“A role still more important than generalizations in dealing with mathematical problems is played, I believe, by specialization. Perhaps in most cases we seek in vain for the answer to a question the failure lies in our having not yet completely solved the problems simpler and easier than the one on hand. Everything depends then on finding the easier problems and solving them by the use of tools as perfect as possible and of concepts susceptible to generalization. This role is one of the most important levers for overcoming mathematical difficulties . . .”

D. Hilbert.

- Natural Configuration
  - Can think of it as a stress-free configuration
  - It is really an equivalence class of configurations.
- Eg: Classical Plasticity

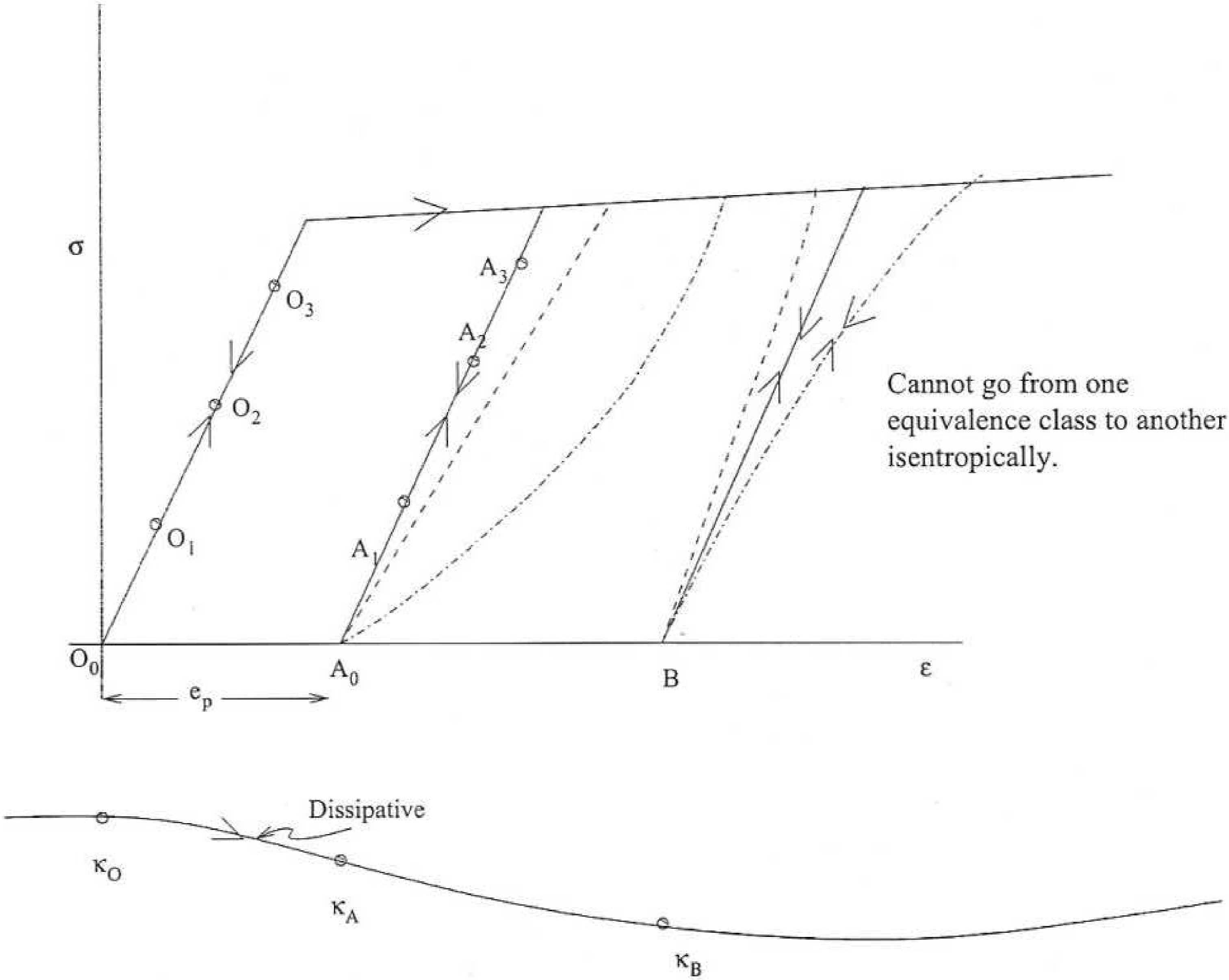


Figure 2: Traditional Plasticity

- Classical metal plasticity involves an infinity of natural configurations, and to determine the stress we require kinematical information from more than one natural configuration.
- The response is elastic from each of these natural configurations and the inelasticity is purely due to the change in the natural configurations.
- Plasticity concerns a “class” of simple materials.

