Variable density Navier-Stokes equations 4

Other control results and related questions

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Prague, May-June 2011

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The problem

$$\begin{cases} \text{Minimize} \quad I(\mathbf{v},\rho,\mathbf{u}) = \frac{1}{2} T^*(\mathbf{v},\mathbf{u};\mathbf{u}_e,\delta)^2 + \frac{b}{2} \iint_{\omega\times(0,T_0)} |\mathbf{v}|^2 \\ \text{Subject to } \mathbf{v} \in \mathcal{U}_{\rm ad}, \quad (\mathbf{v},\rho,\mathbf{u}) \text{ solves (NS)-}\rho \text{ in } \Omega\times(0,T_0) \end{cases}$$

Here: $\mathbf{u}_e \in H$, $\delta > 0$, $b > 0$,
$\mathcal{U}_{\rm ad} \subset L^2(\omega\times(0,T))^3$ is non-empty, closed and convex,
$T^*(\mathbf{v},\mathbf{u};\mathbf{u}_e,\delta) := \inf\{\, \hat{T} : \|\mathbf{u}(\cdot\,,\hat{T}) - \mathbf{u}_e\| \le \delta \,\}$
(eventually, we can have $I(\mathbf{v},\rho,\mathbf{u}) = +\infty$)

$$\begin{cases} (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \mathbf{v}1_\omega, \;\; \nabla\cdot\mathbf{u} = 0, \;\; (\mathbf{x},t)\in\Omega\times(0,T_0) \\ \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x},t)\in\Omega\times(0,T_0) \\ \mathbf{u} = 0, \quad (\mathbf{x},t)\in\partial\Omega\times(0,T_0) \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x}\in\Omega \end{cases}$$

The goals

Existence, optimality (characterization) and algorithms

From now on: $\rho_0 \ge \alpha > 0$ a.e.

A time optimal control problem

Optimizing the time of control: existence

Theorem:

Assume: $\inf I(\mathbf{v}, \rho, \mathbf{u}) < +\infty$ Then: existence

Idea of the proof:

$\mathcal{U}_{\rm ad}$ is weakly closed in $L^2(\omega \times (0, T_0))^N$, I is coercive

The task: prove lower semicontinuity, i.e.

$v^m \to v^*$ weakly $\Rightarrow \liminf_{m\to+\infty} \; T_m^* \geq T^*$

Otherwise: it can be assumed that $T_m^* \to \tilde{T} < T^*$ and then

$(\mathbf{u}^*(\cdot\,, \tilde{T}) - \mathbf{u}_e, \mathbf{z})_{L^2} \leq \delta \|\mathbf{z}\| \; \forall \mathbf{z} \in V$, an absurd

Indeed,

$$|(\mathbf{u}(\cdot\,, \tilde{T}) - \mathbf{u}_e, \mathbf{z})_{L^2}| \leq |(\mathbf{u}(\cdot\,, \tilde{T}) - \mathbf{u}(\cdot\,, T_m^*), \mathbf{z})_{L^2}| + |(\mathbf{u}(\cdot\,, T_m^*) - \mathbf{u}^m(\cdot\,, T_m^*), \mathbf{z})_{L^2}| + |(\mathbf{u}^m(\cdot\,, T_m^*) - \mathbf{u}_e, \mathbf{z})_{L^2}|$$

But $\mathbf{u}^m \to \mathbf{u}$ strongly in $C^0([0, T]; V')$

Also, $\mathbf{u}(\cdot\,, T_m^*) \to \mathbf{u}(\cdot\,, \tilde{T})$

Finally, $|(\mathbf{u}^m(\cdot\,, T_m^*) - \mathbf{u}_e, \mathbf{z})_{L^2}| \leq \|\mathbf{u}^m(\cdot\,, T_m^*) - \mathbf{u}_e\|_{L^2} \|\mathbf{z}\|_{L^2} \leq \delta \|\mathbf{z}\|_{L^2}$

(by the definition of T_m^*)

A time optimal control problem

Optimizing the time of control: optimality

Theorem:

Assume: $\inf I(\mathbf{v}, \rho, \mathbf{u}) < +\infty$, $(T^*, \mathbf{v})$ is an optimal solution with associated state $(\rho, \mathbf{u})$, $t \mapsto \mathbf{u}^*(\cdot\,, t)$ is C^1 in $[T^* - \kappa, T^*]$, $(\mathbf{u}^*(\cdot\,, T^*) - \mathbf{u}_e, \mathbf{u}_t(\cdot\,, T^*))_{L^2} < 0$

