on a 10×10 **grid (left)** and on a 100×100 **grid (right)**: dots represent eigenvalues and the lines the boundaries of the ϵ-pseudispectra for $\epsilon = 10^{-1}, ..., 10^{-10}$.

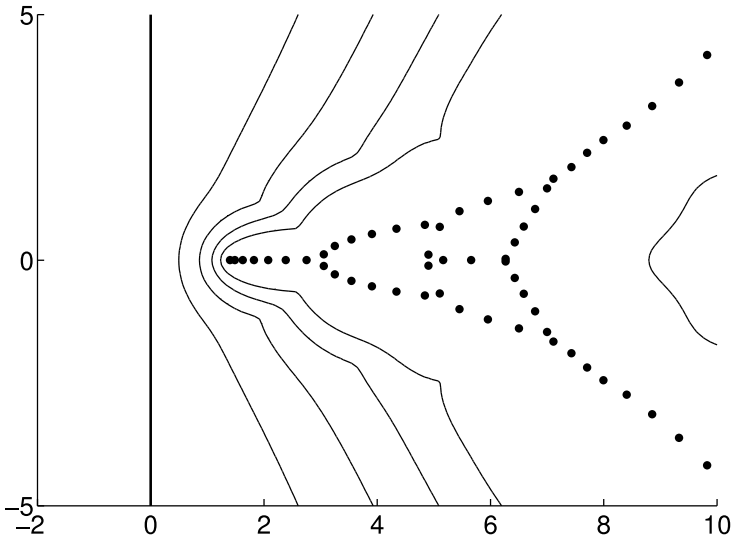




b) Mesh resolution

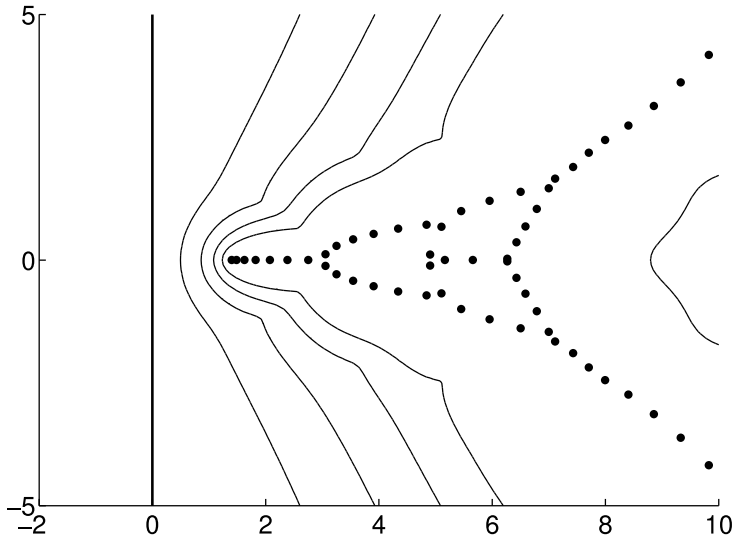
Pseudospectra of the linearized Burgers operator with Dirichlet inflow condition for $\nu = 0.01$ and $h = 2^{-7}$ **(left)** and $h = 2^{-8}$ **(right)** computed by the Arnoldi method with $m = 100$. The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-1}, ..., 10^{-4}$.

