Variable density Navier-Stokes equations 1

Physical origin and first results and questions

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Preliminaries

The outline:

1. Motivation. First results and questions: Existence, ...
2. Weak and strong solutions, regularity
3. Partial uniqueness — Optimal control problems
4. Other control results — Final comments and questions

Some relevant contributors t the theory:

Khazhikhov, Antontsev, Ladyzhenskaya, J-L Lions, Simon, Salvi, P-L Lions, ...

The main references:

[Panton, 1984], [Zeidler, Vol. 4, 1988], [Chorin-Marsden, 1993], [P-L Lions, 1996], [Braz-EFC-Rojas, 2011] and others ...

Contents for today

1. The fundamental problem in fluid mechanics
   - Hypotheses and goal
   - Trajectories and the transport lemma
   - Universal laws
   - Additional comments

2. Weak solutions. Global existence
   - Formulation of the problem and existence result
   - Additional properties and questions

The fundamental problem in fluid mechanics

Hypotheses and goal — The physicist's viewpoint

ASSUMPTIONS:

1. The physicist viewpoint:
2. A medium fills the points of $\Omega \subset \mathbb{R}^3$ (bounded) during $[0, T]$
3. $\exists$ sufficiently regular $\rho \geq 0$, $\mathbf{u} = (u_1, u_2, u_3)$ and $w > 0$ such that

$$m(W,t) = \int_W \rho(\mathbf{x},t)\,d\mathbf{x}$$

$$\mathbf{p}(W,t) = \int_W (\rho\mathbf{u})(\mathbf{x},t)\,d\mathbf{x}$$

$$E(W,t) = \int_W \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho w\right)(\mathbf{x},t)\,d\mathbf{x}$$

for all measurable $W \subset \Omega$ and t.

THE GOAL:

FPM: *Assume: the mechanical state at $t = 0$ and the physical properties are known. Then, determine the mechanical state for all t.*

The fundamental problem in fluid mechanics
Trajectories and the transport lemma

The trajectories:

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{u}(\mathbf{y}, t) \\ \mathbf{y}(0) = \mathbf{x}. \end{cases} \qquad \mathcal{O} = \{\, (\mathbf{x}, t) : 0 \le t < T_*(\mathbf{x}) \,\}$$

Assume (for simplicity) $\mathcal{O} = \Omega \times [0, T)$
Define $\mathbf{Y} : \mathcal{O} \mapsto \mathbb{R}^N$, with $\mathbf{Y}(\mathbf{x}, t) = \mathbf{y}(t) \quad \forall (\mathbf{x}, t) \in \mathcal{O}$
Define $W_t := \{\, \mathbf{Y}(\mathbf{x}, t) : \mathbf{x} \in W \,\}$ for all $W \subset \Omega$ and all t

Lemma:

Assume: $f = f(\mathbf{x}, t)$ is C^1, $W \subset \Omega$ and $F(t) := \int_{W_t} f(\mathbf{y}, t)\, d\mathbf{y}$
Then: F is C^1 and

$$\frac{dF}{dt}(t) = \int_{W_t} (f_t + \nabla \cdot (f\mathbf{u}))(\mathbf{y}, t)\, d\mathbf{y} \quad \forall t$$

The fundamental problem in fluid mechanics

Universal laws

Mass conservation:

$$\frac{d}{dt}\left(\int_{W_t} \rho(\mathbf{x},t)\,d\mathbf{x}\right) = 0 \quad \forall W, \quad \forall t$$

Linear momentum conservation:

$$\frac{d}{dt}\left(\int_{W_t} (\rho\mathbf{u})(\mathbf{x},t)\,d\mathbf{x}\right) = \mathbf{F}(W_t,t)$$

$\mathbf{F} = \mathbf{F}_{\rm ten} + \mathbf{F}_{\rm ext}$, $\mathbf{F}_{\rm ten} = \int_{\partial W_t} \mathbf{T}(W_t;\mathbf{x},t)\,d\Gamma$ and $\mathbf{F}_{\rm ext} = \int_{W_t} (\rho\mathbf{f})(\mathbf{x},t)\,d\mathbf{x}$
We assume: $\mathbf{T} = \sigma \cdot \mathbf{n}$ for some (unknown) $\sigma = \sigma(\mathbf{x},t)$ and $\mathbf{f}$ is known