Then: $\exists \lambda \in \mathbb{R}, \exists (\eta, \mathbf{w}, \beta)$ such that

$$\begin{cases} -\rho \mathbf{w}_t - \rho(\mathbf{u}\cdot\nabla)\mathbf{w} + \rho(\nabla\mathbf{u})^t\mathbf{w} + \nabla\beta \\ \qquad\qquad = \mu\Delta\mathbf{w} + \rho\nabla\eta, \quad \nabla\cdot\mathbf{w} = 0, \quad (\mathbf{x}, t) \in \Omega\times(0, T^*), \\ -\eta_t - \mathbf{u}\cdot\nabla\eta + (\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{w} = 0, \quad (\mathbf{x}, t) \in \Omega\times(0, T^*), \\ \mathbf{w} = 0, \quad (\mathbf{x}, t) \in \partial\Omega\times(0, T^*), \\ \eta|_{t=T^*} = 0, \quad \mathbf{w}|_{t=T^*} = \lambda(\mathbf{u}|_{t=T^*} - \mathbf{u}_e), \quad \mathbf{x} \in \Omega, \end{cases}$$

$$\iint_{\omega\times(0,T^*)} (\mathbf{w} + b\mathbf{v})\cdot(\mathbf{v}' - \mathbf{v}) \geq 0 \quad \forall \mathbf{v}' \in \mathcal{U}_{\rm ad}, \quad \mathbf{v} \in \mathcal{U}_{\rm ad}$$

$$T^* = P_{[0,T_0]}\left(-\lambda(\mathbf{u}(\cdot\,, T^*) - \mathbf{u}_e, \mathbf{u}_t(\cdot\,, T^*))_{L^2}\right)$$

$$\|\mathbf{u}(\cdot\,, T^*) - \mathbf{u}_e\|_{L^2} = \delta$$

A time optimal control problem
Additional comments

- Explanation:
  We use some kind of “partial derivatives” with respect to $\mathbf{v}$ and T^*
  λ and $(\eta, \mathbf{w}, \beta)$ are Lagrange multipliers
- On the assumptions:
  The assumption on T^* serves to discard trivial cases
  The other assumptions play the role of qualification hypotheses (needed in general in this context)
  They can be show to be reasonable
- Work in progress:
  Using the optimality system to deduce iterative algorithms . . .

Controllability
Formulation of the problems

The general controllability question:

Fix Ω, ω, T and a *desired property* at T:

$$\mathbf{u}(\cdot\,,T) \in \mathbf{U}_T \quad \text{for some } \mathbf{U}_T \subset H$$

Then, is it true that, for any $(\rho_0, \mathbf{u}_0)$ we can find $\mathbf{v}$ that produces a state satisfying this desired property?

(more difficult than optimal control)
Attention: we only require something for $\mathbf{u}(\cdot\,,T)$!

Particular choices of $\mathbf{U}_T$ – Comments:

- $\mathbf{U}_T$ = an arbitrary singleton: exact controllability (EC)
  Not to be expected if $\omega \neq \Omega$
- $\mathbf{U}_T$ = an arbitrary ball: approximate controllability (AC)
  True for similar linear systems
- $\mathbf{U}_T$ = an arbitrary final state $(\overline{\rho}, \overline{\mathbf{u}})|_{t=T}$: EC to the trajectories
  Stronger than AC — True for similar linear problems
- $\mathbf{U}_T = \{\mathbf{0}\}$: null controllability (NC)

Controllability

The constant density case

We begin by considering NS fluids

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} - \Delta\mathbf{y} + \nabla p = \mathbf{v}1_\omega, \ \nabla\cdot\mathbf{y} = 0, \ (\mathbf{x},t)\in Q \\ \mathbf{y} = 0, \quad (\mathbf{x},t)\in\Sigma \\ \mathbf{y}\big|_{t=0} = \mathbf{y}_0, \quad \mathbf{x}\in\Omega \end{cases}$$

Essentially, no general global controllability result, even in this case

Theorem (local EC to the trajectories for NS):