- Eg. 2 Twinning
- In twinning there are a finite number. As many as the number of variants.

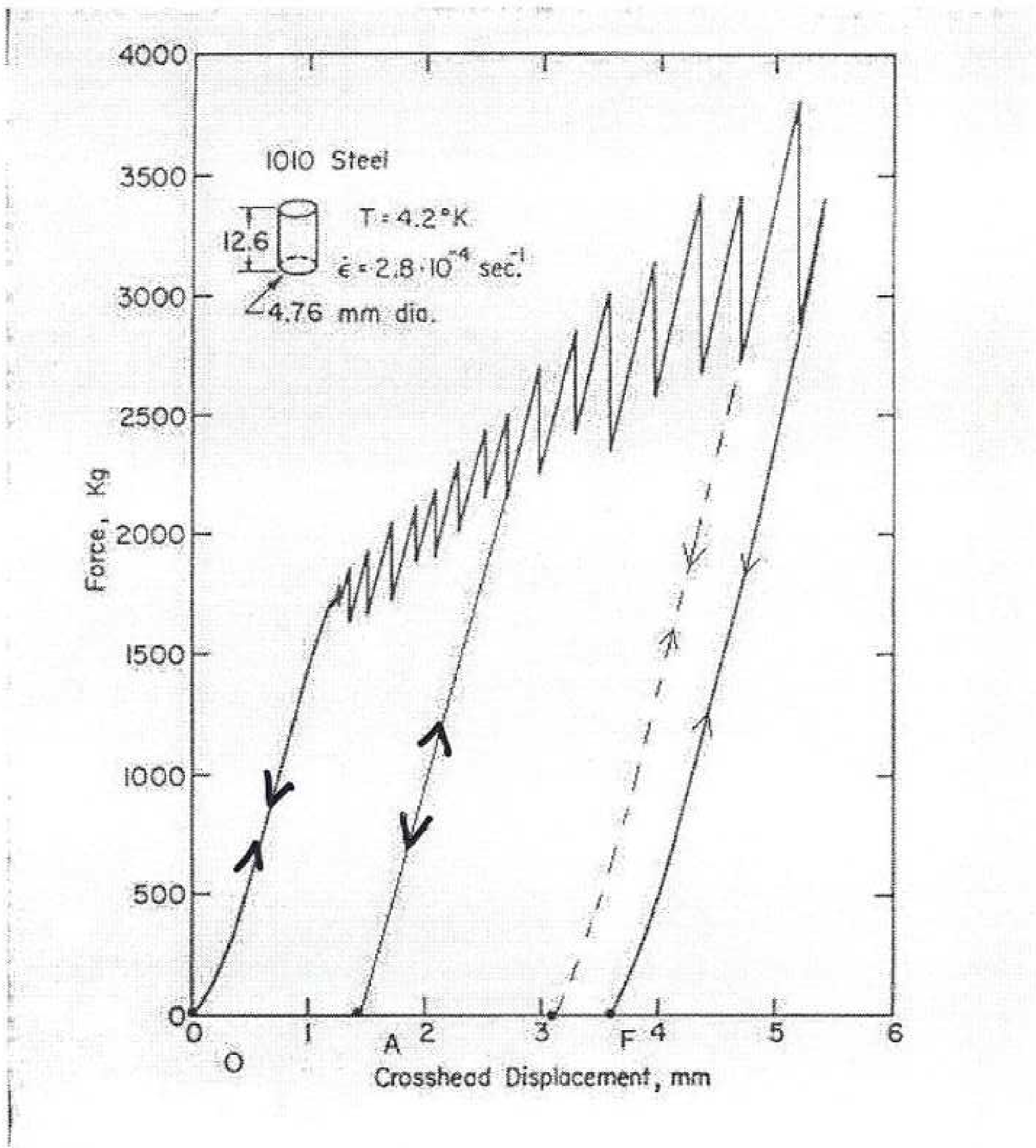
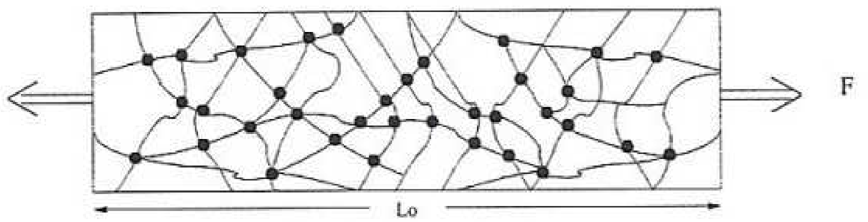
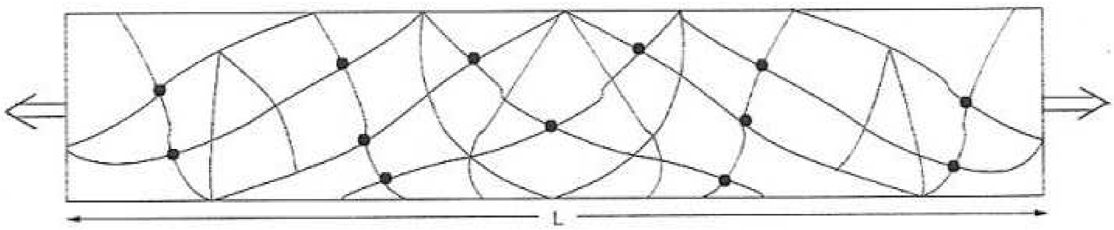


