Variable density Navier-Stokes equations 2

Weak and strong solutions

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Equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Initial conditions:

$$\rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}) \quad \text{in} \quad \Omega$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}) \quad \text{in} \quad \Omega$$

Boundary conditions:

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

For the moment . . .

Weak solutions. Global existence

Formulation of the problem and existence result

The basic Hilbert spaces:
$H = \{ \mathbf{v} \in L^2(\Omega)^3 : \nabla \cdot \mathbf{v} \equiv 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$
$V = \{ \mathbf{v} \in H_0^1(\Omega)^3 : \nabla \cdot \mathbf{v} \equiv 0 \}$
One has $V \hookrightarrow H \cong H' \hookrightarrow V'$, $V' \cong H^{-1}(\Omega)^3/\nabla L^2(\Omega)$
Remember: Ω is bounded

Theorem:

Data: $T > 0$, $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^1(0, T; L^2)$
Then $\exists \rho \in L^\infty(Q), \mathbf{u} \in L^2(0, T; V), p \in W^{-1,\infty}(0, T; L^2)$ such that

- $\rho\mathbf{u} \in L^\infty(0, T; L^2) \cap N^{1/4,2}(0, T; W^{-1,3})$, $\inf_\Omega \rho_0 \leq \rho \leq \sup_\Omega \rho_0$
- The variable density NS equations hold in Q (for some p)
- $\rho\,|_{t=0} = \rho_0$ in H^{-1} and $\rho\mathbf{u}\,|_{t=0} = \rho_0\mathbf{u}_0$ in V'

Weak solutions. Global existence

Additional properties and questions:

Behavior of ρ:

$\rho \in L^\infty(Q) \cap C^0([0,T]; L^p(\Omega)) \quad \forall p < +\infty, \quad \rho_t \in L^\infty(0,T; H^{-1}(\Omega))$

$|\{\mathbf{x} \in \Omega : a \leq \rho(\mathbf{x},t) \leq b\}|$ is independent of t $\forall a, b$

DiPerna-PL Lions, renormalized solution

Thus: ρ_0 takes m (or ∞) values $\Rightarrow$ The same for all $\rho(\cdot\,,t)$

Energy inequalitites:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho \mid \mathbf{u} \mid^2 + \mu \int_\Omega \mid \nabla \mathbf{u} \mid^2 \leq \int_\Omega \rho \mathbf{u} \cdot \mathbf{f}$$

$$\frac{1}{2}\int_\Omega \rho \mid \mathbf{u} \mid^2 + \mu \iint_{\Omega\times(0,t)} \mid \nabla \mathbf{u} \mid^2 \leq \frac{1}{2}\int_\Omega \rho_0 \mid \mathbf{u}_0 \mid^2 + \iint_{\Omega\times(0,t)} \rho \mathbf{u} \cdot \mathbf{f}$$

Weak solutions. Global existence

Sketch of the proof (I): semi-Galerkin

A Schauder basis in V, $\{\mathbf{w}^1, \dots, \mathbf{w}^m, \cdots\}$, orthonormal in H
$V^m := [\mathbf{w}^1, \dots, \mathbf{w}^m]$, $\mathbf{u}_0^m \in V^m$ and $\rho_0^m \in C^1(\overline{\Omega})$ such that
$\frac{1}{m} + \inf_\Omega \rho_0 \le \rho_0^m \le \frac{1}{m} + \sup_\Omega \rho_0$, $\mathbf{u}_0^m \to u_0$ in H, $\rho_0^m \to \rho_0$ weakly-$*$ in L^∞

Semi-Galerkin approximation — The m-th problem:

$\rho^m \in C^1(\overline{Q})$, $\mathbf{u}^m \in C^1([0,T]; V^m)$

$$\rho_t^m + \mathbf{u}^m \cdot \nabla \rho^m = 0 \quad \text{in} \quad Q, \qquad \rho^m |_{t=0} = \rho_0^m$$

$$\int_\Omega \{\rho^m(\mathbf{u}_t^m + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^m - \mathbf{f}^m) \cdot \mathbf{v} + \mu \nabla \mathbf{u}^m : \nabla \mathbf{v}\} = 0 \quad \forall \mathbf{v} \in V^m$$

$$\mathbf{u}^m |_{t=0} = \mathbf{u}_0^m$$

An ODE of dimension m coupled to a PDE
EXISTENCE:
For fixed $\mathbf{w}^m$, the PDE for ρ^m with $\mathbf{u}^m = \mathbf{w}^m$ can be easily solved
$\mathbf{w}^m \mapsto \rho^m \mapsto \mathbf{u}^m$ possesses fixed points (estimates below + compactness)

