Many people would sooner die than think – in fact they do.

Bertrand Russell

..... thinking many things with the clearest conviction to my greatest satisfaction, which I would never have courage to say.

Immanuel Kant

Nor should it be considered rash not to be satisfied with those opinions that have become common. No one should be scorned in physical disputes for not holding to opinions which happens to please other people best

Galileo, Galilei

Gibbs Potential Approach: (G)

- Suppose the Helmholtz potential is a convex function of the logarithmic strain, then the Helmholtz potential and Gibbs potential are related by a Legendre transformation:

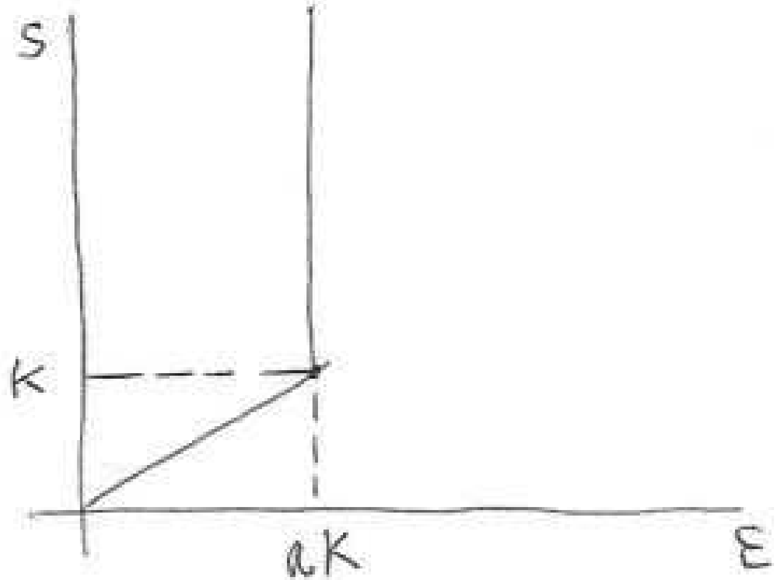
$$\underset{\sim}{T} = \rho \frac{\partial \psi}{\partial (\ln V)} \implies \underset{\sim}{S} = \frac{\partial \psi}{\partial (\ln \underset{\sim}{V})} .$$

Then

$$\psi = G - \frac{\partial G}{\partial \underset{\sim}{S}} \cdot \underset{\sim}{S} \quad , \quad \ln \underset{\sim}{V} = \frac{\partial G}{\partial \underset{\sim}{S}} .$$

- However, there are bodies for which the Helmholtz potential or Gibbs potential can even be defined.

- Cannot define Helmholtz potential corresponding to



- We can however define a Gibbs potential through

$$G = \frac{a}{2}\left\{ (s^2 + k^2 - 2sk) \langle s-k \rangle^{0} - s^2 \right\}$$

where

$$\langle x \rangle^{n} = \left(\max(x, 0) \right)^{n}$$

$$\varepsilon = -\frac{\partial G}{\partial s} = \begin{cases} s & x < k \\ k & x \geq k \end{cases}$$

- We shall now consider the Gibbs potential Approach.

$$G = G(\underset{\sim}{s}, \theta) .$$

- $G(\underset{\sim}{S}, \theta) = G(\underset{\sim}{Q}\underset{\sim}{S}\underset{\sim}{Q}^T, \theta) \quad \forall \; \underset{\sim}{Q} \in \mathcal{O}$

$$\Rightarrow \quad G = G(\hat{I}_1, \hat{I}_2, \hat{I}_3, \theta),$$

$$\hat{I}_1 = \operatorname{tr} \underset{\sim}{S}, \quad \hat{I}_2 = \frac{1}{2}(\operatorname{tr} \underset{\sim}{S})^2, \quad \hat{I}_3 = \frac{1}{3}(\operatorname{tr} \underset{\sim}{S}^3).$$

$$\psi = G - \frac{\partial G}{\partial \underset{\sim}{S}} \cdot \underset{\sim}{S},$$

$$\varepsilon = G - \frac{\partial G}{\partial \underset{\sim}{S}} \cdot \underset{\sim}{S} - \theta \frac{\partial G}{\partial \theta},$$