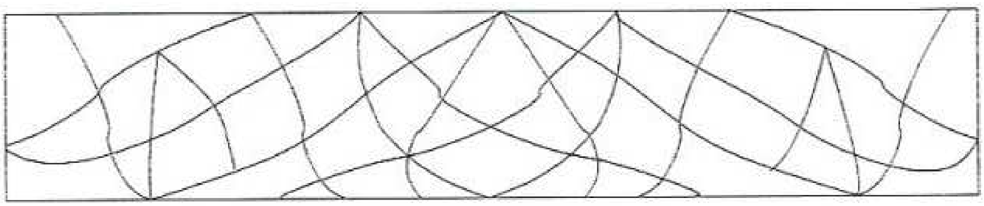
Figure 3: Modulo variants, we have two natural configurations, that corresponding to O and F, and these two natural configurations have different material symmetries.

Armstrong & Co.

MULTI-NETWORK POLYMERS







Other Examples:

- Viscoelasticity
- Superplasticity
- Solid to solid phase transitions
- Crystallization of polymers
- Viscoplasticity

Classical theories are trivial examples:

In classical elasticity the natural configuration does not evolve.

In classical fluids the current configuration is the natural configuration.

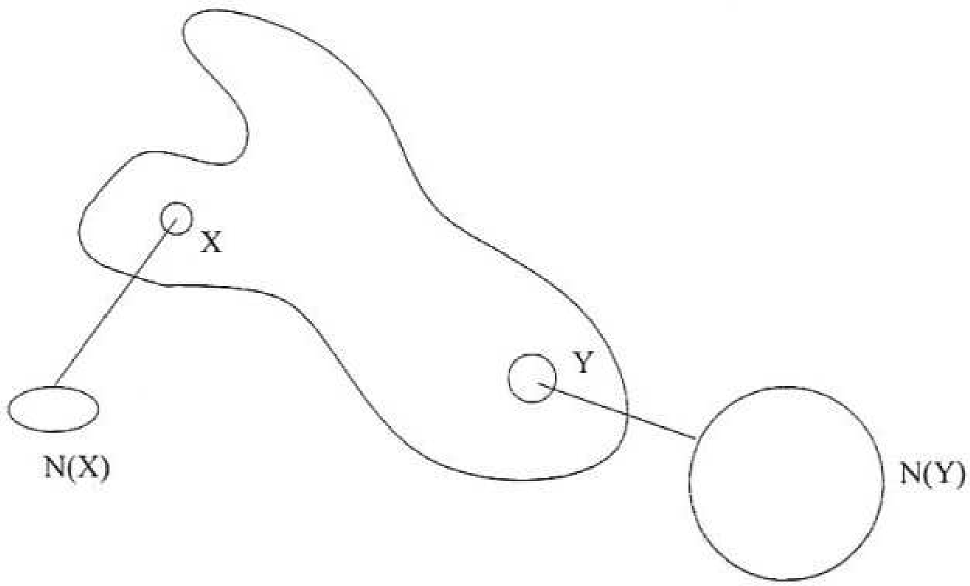


Figure 4: Configuration as a local notion

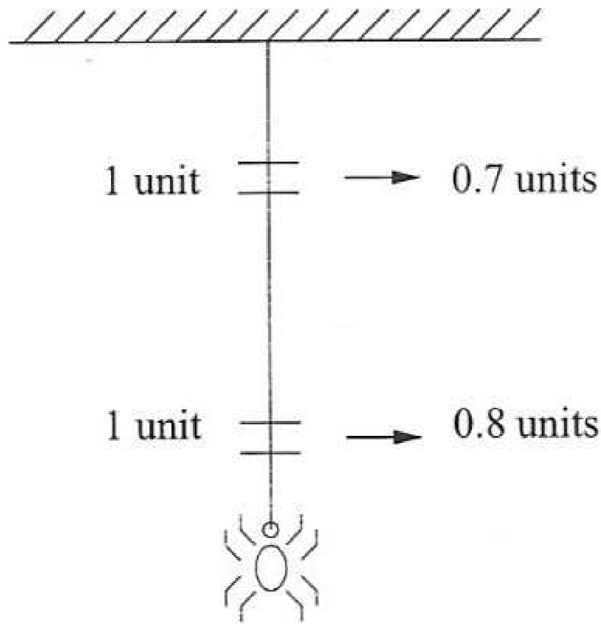


Figure 5: Spider spinning a web

New material is laid in a stressed state. It can have a different natural configuration than the material laid down previously.

- Restrict ourselves to homogeneous deformation.
- Think in terms of Global configurations.