Weak solutions. Global existence

Sketch of the proof (II): uniform estimates

$$\rho^m \text{ is bounded in } L^\infty(Q).$$

Energy estimates ($\mathbf{v} = \mathbf{u}^m$, computations):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^m \mid \mathbf{u}^m \mid^2 + \mu \int_\Omega \mid \nabla \mathbf{u}^m \mid^2 = \int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{f}^m$$

$$\frac{1}{2}\int_\Omega \rho^m \mid \mathbf{u}^m \mid^2 + \mu \iint_{\Omega\times(0,t)} \mid \nabla \mathbf{u}^m \mid^2 \le \frac{1}{2}\int_\Omega \rho_0 \mid \mathbf{u}_0 \mid^2 + \iint_{\Omega\times(0,t)} \rho^m \mathbf{u}^m \cdot \mathbf{f}^m$$

Consequences:

$$\mathbf{u}^m \text{ is bounded in } L^2(0,T;V)$$

$$(\rho^m)^{1/2}\mathbf{u}^m \text{ is bounded in } L^\infty(0,T;L^2) \text{ and } L^2(0,T;L^6)$$

Weak solutions. Global existence

Sketch of the proof (II): uniform estimates

Other estimates:

$$\rho^m_t \text{ is bounded in } L^\infty(0,T;H^{-1}) \text{ and } L^2(0,T;W^{-1,6})$$

$$\begin{cases} \left| \frac{d}{dt}\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v} \right| \leq \; g_m \|\nabla \mathbf{v}\|_{L^2} \quad \forall \mathbf{v} \in V^m \\ \text{with } g_m = C\|\rho^m \mathbf{f}^m\|_{L^2} + C\|\rho^m \mathbf{u}^m \mathbf{u}^m - \mu \nabla \mathbf{u}^m\|_{L^2}, \quad \|g_m\|_{L^1(0,T)} \leq C \end{cases}$$

$$\|\tau_h(\rho^m \mathbf{u}^m) - \rho^m \mathbf{u}^m\|_{L^2(0,T-h;W^{-1,3})} \; \leq C \; h^{1/4}$$

and thus

$$\rho^m \mathbf{u}^m \text{ is bounded in } N^{1/4,2}(0,T;W^{-1,3})$$

Weak solutions. Global existence

Sketch of the proof (II): uniform estimates

All these estimates together with compactness results:

ρ^m is bounded in $L^\infty(Q)$

ρ_t^m is bounded in $L^\infty(0,T;H^{-1}) \cap L^2(0,T;W^{-1,6})$

Hence: $\rho^m \in$ compact set in $C^0([0,T];\tilde{X})$ whenever $L^\infty \hookrightarrow\hookrightarrow \tilde{X}$

$\mathbf{u}^m$ is bounded in $L^2(0,T;V)$

$\rho^m\mathbf{u}^m$ is bounded in $L^\infty(0,T;L^2) \cap L^2(0,T;L^6)$

$\rho^m\mathbf{u}^m$ is bounded in $N^{1/4,2}(0,T;W^{-1,3})$

Hence: $\rho^m\mathbf{u}^m \in$ compact set in $C^0([0,T];\hat{X})$ whenever $L^2 \hookrightarrow\hookrightarrow \hat{X}$

$\rho^m\mathbf{u}^m\mathbf{u}^m$ is bounded in $L^{4/3}(0,T;L^2)$

Weak solutions. Global existence

Sketch of the proof (III): convergent subsequences, taking limits and ...

