Variable density Navier-Stokes equations 3

Uniqueness and optimal control results

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No satisfactory general result for $N = 2$ (???)

Minimal regularity hypotheses?

Uniqueness for $N = 2$ and some particular structure of ρ_0?

Uniqueness

A partial result

Theorem:

Assume: $\rho_0 \geq \alpha > 0$, $(\overline{\rho}, \overline{\mathbf{u}}, \overline{p})$ is a solution with $\nabla\overline{\rho}, \nabla\overline{\mathbf{u}}, \overline{\mathbf{u}}_t \in L^2(L^\infty)$, $\overline{\rho}, \overline{\mathbf{u}} \in C^0(\overline{Q})$. Then $\mathbf{u} = \overline{\mathbf{u}}$ a.e.

Sketch of the proof:

(A) Energy identity for $(\overline{\rho}, \overline{\mathbf{u}})$:

$$\frac{1}{2}\int_\Omega \overline{\rho} \mid \overline{\mathbf{u}} \mid^2 + \mu \iint_{\Omega\times(0,t)} \mid \nabla\overline{\mathbf{u}} \mid^2 = \frac{1}{2}\int_\Omega \rho_0 \mid \mathbf{u}_0 \mid^2 + \iint_{\Omega\times(0,t)} \overline{\rho}\overline{\mathbf{u}} \cdot \mathbf{f}$$

(B) Energy inequality for $(\rho, \mathbf{u})$:

$$\frac{1}{2}\int_\Omega \rho \mid \mathbf{u} \mid^2 + \mu \iint_{\Omega\times(0,t)} \mid \nabla\mathbf{u} \mid^2 \leq \frac{1}{2}\int_\Omega \rho_0 \mid \mathbf{u}_0 \mid^2 + \iint_{\Omega\times(0,t)} \rho\mathbf{u} \cdot \mathbf{f}$$

(C) Motion equation for $(\overline{\rho}, \overline{\mathbf{u}})$:

$$\iint_{\Omega\times(0,t)} \{\overline{\rho}(\overline{\mathbf{u}}_t + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} - \mathbf{f}) \cdot \mathbf{u} + \mu\nabla\overline{\mathbf{u}} : \nabla\mathbf{u}\} = 0$$

(D) Motion equation for $(\rho, \mathbf{u})$:

$$\iint_{\Omega\times(0,t)} \{\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{f}) \cdot \overline{\mathbf{u}} + \mu\nabla\mathbf{u} : \nabla\overline{\mathbf{u}}\} = 0$$

(A) + (B) - (C) -(D) leads to ...

Uniqueness
Sketch of the proof:

. . .

$$\left(\int_\Omega \rho \mid \mathbf{u} - \bar{\mathbf{u}} \mid^2\right)(t) + \mu \iint_{\Omega\times(0,t)} \mid \nabla \mathbf{u} \mid^2$$
$$\leq \iint_{\Omega\times(0,t)} \left(A(s)|\mathbf{u} - \bar{\mathbf{u}}|^2 + B_\varepsilon(s)|\rho - \bar{\rho}|^2\right) + \varepsilon \iint_{\Omega\times(0,t)} |\mathbf{u} - \bar{\mathbf{u}}|^2$$

with $A, B_\varepsilon \in L^\infty(0,T)$.
Transport equations for ρ and $\bar{\rho}$:
$\frac{1}{2}\frac{d}{dt}\int_\Omega \mid \rho - \bar{\rho} \mid^2 = \int_\Omega \nabla\bar{\rho}\cdot(\mathbf{u} - \bar{\mathbf{u}})(\rho - \bar{\rho})$
Consequences:

$$\left(\int_\Omega \mid \rho - \bar{\rho} \mid^2\right)(t) \leq C \iint_{\Omega\times(0,t)} |\mathbf{u} - \bar{\mathbf{u}}|^2 \quad \forall t$$

$$\left(\int_\Omega \rho \mid \mathbf{u} - \bar{\mathbf{u}} \mid^2\right)(t) + \mu \iint_{\Omega\times(0,t)} \mid \nabla \mathbf{u} \mid^2 \leq \iint_{\Omega\times(0,t)} C(s)|\mathbf{u} - \bar{\mathbf{u}}|^2$$

with $C \in L^\infty(0,T)$ and . . .

Uniqueness

Additional comments and questions:

- No satisfactory for $N = 2$!
  Minimal regularity hypotheses on $(\overline{\rho}, \overline{\mathbf{u}})$? Can we have only $\rho_0 \geq 0$?
- Another strategy: Duality
  Let $(\overline{\rho}, \overline{\mathbf{u}})$, $(\rho, \mathbf{u})$ be two solutions. Set $(\sigma, \mathbf{w}) := (\rho - \overline{\rho}, \mathbf{u} - \overline{\mathbf{u}})$

$$\left. \begin{array}{l} E(\rho, \mathbf{u}) = 0 \\ E(\overline{\rho}, \overline{\mathbf{u}}) = 0 \end{array} \right\} \Rightarrow \tilde{E}(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})(\sigma, \mathbf{w}) = 0 \quad (\text{linear in } (\sigma, \mathbf{w}))$$

Task: $N(\tilde{E}(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})) = \{0\}$, equivalent to $R(\tilde{E}(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})^*)$ is dense
Therefore: try to solve a linear problem with RHS in a dense subspace
Same requirements on $(\rho, \mathbf{u}), (\overline{\rho}, \overline{\mathbf{u}})$ ...