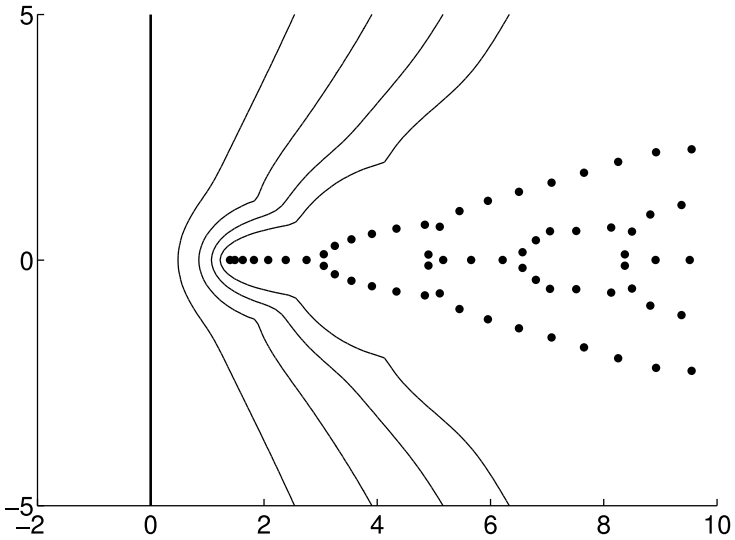




c) Dimension of Krylov space

Pseudospectra of the linearized Burgers operator with Dirichlet inflow condition for $\nu = 0.01$ and $h = 2^{-8}$ computed by the Arnoldi method with $m = 100$ **(left)** and $m = 200$ **(right)**. The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-1}, ..., 10^{-4}$.