$\exists$ convergent subsequences

- Taking limits in the equations:
  $\rho^m \mathbf{u}^m \to \chi_1$ in $L^2(0,T;L^6)$ – weak (for instance)
  $\rho^m \mathbf{u}^m \mathbf{u}^m \to \chi_2$ in $L^{4/3}(0,T;L^2)$ – weak
  From compactness: $\chi_1 = \rho \mathbf{u}$ and $\chi_2 = \rho \mathbf{u}\mathbf{u}$ and all terms converge
- Taking limits in the initial conditions:
  $\rho \in C^0([0,T];H^{-1})$ and $\rho^m|_{t=0} \to \rho|_{t=0}$ in H^{-1} – weak
  (in fact: $\rho \in C^0([0,T];\tilde{X})$ whenever $L^\infty \hookrightarrow\hookrightarrow \tilde{X}$)
  For any $\mathbf{v} \in V$, $\int_\Omega \rho \mathbf{u}\mathbf{v} \in C^0([0,T])$ and $\int_\Omega \rho^m \mathbf{u}^m \mathbf{v}|_{t=0} \to \int_\Omega \rho \mathbf{u}\mathbf{v}|_{t=0}$

CONSEQUENCES:

- $\rho \in L^\infty(Q)$, $\mathbf{u} \in L^2(0,T;V)$, etc.
- The boundary and initial conditions are satisfied
- $\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0$
- $(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} - \rho\mathbf{f} \in W^{-1,\infty}(0,T;H^{-1})$ and

$$\langle (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} - \rho\mathbf{f}, \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in V$$

Weak solutions. Global existence

Sketch of the proof (III): convergent subsequences, taking limits and ...

To find p, we apply De Rham's lemma:
If $S \in \mathcal{D}'(\Omega; E)^3$ *and* $\langle S, \varphi \rangle = 0$ *for all* $\varphi \in \mathcal{D}(\Omega)^3$ *with* $\nabla \cdot \varphi \equiv 0$, *then* S *is a gradient:* $S = \nabla q$. *Furthermore:*

$$S \in W^{r,p}(\Omega; E)^3 \Rightarrow q \in W^{r+1,p}(\Omega; E)$$

(A complete proof in [Simon, 2011])

Set $S := (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \mu \Delta \mathbf{u} - \rho \mathbf{f}$
Then $S \in W^{-1,\infty}(0, T; H^{-1}(\Omega)^3) \cong H^{-1}(\Omega; W^{-1,\infty}(0, T))^3$ and

$$\langle S, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in V$$

Hence $S = -\nabla p$ with $p \in L^2(\Omega; W^{-1,\infty}(0, T)) \cong W^{-1,\infty}(0, T; L^2(\Omega))$
Better regularity for p?

Weak solutions. Global existence

More results and questions:

- Of course, a similar 2D result holds, bounded open $\Omega \subset \mathbb{R}^2$
- Extension to density-dependent C^0_b viscosity: $\mu = \mu(\rho)$: OK
  For $\mu = \mu(\rho, p)$? An open question
- Extension to unbounded Ω (and for $\Omega = \mathbb{R}^3$): OK
  Introduce Ω_R, with $\Omega_R \to \Omega$ as $R \to +\infty$ and solve in $\Omega_R \times (0, T)$
  We get: $\|\rho_R\|_{L^\infty(Q_R)} \leq C$, $\inf \rho_0 \leq \rho_R \leq \sup \rho_0$,

$$\|\rho_R \mathbf{u}_R\|_{L^\infty(0,T;L^2(\Omega_R))} + \|\nabla \mathbf{u}_R\|_{L^2(Q_R)} + \|\rho_R \mathbf{u}_R \mathbf{u}_R\|_{L^{4/3}(0,T;L^2(\Omega_R))} \leq C$$

$$\|(\widetilde{\rho}_R)_t\|_{L^\infty(0,T;H^{-1}(\omega))} + \|\widetilde{\rho}_R \widetilde{\mathbf{u}}_R\|_{N^{1/4,2}(0,T;W^{-1,3}(\omega))} \leq C \quad \forall \omega \subset\subset \Omega$$

Sufficient to take limits ...
Other (better) results for $\Omega = \mathbb{R}^3$ (Next Lecture ...)

Weak solutions. Global existence

More results and questions:

- Other boundary conditions? A delicate point

$$\mathbf{u} = \mathbf{a} \quad \text{on} \quad \partial\Omega \times (0, T), \qquad \rho = \overline{\rho} \quad \text{on} \quad \Sigma_{\rm in}$$

  where $\mathbf{a}$ is sufficiently smooth, $\overline{\rho} \geq 0$ and

$$\Sigma_{\rm in} = \{ (\mathbf{x}, t) \in \partial\Omega \times (0, T) : \mathbf{a}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) < 0 \}$$

- The behavior of two immiscible fluids:
  Assume: $\rho_0 = \alpha$ in Ω_1, $\rho_0 = \beta$ in Ω_2 (a partition of Ω), $\mathbf{f} \equiv$ Const.
  Then: identify the asymptotic limit of ρ and $\mathbf{u}$ (stratified system)
  [P-L Lions, Masmoudi, . . .]