Uniqueness

Additional comments and questions:

- Many other questions can be considered.
  For instance: how "large" is the set of uniqueness data?
  An "entropy condition" ensuring uniqueness
  (maybe together with $N = 2$ and/or $\mathbf{u} \in L^r(0, T; L^s)$)?, etc.

Optimal control

Formulation of the problems

$$\begin{cases} \text{Minimize} \quad J(\mathbf{v}, \rho, \mathbf{u}) \\ \text{Subject to} \quad \mathbf{v} \in \mathcal{U}_{\rm ad}, \quad (\mathbf{v}, \rho, \mathbf{u}) \text{ solves NS-}\rho \end{cases}$$

- The state equation is NS-ρ:

$$\begin{cases} (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \mathbf{v}1_\omega, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t) \in Q, \\ \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x},t) \in Q, \\ \mathbf{u} = 0, \quad (\mathbf{x},t) \in \Sigma, \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x} \in \Omega, \end{cases}$$

- The constraints: $\mathbf{v} \in \mathcal{U}_{\rm ad} \subset L^2(\omega \times (0,T))^N$, closed and convex

Optimal control

Some particular cases:

- Cost:
  $J(\mathbf{v},\rho,\mathbf{u}) = \iint_Q \left(\frac{a}{2}|\mathbf{u}-\mathbf{u}_d|^2 + \frac{a'}{2}|\rho-\rho_d|^2 \right) + \frac{b}{2}\iint_{\omega\times(0,T)} |\mathbf{v}|^2$

$J(\mathbf{v},\rho,\mathbf{u}) = \int_\Omega \left(\frac{a}{2}|\mathbf{u}(\mathbf{x},T)-\mathbf{u}_e|^2 + \frac{a'}{2}|\rho(\mathbf{x},T)-\rho_e|^2 \right) + \frac{b}{2}\iint_{\omega\times(0,T)} |\mathbf{v}|^2$

- Constraints:
  $\mathcal{U}_{\rm ad} = L^2(\omega\times(0,T))^N$
  $\mathcal{U}_{\rm ad} = \{\, \mathbf{v}\in L^2(\omega\times(0,T))^N : |\mathbf{v}|\le M \ \text{ a.e.} \,\}$
  $\mathcal{U}_{\rm ad} = \{\, \mathbf{v}\in L^2(\omega\times(0,T))^N : \mathbf{v} = \sum_{i=1}^I \mathbf{v}^i(\mathbf{x}) 1_{(t_i,\tau_i)}(t) \ \text{ a.e.,} \ \ \mathbf{v}^i\in L^2 \,\}$
  where $M>0$ and the t_i and τ_i satisfy $0\le t_1<\tau_1<\cdots<t_I<\tau_I\le T$
- Goals:

1. Existence, uniqueness
2. Characterization: find a system necessarily satisfied by any optimal solution
3. Computation: provide iterative algorithms that produce sequences of controls $\mathbf{v}^m$ that converge to a solution

Optimal control

Existence — A general result

Theorem:

Assume:

1. $\mathcal{U}_{\rm ad} \subset L^2(\omega \times (0,T))^N$ is non-empty, closed and convex
2. J is sequentially weakly lower semi-continuous
3. Either $\mathcal{U}_{\rm ad}$ is bounded or J is coercive in $\mathbf{v}$

Then: existence

Can be applied to the particular cases above . . .
Uniqueness?

Optimal control

Characterization of optimality:

Theorem:

Assume: $\mathcal{U}_{\rm ad} \subset L^2(\omega \times (0,T))^N$ is non-empty, closed and convex, $J \equiv \iint_Q \left(\frac{a}{2}|\mathbf{u} - \mathbf{u}_d|^2 + \frac{a'}{2}|\rho - \rho_d|^2 \right) + \frac{b}{2} \iint_{\omega \times (0,T)} |\mathbf{v}|^2$, $(\mathbf{v}, \rho, \mathbf{u})$ is optimal

Then: $\exists (\eta, \mathbf{w}, \beta)$ such that:

$$\left\{ \begin{array}{l} -\rho \mathbf{w}_t - \rho(\mathbf{u} \cdot \nabla)\mathbf{w} + \rho(\nabla \mathbf{u})^T \mathbf{w} + \nabla \beta \\ \quad = \mu \Delta \mathbf{w} + \rho \nabla \eta + a(\mathbf{u} - \mathbf{u}_d), \quad \nabla \cdot \mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q \\ -\eta_t - \mathbf{u} \cdot \nabla \eta + (\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{w} = a'(\rho - \rho_d), \quad (\mathbf{x}, t) \in Q \\ \mathbf{w} = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \eta|_{t=T} = 0, \quad (\eta \mathbf{w})|_{t=T} = 0, \quad \mathbf{x} \in \Omega \end{array} \right.$$

$$\iint_{\omega \times (0,T)} (\mathbf{w} + b\mathbf{v}) \cdot (\mathbf{v}' - \mathbf{v}) \geq 0 \quad \forall \mathbf{v}' \in \mathcal{U}_{\rm ad}, \quad \mathbf{v} \in \mathcal{U}_{\rm ad}$$