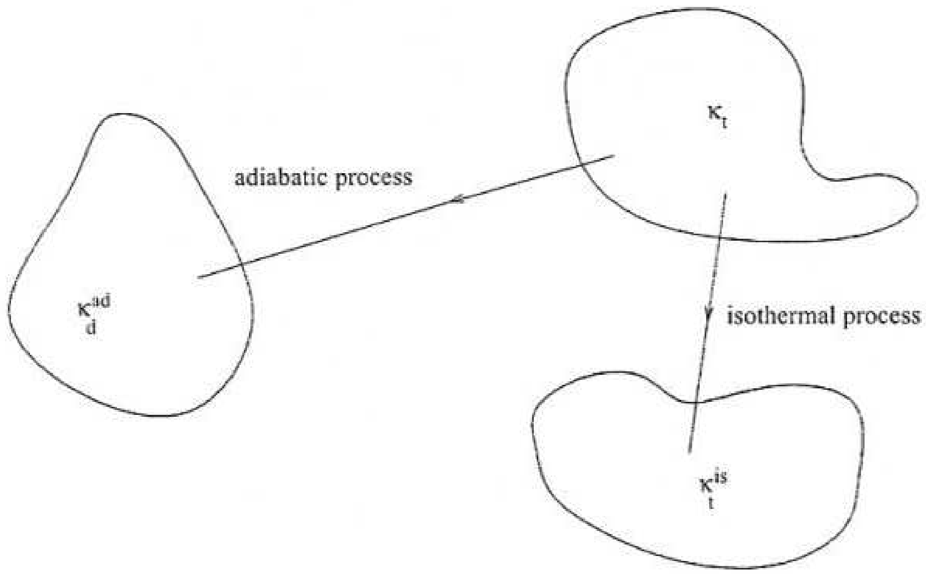


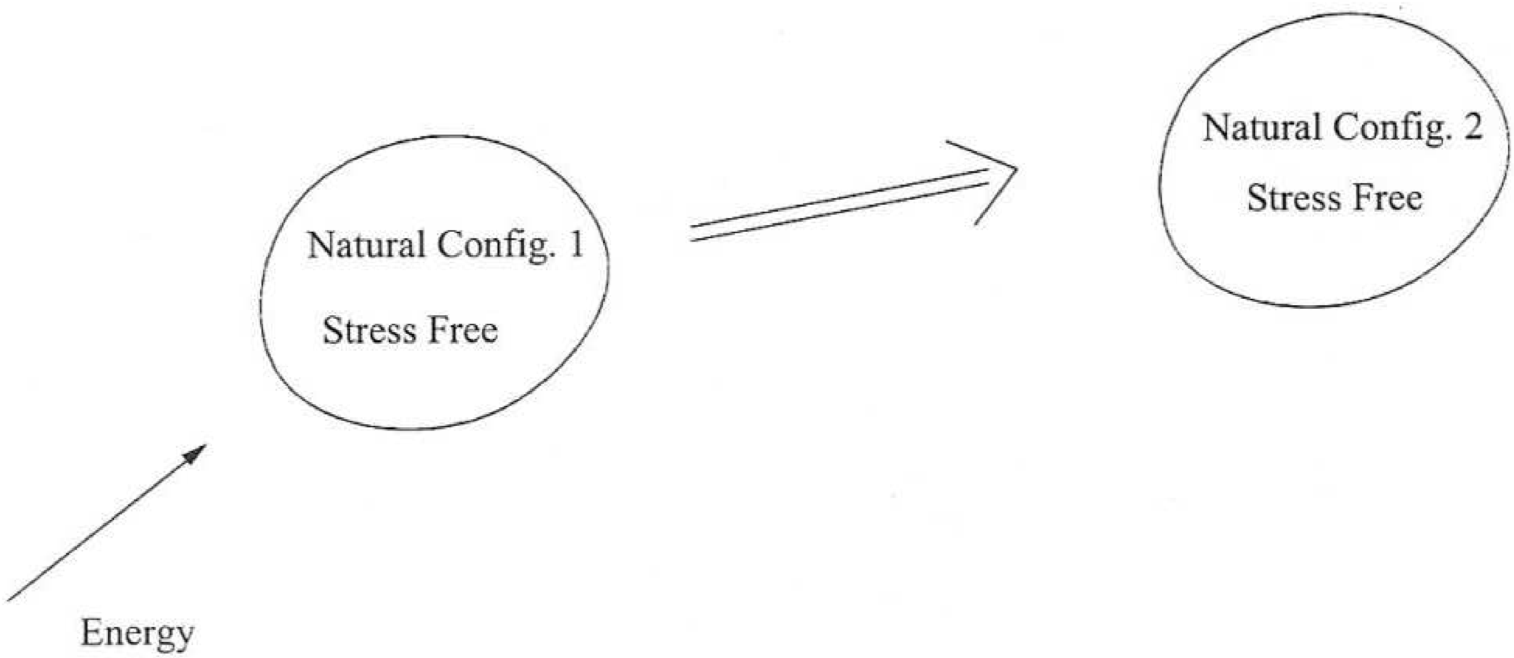
Figure 6: Non-uniqueness of stress-free state (Modulo rigid motion)

- More than one Natural Configuration can be associated with the current deformed configuration .
- Example: Consider a Viscoelastic body capable of instantaneous elastic response κ_t.
  - Natural Configuration reached by instantaneous unloading—An adiabatic process.
  - Natural Configuration reached in an isothermal stress-relaxation process.
- Thus, we need to know the process class under consideration.

The Eshelby Energy-Momentum tensor is one of the various driving forces that arise as a natural consequence of a body's ability to exist in more than one natural configuration.

There is no need to postulate additional balance laws and introduce quantities that appear in such laws that have never been measured.

"Configurational forces" are related to the energy supply that changes the "natural configurations".



These forces provide only partial information concerning the process a body is subject to. They provide no information concerning the entropy production.

I have resolved to quit abstract geometry, that is to say, the consideration of questions which serve only to exercise the mind, and this in order to study another kind of geometry, which has for its object the explanation of the phenomena of nature.

R. Descartes.

Q : Can the viscosity of a fluid depend on the pressure?

A : Yes.

Q : Is it reasonable to assume that a liquid is incompressible and its viscosity depends on the pressure (normal stress)?