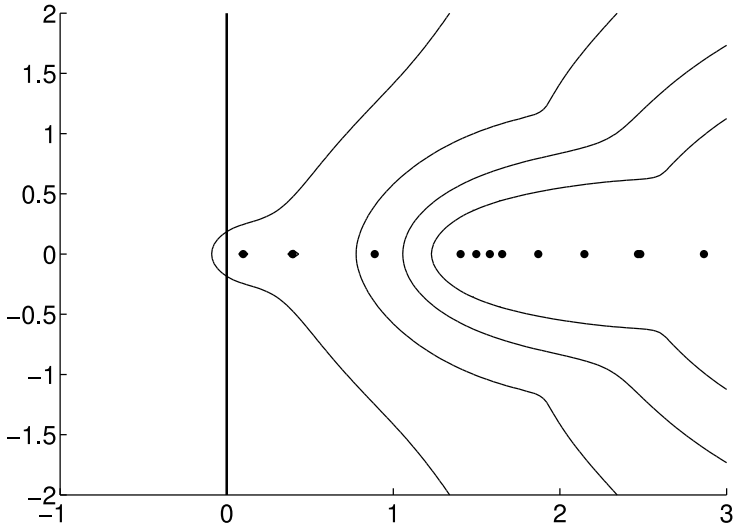


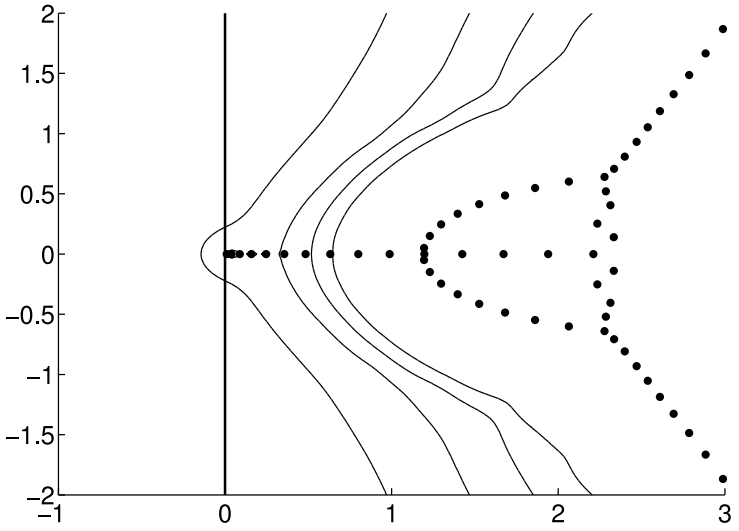


4.4.2 Burgers versus Navier-Stokes pseudospectra

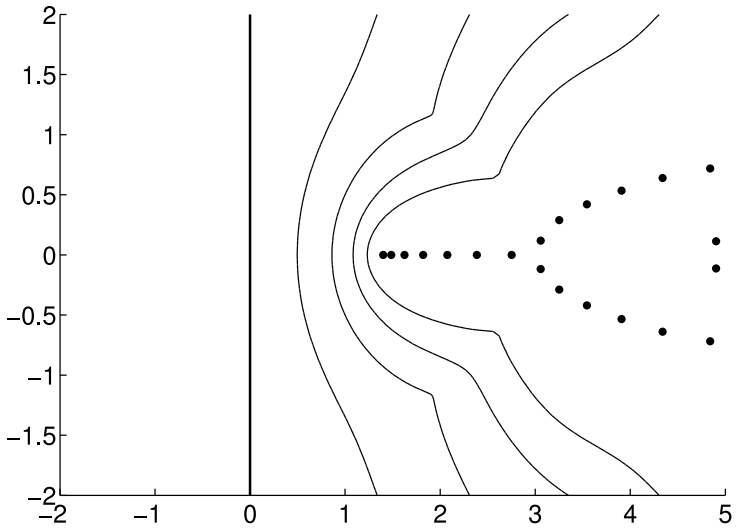
a) Couette flow

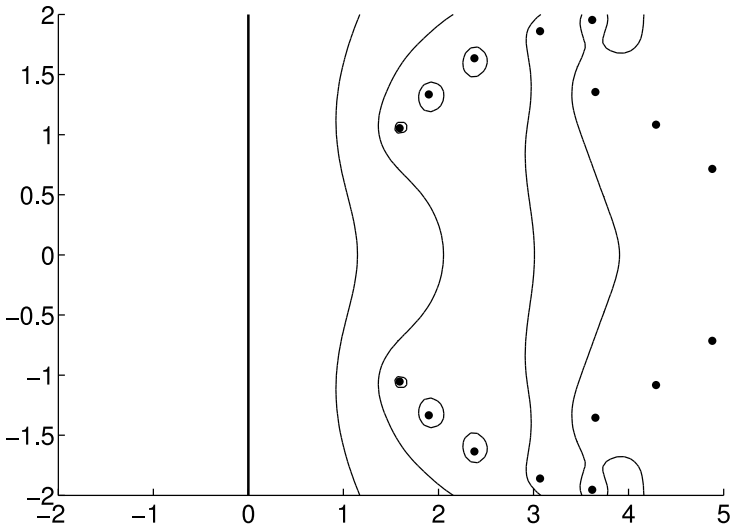
Pseudospectrum of the linearized (about Couette flow) Burger operator with Neumann inflow conditions for viscosities $\nu = 0.01$ **(left)** and $\nu = 0.001$ **(right)**: The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-1}, \ldots, 10^{-4}$.



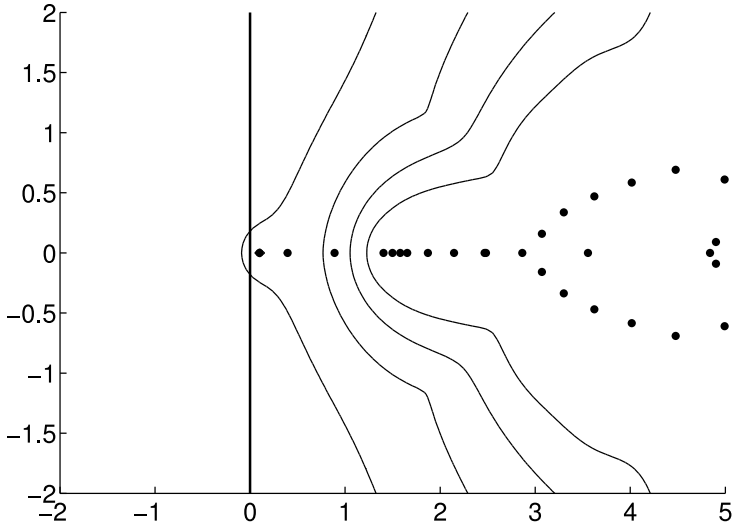


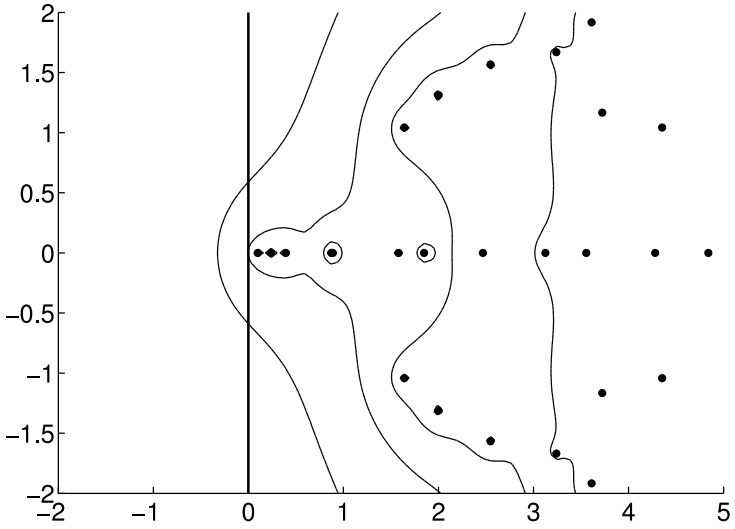
Pseudospectra of the linearized (about Couette flow) **Burgers (left)** and **Navier-Stokes operator (right)** for $\nu = 0.01$ and Dirichlet inflow condition: The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-1}, \ldots, 10^{-4}$.