Assume: $(\bar{\mathbf{y}},\bar{p})$ is an uncontrolled L^∞ solution to Navier-Stokes:

$$\begin{cases} \bar{\mathbf{y}}_t - \Delta\bar{\mathbf{y}} + (\bar{\mathbf{y}}\cdot\nabla)\bar{\mathbf{y}} + \nabla\bar{p} = 0 & \text{in } Q \\ \nabla\cdot\bar{\mathbf{y}} = 0 & \text{in } Q, \dots \end{cases}$$

Then: $\exists\varepsilon > 0$ such that, whenever $\|\mathbf{y}^0 - \bar{\mathbf{y}}(\cdot\,,0)\|_{L^{2N-2}} \le \varepsilon$,
$\exists(\mathbf{v},\mathbf{y},p)$ with $\mathbf{y}(\mathbf{x},T) = \bar{\mathbf{y}}(\mathbf{x},T)$

Controllability
The constant density case

The global AC was conjectured by J.-L. Lions in 1990

To our knowledge: global results only known for modified systems

- Global AC and EC to the trajectories for linearized NS:

$$\begin{cases} \mathbf{y}_t + (\bar{\mathbf{y}} \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\bar{\mathbf{y}} - \Delta \mathbf{y} + \nabla p = \mathbf{v}1_\omega, \ (\mathbf{x}, t) \in Q \\ \nabla \cdot \mathbf{y} = 0, \ (\mathbf{x}, t) \in Q \\ \mathbf{y} = 0, \quad (\mathbf{x}, t) \in \Sigma \\ \mathbf{y}|_{t=0} = \mathbf{y}_0, \quad \mathbf{x} \in \Omega \end{cases}$$

- AC for some nonlinear approximations [Fabre, 1996]:

$$\begin{cases} \mathbf{y}_t + (T_M(\mathbf{y}) \cdot \nabla)\mathbf{y} - \Delta \mathbf{y} + \nabla p = \mathbf{v}1_\omega, \ \nabla \cdot \mathbf{y} = 0, \ (\mathbf{x}, t) \in Q \\ \mathbf{y} = 0, \quad (\mathbf{x}, t) \in \Sigma \\ \mathbf{y}|_{t=0} = \mathbf{y}_0, \quad \mathbf{x} \in \Omega \end{cases}$$

- AC with other (slip) boundary conditions and $N = 2$ [Coron, 1996]
  NC for $N = 2$, manifold without boundary, [Coron-Fursikov, 1996]

Controllability

Idea of the proof of the local result (I): Stokes

$$\begin{cases} \mathbf{y}_t + (\mathbf{a}\cdot\nabla)\mathbf{y}) + (\mathbf{y}\cdot\nabla)\mathbf{a} - \Delta\mathbf{y} + \nabla p = \mathbf{v}1_\omega, \ \ \nabla\cdot\mathbf{y} = 0, \\ \mathbf{y}(\cdot\,,0) = \mathbf{y}_0 \quad \dots \end{cases}$$

$$\begin{cases} -\varphi_t - D\varphi\cdot\mathbf{a} - \Delta\varphi + \nabla q = 0, \quad \nabla\cdot\varphi = 0 \\ \varphi(\cdot\,,T) = \varphi^T \quad \dots \quad \text{with } \mathbf{a}\in L^\infty \end{cases}$$

For this linear problem:

NC $\Leftrightarrow$ EC to the trajectories $\Leftrightarrow$ Observability:

$$\int_\Omega |\varphi(\mathbf{x},0)|^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2 \, d\mathbf{x}\, dt \quad \forall\varphi^T\in H$$

The main tool for observability: Carleman estimates

$$\iint_Q \rho^{-2}|\varphi|^2 \, d\mathbf{x}\, dt \le C \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2 \, d\mathbf{x}\, dt \quad \forall\varphi^T\in H$$

$\rho(x,t) \sim e^{\rho_0(x)/(T-t)}$, depending on ω, Ω, T

Carleman + Dissipativity $\Rightarrow$ Observability (hence, Stokes is NC)

Controllability

Idea of the proof of the local result (II): Carleman

Proof of Carleman:

Technical; several difficulties: heat-like, pressure-like, ...

$$\begin{cases} -\varphi_t - D\varphi \cdot \mathbf{a} - \Delta\varphi + \nabla q = g, \quad \nabla \cdot \varphi = 0 \\ \varphi(\cdot\,, T) = \varphi^T \quad + \text{Dirichlet conditions} \end{cases}$$