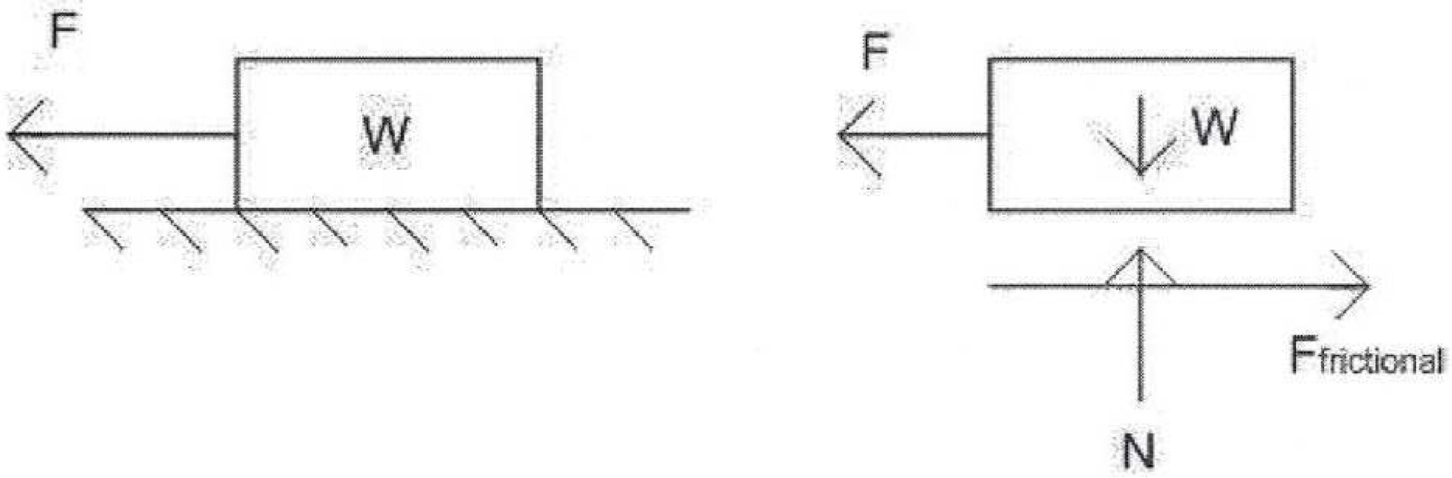
A : Yes.

Density changes in liquids in certain applications(wherein the pressure (normal stresses) changes by several orders of magnitude) are of the order of a few percent, while the viscosity changes by factor of 10^7-10^8 !!!

- Elastohydrodynamic Lubrication

Szeri (1998)

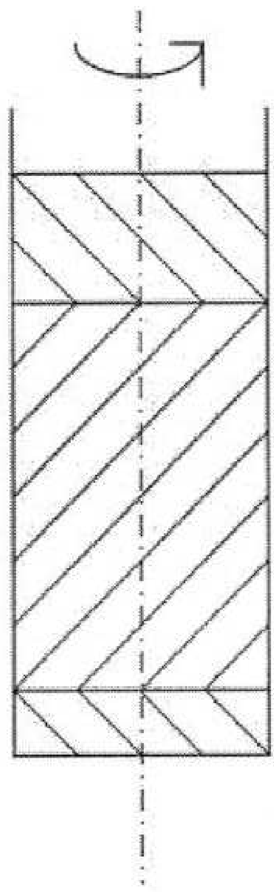
Digression: Consider the sliding of a rough block on a frictional plane surface.

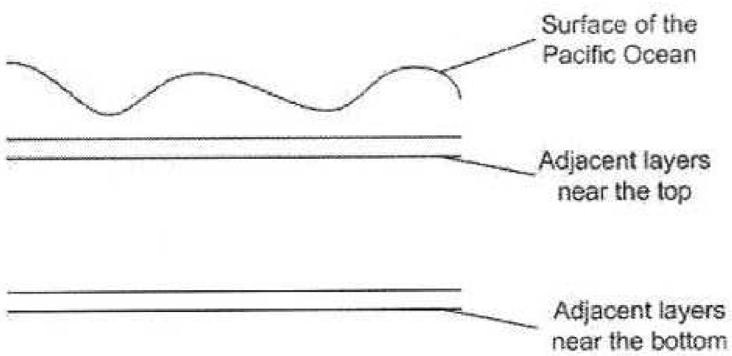


$$F_{\text{frictional}} = \bar{\mu} N \tag{11}$$

Frictional force definitely depends on the normal force for solids. Why should it be any different for fluids?

Coulomb's erroneous conclusions on the basis of his experiments:





All fluids are compressible:

- J.Canton (1762),(1764)
- J. Perkins (1819-20),(1826) $\sim 100 kg/cm^2$
- J. Jamin (1857-58): How optical properties change with pressure.
- E. Wartman (1859): How electrical resistance changes with pressure
- L. Cailletet (1870-1880) $\sim 1000 kg/cm^2$
- E. H. Amagat (1869-1893) $\sim 3000 kg/cm^2$
- G. Tammann (1893-1928)
- P. W. Bridgman (1909-1930)
- E.C. Andrade (1925-1931).