Pseudospectra of the linearized (about Couette flow) **Burgers (left)** and **Navier-Stokes operator (right)** with Neumann inflow conditions for $\nu = 0.01$: The dots repreent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-1}, \ldots, 10^{-4}$.

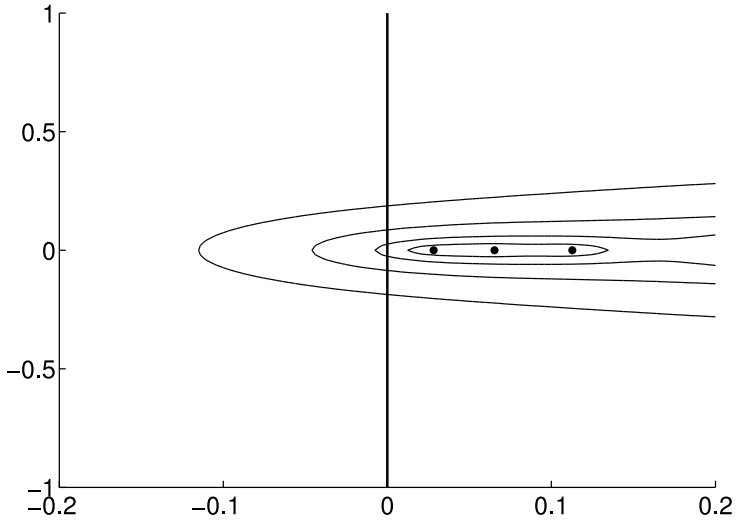


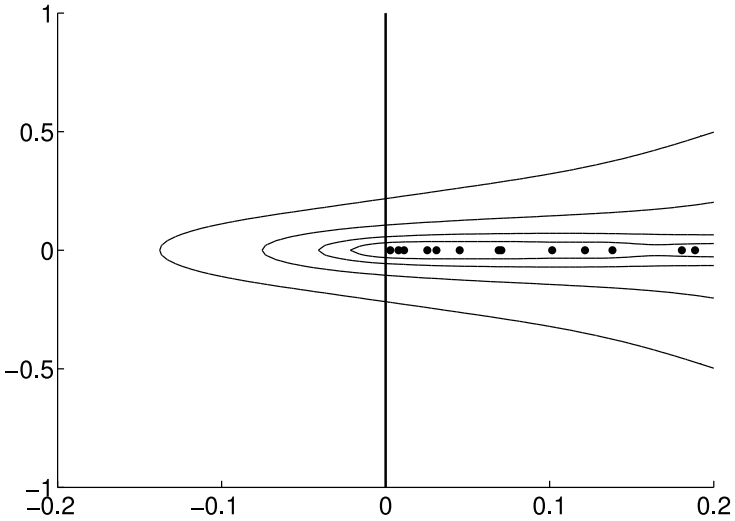


4.4.3 Pseudospectra of Navier-Stokes operator

a) Navier-Stokes Couette flow

Pseudospectra of the linearized (about Couette flow) Navier-Stokes operator with Neumann inflow conditions for $\mathrm{Re} = 350$ **(left)** and $\mathrm{Re} = 3500$ **(right)**: The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}$.





Development of the "test quantity" $|(v_h, v_h^*)|$ under mesh refinement for the critical eigenvalues of the linearized (about Couette flow) Navier-Stokes operator with Neumann inflow conditions for different Reynolds numbers $\mathrm{Re} = 100, 350, 500, 1000, 3500, 5000, 10000$.