Optimal control

Characterization of optimality:

Idea of the proof (I): the key idea

Set $\mathbf{y}(\mathbf{v}) := (\rho, \mathbf{u})$ (the state) and assume regularity. Then
$j(\mathbf{v}) := J(\rho, \mathbf{u}, \mathbf{v}) = F(\mathbf{y}(\mathbf{v}), \mathbf{v})$ and

$$\langle j'(\mathbf{v}), \mathbf{v}' - \mathbf{v} \rangle \geq 0 \quad \forall \mathbf{v}' \in \mathcal{U}_{\mathrm{ad}}$$

One has:

$$\begin{aligned} &\langle j'(\mathbf{v}), \mathbf{v}' - \mathbf{v} \rangle \\ &\quad = \langle F_{\mathbf{y}}(\mathbf{y}(\mathbf{v}), \mathbf{v}), \tfrac{d\mathbf{y}}{d\mathbf{v}}(\mathbf{v})(\mathbf{v}' - \mathbf{v}) \rangle + \langle F_{\mathbf{v}}(\mathbf{y}(\mathbf{v}), \mathbf{v}), \mathbf{v}' - \mathbf{v} \rangle \\ &\quad = \langle \mathbf{p}(\mathbf{v}) + F_{\mathbf{v}}(\mathbf{y}(\mathbf{v}), \mathbf{v}), \mathbf{v}' - \mathbf{v} \rangle \end{aligned}$$

with $\mathbf{p}(\mathbf{v}) = \frac{d\mathbf{y}}{d\mathbf{v}}(\mathbf{v})^* \cdot F_{\mathbf{y}}(\mathbf{y}(\mathbf{v}), \mathbf{v})$ (the so called adjoint state)
Hence:

$$\langle \mathbf{p}(\mathbf{v}) + F_{\mathbf{v}}(\mathbf{y}(\mathbf{v}), \mathbf{v}), \mathbf{v}' - \mathbf{v} \rangle \geq 0 \quad \forall \mathbf{v}' \in \mathcal{U}_{\mathrm{ad}}$$

Optimal control

Characterization of optimality:

Idea of the proof (II): the rigorous computations

Take $\mathbf{v}' = \mathbf{v} + \alpha \mathbf{h}$ with $\alpha \in \mathbb{R}_+$ (small), $\mathbf{h} \in L^2(\omega \times (0,T))^N$, $\mathbf{v}' \in \mathcal{U}_{\rm ad}$. Let $(\rho', \mathbf{u}')$ be a state associated to $\mathbf{v}'$. Then

$$
\begin{aligned}
&J(\mathbf{v}', \rho', \mathbf{u}') - J(\mathbf{v}, \rho, \mathbf{u}) \\
&= \alpha \Big(\iint_Q [a(\mathbf{u} - \mathbf{u}_d) \cdot \mathbf{y} + a'(\rho - \rho_d)\sigma] + b \iint_{\omega \times (0,T)} \mathbf{v} \cdot \mathbf{h} \, dx \, dt \Big) + \alpha Z_\alpha
\end{aligned}
$$

with $Z_\alpha \to 0$.
After integration by parts:

$$
\iint_Q (a(\mathbf{u} - \mathbf{u}_d) \cdot \mathbf{y} + a(\rho - \rho_d)\sigma) = \iint_{\omega \times (0,T)} \mathbf{w} \cdot \mathbf{h}
$$

Consequence:

$$
\iint_{\omega \times (0,T)} (\mathbf{w} + b\mathbf{v}) \cdot \mathbf{h} \geq 0.
$$

Optimal control

Further comments and questions:

- A similar result holds when
$J \equiv \int_\Omega \left(\frac{a}{2}|\mathbf{u}(\mathbf{x},T) - \mathbf{u}_e|^2 + \frac{a'}{2}|\rho(\mathbf{x},T) - \rho_e|^2 \right) + \frac{b}{2} \iint_{\omega\times(0,T)} |\mathbf{v}|^2$
Another interesting optimal control problem: Next Lecture
- The interest of the optimality system:
can be used to introduce iterative algorithms several ways
The "gradient" of J at $\mathbf{v}$ is given by

$$\langle G(\mathbf{v}), \mathbf{v}' \rangle = \iint_{\omega\times(0,T)} (\mathbf{w} + b\mathbf{v}) \cdot \mathbf{v}'$$

where $\mathbf{w}$ solves, together with η and β ...
- Boundary control: similar ideas work (up to technical details)
- Density-dependent fluids and optimal control?

THANK YOU VERY MUCH
SEE YOU TOMORROW AGAIN!