However many liquids can be approximated as incompressible, even when subject to a large range of pressures.

- W.G. Cutler et al., (1958).
- E.M. Griest et al., (1958).
- D. Dowson (1966)
- K.L. Johnson and R. Cameron, (1967).
- J.B. Irving (1971)
- K.L. Johnson and J.A. Greenwood, (1980).
- K.L. Johnson and J.L. Tevaarwerk, (1977).
- T.W. Bates et al., (1986).
- J.H. Hutton et al., (1983).
- A.Z. Szeri, (1998)
- J.A. Greenwood (2000)
- S. Bair and F. Qureshi (2002).
- S. Bair and F. Qureshi (2003).
- S. Bair (2004)

Barus (1891)

$$\mu = A\exp(\alpha p), \alpha - \text{constant}, \alpha > 0. \qquad (12)$$

Andrade (1930)

$$\mu = A\rho^{\frac{1}{2}} exp\left[\left(p + \rho^2 r\right)\frac{s}{\theta}\right] \qquad (13)$$

ρ - density, p - pressure, θ - temperature.

A, r, s - constants.

There has been a considerable amount of experimental work on the variation of the viscosity with pressure for liquids.

Bridgman (1931): The Physics of High Pressure.

Bridgman (1926): The effect of the pressure on the viscosity of forty three pure liquids.

Table 6. Influence of pressure on the stiffness of different asphalts at temperatures between 20°C and 60°C

Asphalt: Origin	Type	Pen./25°C	Temp R&B, °C	P.I.	Asphaltenes, % wt	Measuring temp, °C	Pressure P, kg/cm²: 0	100	200	300	400	500
							Ratio $\frac{\text{Stiffness at pressure P}}{\text{Stiffness at pressure P=0}}$					
Borneo	Newton-ian type	47	47	−2.2	1.2	25.0	1	4.0	13	45	167	597
						40.0	1	-	-	20	59	170
						49.0	1	2.2	5.8	15	37	90
California	Sol type	54	48	−1.8	5.1	20.0	1	2.5	6.6	17	44	115
						40.0	1	2.1	4.5	9.2	19	40
						49.4	1	1.8	3.7	7.1	14	27
Venezuela	Sol type	44	55	−0.3	15.5	20.9	1	2.2	4.9	11	24	54
						30.0	1	2.0	4.2	8.5	18	36
						40.0	1	1.9	3.6	7.0	14	26
						50.0	1	1.8	3.1	5.5	10	19
Venezuela (blown)	Gel type	35	86.5	+4.4	28.9	60.0	1	1.7	3.0	5.2	9.1	16

From: R. N. J. Saal and J. W. A. Labout, Rheological Properties of Asphalts, in Rheology: Theory and Experiments, Vol II, F. R. Eirich (ed.), Academic Press, New York, 363-400.

Stokes recognized that the viscosity can depend on the pressure for incompressible liquids:

"If we suppose μ to be independent of pressure also, and substitute ..."

"Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Du Buat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure... I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure."

Du Buat, Principes D′Hydraulique Et De Pyrodynamique, Didot, 1786.

Some related questions:

Q: Can the material moduli depend on the Lagrange multiplier?

A: Yes.

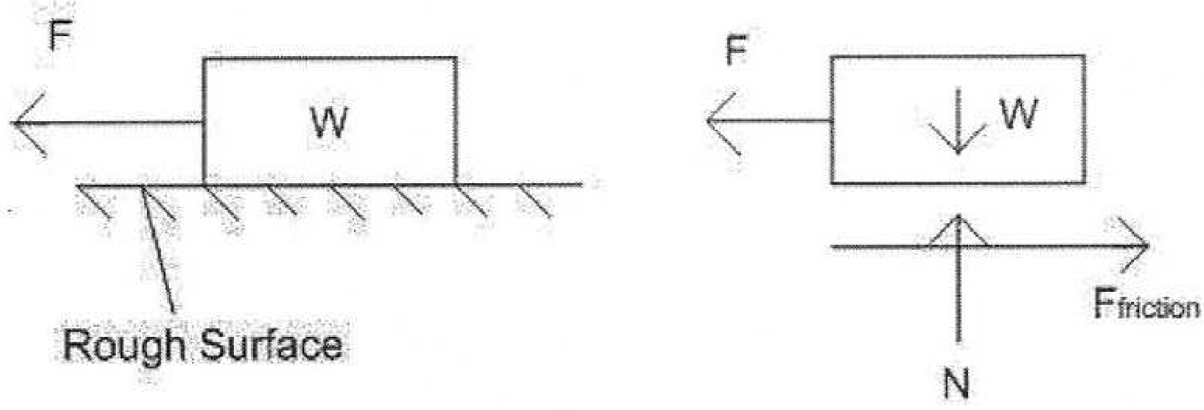
$$\mathbf{T} = -p\mathbf{I} + \hat{\alpha_1}\mathbf{D} + \hat{\alpha_2}\mathbf{D}^2, \tag{14}$$