Weak solutions. Global existence

Stationary solutions

The problem:

$$\begin{cases} \nabla \cdot (\rho \mathbf{u}) = 0, \quad \nabla \cdot \mathbf{u} = 0 \\ \rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} \\ \mathbf{u}|_{\partial\Omega} = 0 \end{cases}$$

Comments and questions:

- $\exists$ many solutions
- The goal: add a condition that provides uniqueness for large μ (?)
- Something on ρ. The mass distribution function M with

$$M(s) = |\{ \mathbf{x} \in \Omega : \rho(\mathbf{x}) \leq s \}|?$$

($\nearrow$, right-continuous, $M(s) = 0 \ \forall s < 0$, $M(\lambda) = |\Omega|, \lambda > 0$)
An interesting open question

Strong solutions and regularity results

A global-local result for $N = 2/N = 3$

Theorem:

As before, with $0 < \alpha \leq \rho_0 \leq \beta$ a.e., $\mathbf{u}_0 \in V$, $\mathbf{f} \in L^2(Q)$
Then: $\mathbf{u} \in L^2(0, T_*; H^2) \cap C^0([0, T_*]; V)$, $\mathbf{u}_t \in L^2(0, T_*; H)$
Moreover: $T_* = T$ if $N = 2$. Also, $p \in L^2(H^1) \ldots$

Estimates of $\mathbf{u}_t$ for $N = 2$:

$$\begin{aligned}
\|\rho^{1/2}\mathbf{u}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 &\leq C\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq C\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|\|D^2\mathbf{u}\|^{1/2}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq \varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^4 + \varepsilon\|D^2\mathbf{u}\|^2 + C_\varepsilon\|\mathbf{f}\|^2
\end{aligned}$$

Estimates of $\mathbf{u}_t$ for $N = 3$:

$$\begin{aligned}
\|\rho^{1/2}\mathbf{u}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 &\leq C\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq C\|\nabla\mathbf{u}\|^{3/2}\|D^2\mathbf{u}\|^{1/2}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\
&\leq \varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^6 + \varepsilon\|D^2\mathbf{u}\|^2 + C_\varepsilon\|\mathbf{f}\|^2
\end{aligned}$$

Strong solutions and regularity results

A global/local result for $N = 2/N = 3$

Estimates of $D^2\mathbf{u}$ for $N = 2$:

$$-\mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0$$

Regarding this as a Stokes system at each t:

$$\begin{aligned}\|D^2\mathbf{u}\| &\leq C\|\rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t\| \\ &\leq C\|\mathbf{f}\| + \|\mathbf{u}_t\| + C_\delta\|\nabla\mathbf{u}\|^2 + \delta\|D^2\mathbf{u}\|\end{aligned}$$

Estimates of $D^2\mathbf{u}$ for $N = 3$:

Arguing similarly:

$$\begin{aligned}\|D^2\mathbf{u}\| &\leq C\|\rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t\| \\ &\leq C\|\mathbf{f}\| + \|\mathbf{u}_t\| + C_\delta\|\nabla\mathbf{u}\|^3 + \delta\|D^2\mathbf{u}\|\end{aligned}$$

Conclusion:

$$\begin{aligned}&\|\rho^{1/2}\mathbf{u}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 \\ &\qquad \leq C\varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^\sigma + C_\varepsilon\|\mathbf{f}\|^2\end{aligned}$$

where $\sigma = 4$ if $N = 2$ and $\sigma = 6$ if $N = 3$. Therefore ...

Strong solutions and regularity results

Other results and open questions

- Regularity results for $\mu = \mu(\rho)$ and/or $\inf \rho_0 = 0$?
- Minimal hypotheses for (global) regularity of the kind

$$\mathbf{u} \in L^r(0, T; L^s), \quad \frac{2}{r} + \frac{3}{s} \leq 1?$$

- Minimal hypotheses for energy identity?

  All these questions are open

THANK YOU VERY MUCH
SEE YOU TOMORROW!