$$\eta = -\frac{\partial G}{\partial \theta}.$$

- Energy equation reduces to

$$-\rho\theta \frac{d}{dt}\left[\frac{\partial G}{\partial \theta}\right] = \rho r - \operatorname{div} \underset{\sim}{q} + \rho\xi ,$$

$$\xi = \underset{\sim}{T} \cdot \underset{\sim}{D} + \underset{\sim}{S} \cdot \left\{ \frac{\partial^2 G}{\partial \underset{\sim}{S}^2} [\dot{\underset{\sim}{S}}] \right\} .$$

- It can then be shown that

$$\xi = \underset{\sim}{S} \cdot \left(\underset{\sim}{D} - \underset{\sim}{\mathbb{C}} \dot{\underset{\sim}{S}} \right)$$

$$\underset{\sim}{\mathbb{C}} = -\frac{\partial^2 G}{\partial \underset{\sim}{S}^2} = \sum_{i,j=1}^{3} \left\{ G_{,ij}\, \underset{\sim}{S}^{i-1} \otimes \underset{\sim}{S}^{j-1} \right\} - G_{,2}\, \underset{\sim}{\mathbb{I}} + G_{,3}\, \underset{\sim}{\mathbb{A}}$$

$$G_{,i} = \frac{\partial G}{\partial \hat{I}_i},$$

$$(\underset{\sim}{\mathbb{I}})_{ijkl} = \delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl},$$

$$(\underset{\sim}{\mathbb{A}})_{ijkl} = \delta_{ik}\,S_{jl} + \delta_{ij}\,S_{kl}.$$

- $\underset{\sim}{D}$ and $\overset{\circ}{\underset{\sim}{\mathbb{C}}}$ are frame-indifferent

$\dot{\underset{\sim}{S}}$ is not frame-indifferent.

However $\underset{\sim}{S} \cdot \underset{\sim}{\mathbb{C}}\,\dot{\underset{\sim}{S}}$ is frame-indifferent.

- It can be shown that

$$\underset{\sim}{S} \cdot \underset{\sim}{\mathbb{C}} \left[\underset{\sim}{S}\underset{\sim}{A} - \underset{\sim}{A}\underset{\sim}{S} \right] = 0$$

for any tensor $\underset{\sim}{A}$.

- So

$$\xi = \underset{\sim}{S} \cdot \left\{ \underset{\sim}{D} - \underset{\sim}{\mathbb{C}} \left[\dot{\underset{\sim}{S}} - (\underset{\sim}{A}\underset{\sim}{S} - \underset{\sim}{S}\underset{\sim}{A}) \right] \right\}.$$

- Pick

$$\underset{\sim}{A} = \underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{L} - \underset{\sim}{L}^T).$$

$$\xi = \underset{\sim}{S} \cdot \left\{ \underset{\sim}{D} - \underset{\sim}{\mathbb{C}} \overset{\nabla}{\underset{\sim}{S}} \right\}$$

$$\overset{\nabla}{\mathbf{S}} := \dot{\mathbf{S}} - (\mathbf{W}\mathbf{S} - \mathbf{S}\mathbf{W})$$

∇ – Jaumann Derivative.

- Let

$$\mathbf{D}_p := \mathbf{S} \cdot \{\mathbf{D} - \mathbb{C}\overset{\nabla}{\mathbf{S}}\}.$$

- Then

$$\xi = \mathbf{S} \cdot \mathbf{D}_p \geq 0.$$

- We need to pick constitutive equations for $\underset{\sim}{D}_p$.

Suppose we pick constitutive equations of the form

$$\underset{\sim}{D}_p = \underset{\sim}{f}(\underset{\sim}{S}, \underset{\sim}{D}).$$

These constitutive equations are thermodynamically admissible.

- Suppose the body is incompressible; i.e.,

$$\operatorname{tr}\underset{\sim}{D} = 0.$$

Then we need to pick

$$G = G(\underset{\sim}{\tau}, \theta)$$

$\underset{\sim}{\tau}$ is the deviatoric part of $\underset{\sim}{S}$.