$$\hat{\alpha}_i = \hat{\alpha}_i \left(p, \mathtt{II}_{\mathbf{D}}, \mathtt{III}_{\mathbf{D}}\right). \tag{15}$$

Q: Does the constraint response do no work? (D'Alembert, Bernoulli, Lagrange).

A: It is not correct to make such an assumption. Moreover, it depends on what one means by the constraint response.

Let us consider the constraint that a block moves on a planar **rough** surface.



$$F_{\text{friction}} = \mu N. \qquad (16)$$

While the normal reaction N does no work, the frictional force μN does work. The force μN is also a consequence of the constraint, and it reflects the word **rough** that appears in the description of the constraint. In any event the work done depends on N.

Boundary Conditions:

Daniel Bernoulli (Hydrodynamica): "I attribute these enormous differences to the adhesion of water to the walls of the tube, which, in cases of this kind, can exert an incredible effect by adhesion." (Translation due to P. Villagio)

Du Buat (1786) . . . no-slip.

Navier (1823) . . . slip (molecular arguments).

Girard

Poisson

Stokes

⋮

Fujita

Tani

⋮

Goldstein (1981):

We now restrict ourselves to systems for which the net virtual work of forces of constraint is zero. We have seen that this condition holds for rigid bodies and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. This is no longer true if sliding friction forces are present, and **we must exclude such systems from our formulation**.

Gauss (1829 - Translated into english and published in the Philosophical Magazine in 1841): " The motion of a system of material points connected together in any manner whatsoever, whose motions are modified by any external restraints whatsoever, proceeds in every instance in the greatest possible accordance with free motion, or under the least possible constraint; the measure of the constraint which the whole system suffers in every particle of time being considered equal to the sum of products of the square of the deviation of every point from its free motion into its mass. Let m, m', m'' $\&c$. be the masses of the points a, a', a'' $\&c$. their places at time t; b, b', b'' $\&c$. the places which they would occupy if entirely free in their motion after the infinitely small particle of time dt, in consequence of the forces acting upon them during this time, and of the velocities and directions acquired by them at the time t. Their real places c, c', c'' $\&c$. will then be those of which of all places compatible with the conditions of the systems the quantity $m(bc)^2 + m'(b'c')^2 + m''(b''c'')^2 \&c$. is a minimum.

The equilibrium is evidently a particular case only of the general law, and the condition for this case is, that $m(bc)^2 + m'(b'c')^2 + m''(b''c'')^2 \&c.$ itself is a minimum, or that the continuance of the system in a state of rest more accords with the free motion of the single points than any possible change the system could undergo."

SIMPLY PUT: The constraint force ought to be the least force to enforce the constraint.

Rajagopal & Srinivasa, Proc. Roy. Soc. London (2004): Implications for Continua.

Rajagopal, Applications of Mathematics (2003): Constraints and their consequences for implicit constitutive theories.

- Dettmann and Morriss (1990)
- Evans and Morriss (1990)
- O'Reilly and Srinivasa (2001)
- Udwadia and Kalaba (2002)
- Carlson, Fried and Tortorelli (2003) – Holonomic Constraints
- Rajagopal and Srinivasa (2004) – Non-Holonomic Constraints

There are several liquids that can shear thin or shear thicken. For such liquids, when subject to a high range of pressures, the viscosity would also depend on the pressure. It would thus be reasonable to consider models of the form:

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p, |\mathbf{D}|^2)\mathbf{D}, \quad \text{(A)}$$

with $\text{tr}\mathbf{D} = 0$. Thus $p = -\frac{1}{3}\text{tr}\mathbf{T}$ and (A) takes the form:

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}.$$