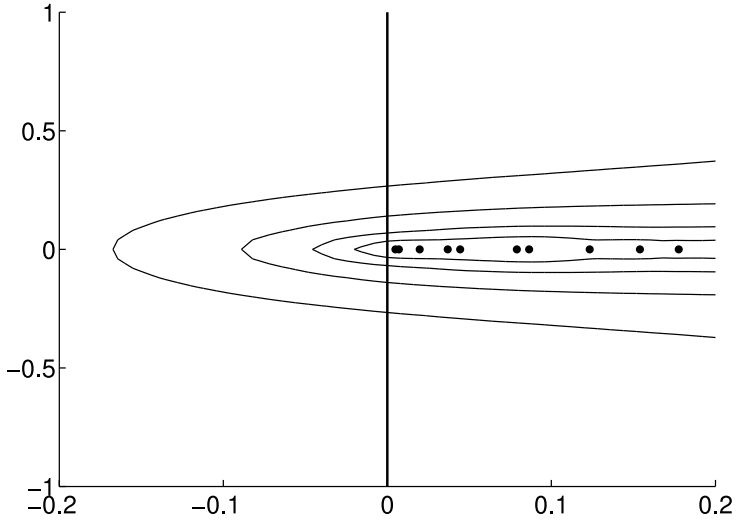
Re	100	**350**	500	1000	**3500**	5000	10000
h	$\lvert(v_h, v_h^*)\rvert$						
2^{-6}	0.0214	0.0027	0.0016	0.00075	0.00040	0.00040	0.000488
2^{-7}	0.0209	0.0024	0.0014	0.00052	0.00017	0.00014	0.000116
2^{-8}	0.0207	0.0023	0.0013	0.00045	0.00009	0.00007	0.000047
λ_{crit}	0.0987	0.0282	0.0197	0.00987	0.00282	0.00197	0.000987

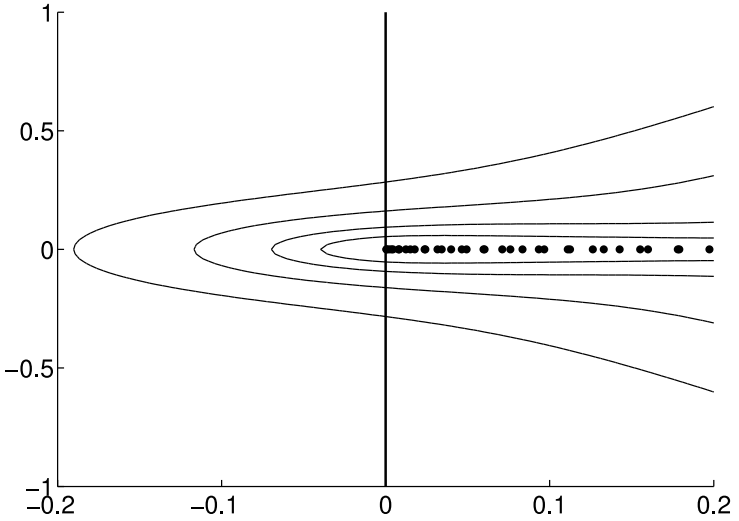
b) Navier-Stokes Poiseuille flow

Eigenvalue with smallest real part of the linearized (about Poiseuille flow) Navier-Stokes operator with Neumann inflow conditions for different Reynolds numbers.

$h \setminus \mathrm{Re}$	1000	2000	4000	6000	10000
2^{-6}	0.004936	0.002468	0.001234	0.0008226	0.0003857
2^{-7}	0.004935	0.002468	0.001234	0.0008225	0.0002225
2^{-8}	0.004935	0.002467	0.001234	0.0008225	0.0003843

Pseudospectra of the linearized (about Poiseuille flow) Navier-Stokes operator with Neumann inflow conditions for $\mathrm{Re} = 1000$ **(left)** and $\mathrm{Re} = 10000$ **(right)** computed by the Taylor-Hood element with GLS transport stabilization: The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}$.