Set

$$I(\varphi) := \iint_Q \rho^{-2}|\varphi|^2 + \iint_Q \rho^{-2}|\nabla\varphi|^2 + \iint_Q \rho^{-2}\left(|\nabla\varphi_t|^2 + |\Delta\varphi|^2\right)$$

Then:

$$\begin{aligned} I(\varphi) &\leq C \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2 + C \iint_Q \rho^{-2}|\nabla q|^2 + \dots \\ &\leq C \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2 + C \iint_{\omega\times(0,T)} \rho^{-2}|q|^2 + \varepsilon I(\varphi) + \dots \\ &\leq C \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2 + 2\varepsilon I(\varphi) + \dots \end{aligned}$$

Here we use $\mathbf{a} \in L^\infty$

Controllability

Idea of the proof of the local result (III): Reformulation and Liusternik's theorem

Reformulation of the problem, $\mathbf{y} = \bar{\mathbf{y}} + \mathbf{u}$:

$$\begin{cases} \mathbf{u}_t + (\bar{\mathbf{y}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\bar{\mathbf{y}} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{v}1_\omega, \text{ etc.} \\ \mathbf{u}(\cdot\,,0) = \mathbf{u}^0 := \mathbf{y}^0 - \hat{\mathbf{y}}(\cdot\,,0), \quad \mathbf{u}(\cdot\,,T) = 0 \end{cases}$$

That is to say:

$\mathcal{A}(\mathbf{v},\mathbf{u},p) := (\mathbf{u}_t + \cdots - \mathbf{v}1_\omega, \mathbf{u}(\cdot\,,0)) = (\mathbf{0},\mathbf{u}^0)$, $(\mathbf{v},\mathbf{u},p) \in Y$

Here, Y is a space of triplets $(\mathbf{v},\mathbf{u},p)$ that vanish at $t = T$ and ...

The tasks:

- Prove that $\mathcal{A} : Y \mapsto Z$ is well defined and C^1 near zero (good definitions of Y and Z) and
- $\mathcal{A}'(0,0,0)$ is onto (solve a NC problem for the linearized problem)

OK, provided $\bar{\mathbf{y}} \in L^\infty$ and $\|\mathbf{y}^0 - \bar{\mathbf{y}}(\cdot\,,0)\|_{L^{2N-2}}$ is small

Notice:

For a global result, we need $\mathcal{A}'(\mathbf{v},\mathbf{u},p)$ bijective for all $(\mathbf{v},\mathbf{u},p)$, with $\|(\mathcal{A}')^{-1}\|$ uniformly bounded (global inversion). This is unknown

Controllability

Ideas for the variable density case

$$\begin{cases} (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \mathbf{v}1_\omega, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t)\in Q, \\ \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x},t)\in Q, \\ \mathbf{u} = 0, \quad (\mathbf{x},t)\in\Sigma, \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x}\in\Omega, \end{cases}$$

With similar arguments:

Theorem (local EC to stationary solutions):

Assume: $(\overline{\rho}, \overline{\mathbf{u}}, \overline{p})$ is a stationary solution with $\overline{\rho} \geq \alpha > 0$, $\overline{\rho}, \overline{\mathbf{u}} \in W^{1,\infty}$
Then: $\exists\varepsilon > 0$ such that, whenever $\|\rho^0 - \overline{\rho}\|_{L^\infty} + \|\mathbf{u}^0 - \overline{\mathbf{u}}\|_V \leq \varepsilon$,
$\exists(\mathbf{v}, \rho, \mathbf{u}, p)$ with $\mathbf{u}(\cdot\,, T) = \overline{\mathbf{u}}(\cdot\,, T)$

Work in progress: EC to nonstationary trajectories . . .