Energy conservation:

$$\frac{d}{dt}\left(\int_{W_t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho w\right)\,d\mathbf{x}\right) = P(W_t,t),$$

$P = \int_{\partial W_t} (\mathbf{T}\cdot\mathbf{u})\,d\Gamma + \int_{W_t} (\rho\mathbf{f}\cdot\mathbf{u})\,d\mathbf{x} + \int_{\partial W_t} (-\mathbf{q}\cdot\mathbf{n})\,d\mathbf{x}$
We assume: $\mathbf{q}$ is unknown

The fundamental problem in fluid mechanics

Universal laws - rewritten

Mass:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{in} \quad \Omega \times (0, T)$$

Linear momentum:

$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \sigma + \rho \mathbf{f} \quad \text{in} \quad \Omega \times (0, T)$$

Energy:

$$(\rho w)_t + \nabla \cdot (\rho w \mathbf{u}) = \sigma : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} \quad \text{in} \quad \Omega \times (0, T)$$

$N + 2$ PDE's for ρ, the u_i, the σ_{ij}, the q_i ($N^2 + 2N + 1$ unknowns)

The fundamental problem in fluid mechanics

Constitutive laws – Non-homogeneous incompressible viscous Newtonian fluids

Incompressible non-homogeneous Newtonian viscous fluids:
Newtonian fluid: $\sigma = -p\,\mathbf{Id}. + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t) - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\,\mathbf{Id}.$
Incompressible:

$$\frac{d}{dt}\left(\int_{W_t} d\mathbf{x}\right) = 0 \quad \forall W \subset \Omega, \quad \forall t$$

From the transport lemma and the universal laws:

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

Simplifications: Navier-Stokes, Euler, ...

Incompressible non-homogeneous Newtonian viscous fluids

Variable density Navier-Stokes equations

Equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Initial conditions:

$$\rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}) \quad \text{in} \quad \Omega$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}) \quad \text{in} \quad \Omega$$

Boundary conditions:

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

For the moment . . .

Incompressible non-homogeneous Newtonian viscous fluids

Now: The mathematician's viewpoint

- We do not know whether $\exists$ ρ, $\mathbf{u}$ and p satisfying this
  FIRST TASK: prove this
  A confirmation of the facts that we were not wrong before, the problem was not overdetermined, we did not introduce contradictory hypotheses, etc.
- Even if solutions exist, we do not know how many there are.
  SECOND TASK: prove that the solution is unique.
  This will show that the set of assumptions is complete.
- The solution can be *unstable* with respect to the data.
  THIRD TASK: prove *stability* (continuous dependence)
  This will show that the model is useful and its (numerical) resolution is meaningful
- It may be interesting to interact and get "desirable" solutions
  FOURTH TASK: control the system (for instance through $\mathbf{f}$)
  This will lead to a system with good behavior

Additional comments

Justification of Newton's law

$$\sigma = -p\,\mathbf{Id.} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t) - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})\,\mathbf{Id.}$$

- $\mathbf{F}_{\rm ten}(W_t, t) = \int_{\partial W_t} \mathbf{T}(W_t; \mathbf{x}, t)\, d\Gamma$ by assumption
  Particular flows $\Rightarrow \exists \sigma$ with $\mathbf{T} = \sigma(\mathbf{x}, t)\mathbf{n}$
  Conservation of angular momentum $\Rightarrow \sigma \equiv \sigma^T$
- Viscosity means friction of particles.
  Relevant when close particles have different velocities $\Rightarrow \sigma$ depends on ρ, w and $\nabla \mathbf{u}$
  $\frac{\partial \sigma}{\partial w_{ij}} \equiv \frac{\partial \sigma}{\partial w_{ji}} \quad \forall i, j \Rightarrow \sigma$ depends on ρ, w and $D = \nabla \mathbf{u} + \nabla \mathbf{u}^T$
- Frame invariance $\Rightarrow \sigma(R \cdot D \cdot R^{-1}) \equiv R \cdot \sigma(D) \cdot R^{-1}\ \forall$ orthogonal R
  $\Rightarrow \sigma = a_0\mathbf{Id.} + a_1 D + a_2 D^2$ for some $a_i = a_i(\rho, w, d_1, d_2, d_3)$
  (Rivlin-Erickssen's theorem)
- Assume that $D \mapsto \Sigma(\rho, w, D)$ is affine. Then:
  $\sigma = -p\,\mathbf{Id.} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t) + \lambda(\nabla \cdot \mathbf{u})\,\mathbf{Id.}$
  Particular flows $\Rightarrow 3\lambda + 2\mu = 0$ (Stokes). Hence, ...