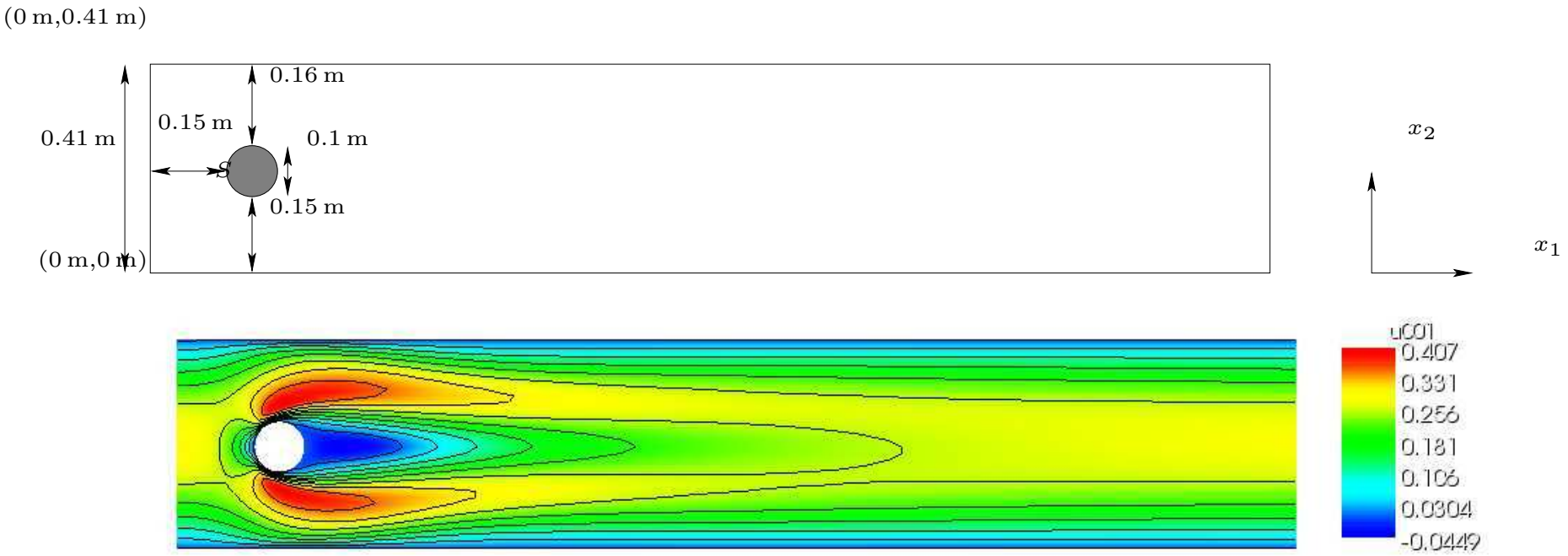
Development of the "test quantity" $|(v_h, v_h^*)|$ under mesh refinement for the critical eigenvalues of the linearized (about Poiseuille flow) Navier-Stokes operator with Neumann inflow conditions for different Reynolds numbers $\mathrm{Re} = 100, 350, 500, 1000, 3500, 5000, 10000$.

Re	100	350	500	**1000**	3500	5000	**10000**
h	$\lvert(v_h, v_h^*)\rvert$						
2^{-6}	0.0140	0.0024	0.0017	0.00116	0.00118	0.00130	0.001459
2^{-7}	0.0132	0.0016	0.0010	0.00054	0.00031	0.00029	0.000331
2^{-8}	0.0130	0.0015	0.0008	0.00033	0.00013	0.00011	0.000095
$\lambda_h^{\mathrm{crit}}$	0.0494	0.0141	0.0099	0.00494	0.00141	0.00099	0.000384

4.4.4 Pseudospectra in the channel flow benchmark

a) Configuration

Configuration of the "channel flow" benchmark and and x_1-component of the velocity for $\mathrm{Re} = 40$.