Other results and questions

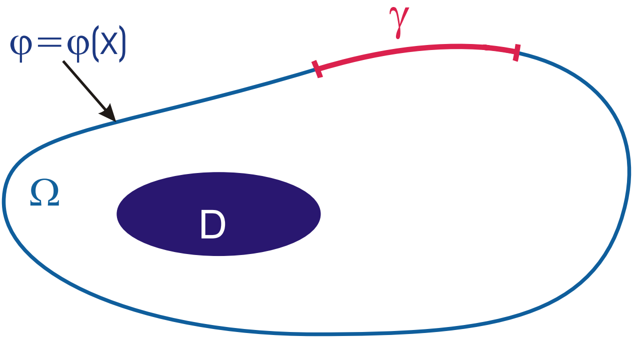
- An inverse problem: Identification of the shape of a body
  Find D from Ω, (φ, ψ) and α, with

$$\begin{cases} (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla p = \mu \Delta \mathbf{u}, \quad (\mathbf{x}, t) \in (\Omega \setminus \overline{D}) \times (0, T) \\ \nabla \cdot \mathbf{u} = 0, \quad (\mathbf{x}, t) \in (\Omega \setminus \overline{D}) \times (0, T) \\ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (\mathbf{x}, t) \in (\Omega \setminus \overline{D}) \times (0, T) \\ \mathbf{u} = \varphi, \quad (\mathbf{x}, t) \in \Sigma, \quad \rho = \psi, \quad (\mathbf{x}, t) \in \Sigma_{in} \\ \mathbf{u} = \mathbf{0}, \quad (\mathbf{x}, t) \in \partial D \times (0, T) \\ \rho|_{t=0} = \rho_0, \quad (\rho \mathbf{u})|_{t=0} = \rho_0 \mathbf{u}_0, \quad \mathbf{x} \in \Omega \setminus \overline{D} \end{cases}$$

$$\sigma(u, p) \cdot \mathbf{n} := (-p \,\mathrm{Id} + \nu(\nabla u + {}^t\nabla u)) \cdot \mathbf{n} = \alpha \quad \text{on } \gamma \subset \Omega$$

Main questions: uniqueness, stability, reconstruction
Work in progress, several satisfactory results for NS . . .

Geometric inverse problem for fluids



Inverse problem

Find a rigid body *D*

Other results and questions

- Besov spaces and Kato-like results (another approach)
  Based on heat kernel, contractive mappings, [Fujita-Kato, 1964]
  Suitable for $\Omega = \mathbb{R}^N$, scale invariant spaces E. Typical result for NS:

Theorem:

Asume: $\Omega = \mathbb{R}^N$, $\mathbf{u}_0 \in E$, $\mathbf{f} \equiv \mathbf{0}$
$\exists T_* > 0$ and $\dot{\exists}$ $\mathbf{u} \in C^0([0, T_*]; E)$ and $\|\mathbf{u}\|_E$ small $\Rightarrow$ $T_* = T$

OK with $E = \dot{H}^{1/2}, L^N, \dot{B}^{N/2-1}_{2,1}, \dots$ In particular:
$E = BMO^{-1}$ [Koch-Tataru]
Other "very weak" spaces [Chemin-Ghallager, 2009]
A result from [Danchin, 2007] for NS-ρ (still not optimal):

Theorem:

Take $F := \dot{B}^{N/2}_{2,1} \times \dot{B}^{N/2-1}_{2,1}$, $(\rho_0^{-1}, \mathbf{u}_0) \in F$, with $\|\rho_0^{-1}\|_{\dot{B}^{N/2}_{2,1}} \le \kappa(N)$
$\exists T_* > 0$ and $\dot{\exists}$ $(\rho^{-1}, \mathbf{u}) \in C^0([0, T_*]; F)$; $\|(\rho_0, \mathbf{u}_0)\|_F$ small $\Rightarrow$ $T_* = T$

Recall: $\dot{B}^{N/2}_{2,1} \hookrightarrow \dot{H}^{N/2} \cap L^\infty$. For large $\|\rho_0^{-1}\|_{\dot{B}^{N/2}_{2,1}}$, open question!

Some references:

- For the physical motivation, basic tools and NS results:
  [Panton, 1984], [Zeidler, 1988], [Chorin-Marsden, 1993]
  [Temam, 2001], [Constantin-Foias, 1989], [Tartar, 2006],
  [Lemarié-Rieusset, 2002]*, [Simon, 2011]*

- For NS-ρ theory (existence, uniqueness, regularity):
  [P-L Lions, 1996], [Braz-EFC-Rojas, 2011]*

- For results concerning control and inverse problems:
  [J-L Lions, 1969], [Gunzburger, 2003], [Glowinski-Lions-He, 2008],
  [Fursikov, 2005], [Fursikov-Imanuvilov, 1995], [EFC-Guerrero, 2006]*,
  [Coron, 2007]*

* . . . and the references therein

THANK YOU VERY MUCH
BYE!