- $$\xi = \underset{\sim}{\tau} \cdot \underset{\sim}{D} - \underset{\sim}{\tau} \cdot \left(\underset{\sim}{\mathbb{C}}_1 [\overset{\circ}{\underset{\sim}{\tau}}] \right).$$
- Let $J_1 = \operatorname{tr} \underset{\sim}{\tau}$, $J_2 = \frac{1}{2}(\operatorname{tr} \underset{\sim}{\tau}^2)$, $J_3 = \frac{1}{3}(\operatorname{tr} \underset{\sim}{\tau}^3)$.

$$\underset{\sim}{\mathbb{C}}_1 = -\frac{\partial^2 G}{\partial \underset{\sim}{\tau}^2}.$$

- $$\underset{\sim}{\mathbb{C}}_1 = -\Big\{ G_{1,22}\, \underset{\sim}{\tau} \otimes \underset{\sim}{\tau} + G_{1,23}\left[\underset{\sim}{\tau} \otimes \underset{\sim}{\tau}^2_{dev} + \underset{\sim}{\tau}^2_{dev} \otimes \underset{\sim}{\tau} \right] G_{1,33}\left(\underset{\sim}{\tau}^2_{dev} \otimes \underset{\sim}{\tau}^2_{dev} \right)$$

$$+ G_{1,2}\, \hat{\underset{\sim}{\mathbb{I}}} + G_{1,3}\, \hat{\underset{\sim}{\mathbb{A}}} \quad ,$$

$$\left(\hat{\underset{\sim}{\mathbb{I}}}\right)_{ijkl} = \left(\underset{\sim}{\mathbb{I}}\right)_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

$$\left(\hat{\underset{\sim}{\mathbb{A}}}\right)_{ijkl} = \tau_{lj}\delta_{ik} - 2\left(\hat{\underset{\sim}{\mathbb{I}}}\right)_{ijkl} - \frac{1}{3}\tau_{kl}\delta_{ij}$$

$\underset{\sim}{\tau}^2_{dev}$ is the deviatoric part of $\underset{\sim}{\tau}^2$.

- Then

$$\xi = \underset{\sim}{\tau} \cdot \left(\underset{\sim}{D} - \underset{\sim}{\mathbb{C}}_1\left[\overset{\nabla}{\underset{\sim}{\tau}}\right] \right)$$

- Define

$$\mathbf{D}_p = \mathbf{D} - \mathbb{C}_1[\overset{\nabla}{\boldsymbol{\tau}}] = \mathbf{f}(\boldsymbol{\tau}, \mathbf{D}).$$

$$\boldsymbol{\tau} \cdot \mathbf{D}_p \geq 0.$$

- EXAMPLES.

Suppose $G_1 = -aJ_2$, $\mathbf{f} = b\boldsymbol{\tau} + c\boldsymbol{\tau}^2$.

Then the constitutive equation reduces to

$$\mathbf{D} = a\overset{\nabla}{\boldsymbol{\tau}} + b\boldsymbol{\tau} + c\boldsymbol{\tau}^2$$ — GIESEKUS MODEL.

- Suppose

$$\underset{\sim}{f} = \tau Y(p)$$

p – mean normal stress

The , the constitutive equation reduces to

$$\underset{\sim}{D} = a\overset{\nabla}{\underset{\sim}{\tau}} + Y(p)\tau$$ – PHAN-THIEN & TANNER Model.

- Suppose

$$\underset{\sim}{f} = \frac{\underset{\sim}{\tau}}{Y(I_2(\underset{\sim}{D}))}$$

In this case the constitutive model reduces to

$$\underset{\sim}{D} = a \overset{\nabla}{\underset{\sim}{\tau}} + \frac{\underset{\sim}{\tau}}{Y(I_2(\underset{\sim}{D}))}$$

$$\Rightarrow [Y(I_2(\underset{\sim}{D}))]\underset{\sim}{D} = a Y_2(I_2(\underset{\sim}{D})) \overset{\nabla}{\underset{\sim}{\tau}} + \underset{\sim}{\tau}$$

METZNER - DENN - WHITE Model.