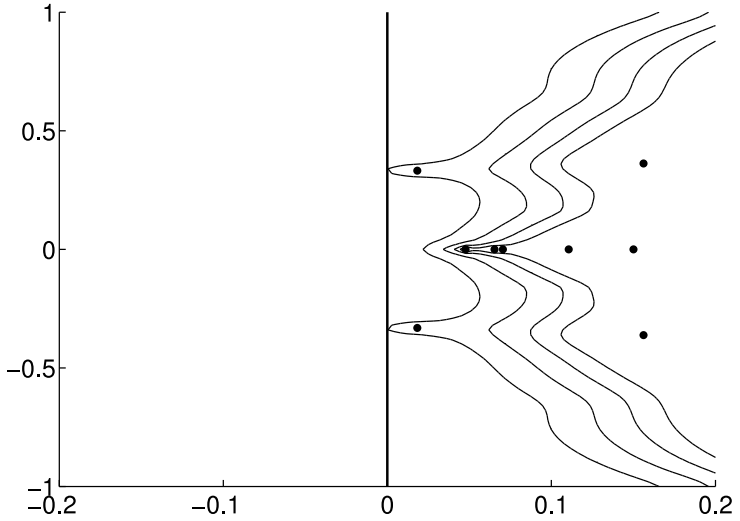


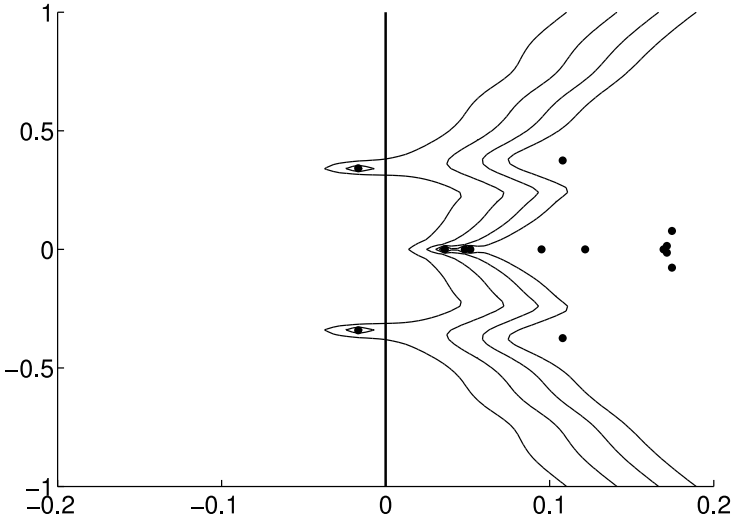
b) Eigenvalues and pseudospectra

Computed eigenvalue with smallest real part of the linearized ("channel flow") Navier-Stokes operator for Dirichlet inflow conditions and different Reynolds numbers $\mathrm{Re} = 20, 40, 45, 50, 60$ and increasing refinement level.

Re	20	40	45	50	60
4	0.062	0.0200±0.33i	0.0097±0.33	0.0011±0.33	-0.0125±0.33i
5	0.062	0.0187±0.33i	0.0080±0.33	-0.0010±0.34	-0.0157±0.34i
6	0.062	0.0186±0.33i	0.0076±0.33	-0.0016±0.34	-0.0165±0.34i
ref.	0.062	0.0185±0.33i	0.0075±0.33	-0.0018±0.34	-0.0165±0.34i

Pseudospectra of the linearized Navier-Stokes operator ("flow channel" benchmark) for different Reynolds numbers, $\mathrm{Re} = 40$ **(left)** and $\mathrm{Re} = 60$ **(right)**, with Dirichlet inflow conditions: Thedots represent eigenvalues and the lines the boundaries of ϵ-pseudospectra for $\epsilon = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}$.





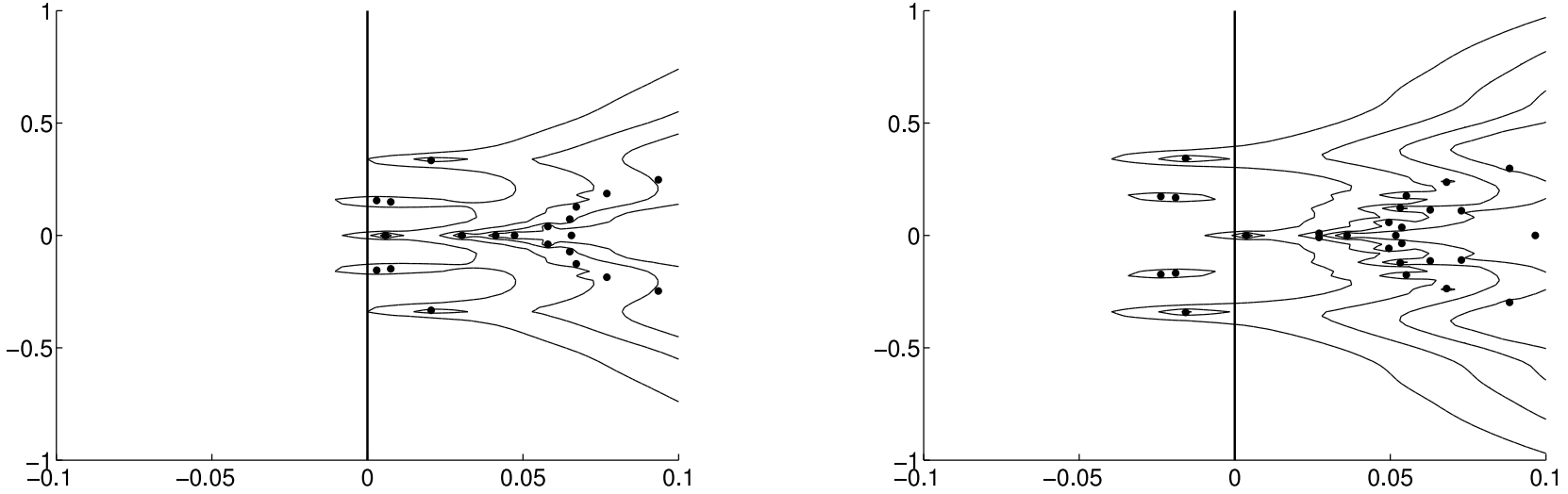
Development of the "test quantity" $|(v_h, v_h^*)|$ under mesh refinement for the critical eigenvalues of the linearized (about the channel flow) Navier-Stokes operator with Dirichlet inflow conditions for different Reynolds numbers $\mathrm{Re} = 20, 40, 45, 50, 60$ and increasing level of refinement.

Re	20	40	45	50	60
level	$\lvert(v_h, v_h^*)\rvert$				
4	0.032	0.086	0.092	0.096	0.099
5	0.032	0.078	0.084	0.088	0.092
6	0.032	0.076	0.082	0.085	0.089
$\mathrm{Re}\lambda_h^{\mathrm{crit}}$	0.0624	0.0186	0.0076	-0.00106	-0.0165

Eigenvalues with smallest real part of the linearized ("channel flow") Navier-Stokes operator with Neumann inflow conditions for $\mathrm{Re} = 20, 40, 50, 60$ and increasing level of refinement.

Re	20	40	45	50	60
4	0.0150	0.005532	0.0002±0.16	-0.0054±0.16	-0.0140±0.17i
5	0.0152	0.004±0.16i	-0.0043±0.16	-0.0110±0.16	-0.0217±0.17i
6	0.0153	0.003±0.16i	-0.0049±0.16	-0.0119±0.16	-0.0232±0.17i
ref.	0.0154	0.003±0.16i	-0.0052±0.16	-0.0120±0.16	-0.0240±0.17i

Pseudospectra of the linearized Navier-Stokes operator ("channel flow") with Neumann inflow conditions for different Reynolds numbers, $\mathrm{Re} = 40$ (left) and $\mathrm{Re} = 60$ (right): The dots represent eigenvalues and the lines the boundaries of the ϵ-pseudospectra for $\epsilon = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}$.



Development of the "test quantity" $|(v_h, v_h^*)|$ under mesh refinement for the critical eigenvalues of the linearized (about the channel flow) Navier-Stokes operator with Neumann inflow conditions for different Reynolds numbers $\mathrm{Re} = 20, 40, 45, 50, 60$ and increasing level of refinement.

Re	20	40	45	50	60
level	(v_h, v_h^*)				
4	0.063	0.033	0.036	0.034	0.031
5	0.064	0.040	0.037	0.037	0.033
6	0.065	0.041	0.040	0.039	0.035
$\mathrm{Re}\lambda_h^{\mathrm{crit}}$	0.015	0.003	-0.005	-0.012	-0.024

Summary

Our computational results lead us to the following conclusions:

- The computed pseudospectra turn out to be reliable for moderately refined meshes with $h \approx 2^{-7} - 2^{-8}$ and dimensions $m = 100 - 200$ of Krylov spaces in the Arnoldi method.
- The computed pseudospectra are not very sensitive with respect to the stabilization of pressure and transport used in the finite element discretizaton. However, for reaching the same accuray using GLS requires about one refinement level less than LPS.
- The "deficiency test" $\limsup_{h \to 0} |(v_h, v_h^*)| \ll 1$ can be used for predicting the presence of a critical pseudospectrum.
- Generally, the base flows considered are much less stable with respect to perturbations satisfying Dirichlet inflow conditions than Neumann ("free") inflow conditions.
- The linearized Burgers operator has significantly different stability properties than the linearized Navier-Stokes operator.

End of Lecture Series