Weak solutions. Global existence
Formulation of the problem and existence result

The basic Hilbert spaces:
$H = \{ \mathbf{v} \in L^2(\Omega)^3 : \nabla \cdot \mathbf{v} \equiv 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$
$V = \{ \mathbf{v} \in H_0^1(\Omega)^3 : \nabla \cdot \mathbf{v} \equiv 0 \}$
One has $V \hookrightarrow H \cong H' \hookrightarrow V'$, $V' \cong H^{-1}(\Omega)^3 / \nabla L^2(\Omega)$
Remember: Ω is bounded

Theorem:

Data: $T > 0$, $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^1(0, T; L^2)$
Then $\exists \rho \in L^\infty(Q), \mathbf{u} \in L^2(0, T; V), p \in W^{-1,\infty}(0, T; L^2)$ such that

- $\rho\mathbf{u} \in L^\infty(0, T; L^2) \cap N^{1/4,2}(0, T; W^{-1,3})$ $\inf_\Omega \rho_0 \leq \rho \leq \sup_\Omega \rho_0$ a.e.
- The variable density NS equations are satisfied in Q (for some p)
- $\rho \,|_{t=0} = \rho_0$ in H^{-1} and $\rho\mathbf{u} \,|_{t=0} = \rho_0 \mathbf{u}_0$ in V'

Weak solutions. Global existence

Additional properties and questions:

Behavior of ρ:

$\rho \in L^\infty(Q) \cap C^0([0,T]; L^p(\Omega)) \quad \forall p < +\infty, \quad \rho_t \in L^\infty(0,T; H^{-1}(\Omega))$

$|\{\mathbf{x} \in \Omega : a \le \rho(\mathbf{x},t) \le b\}|$ is independent of t $\forall a, b$

DiPerna-PL Lions, renormalized solution

Thus: ρ_0 takes m (or ∞) values $\Rightarrow$ The same for all $\rho(\cdot\,, t)$

Energy inequalitites:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho \,|\,\mathbf{u}\,|^2 + \mu \int_\Omega |\,\nabla \mathbf{u}\,|^2 \le \int_\Omega \rho \mathbf{u} \cdot \mathbf{f}$$

$$\frac{1}{2}\int_\Omega \rho \,|\,\mathbf{u}\,|^2 + \mu \iint_{\Omega\times(0,t)} |\,\nabla \mathbf{u}\,|^2 \le \int_\Omega \rho \,|\,\mathbf{u}\,|^2 + \iint_{\Omega\times(0,t)} \rho \mathbf{u} \cdot \mathbf{f}$$

Weak solutions. Global existence

More properties and questions:

- What about p? Next Lecture ...
- Of course, a similar 2D result holds, bounded open $\Omega \subset \mathbb{R}^2$
- Extension to density-dependent viscosity: $\mu = \mu(\rho)$ (C^0_b, strictly positive)

  For $\mu = \mu(\rho, p)$? (an open question)
- Extension to unbounded Ω (and for $\Omega = \mathbb{R}^3$). Next Lecture ...
- Other boundary conditions?

$$\mathbf{u} = \mathbf{a} \quad \text{on} \quad \partial\Omega \times (0, T)$$

$$\rho = \overline{\rho} \quad \text{on} \quad \Sigma_{\rm in}$$

where $\mathbf{a}$ is a (sufficiently smooth) vector-valued prescribed function and

$$\Sigma_{\rm in} = \{ (\mathbf{x}, t) \in \partial\Omega \times (0, T) : \mathbf{a}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) < 0 \}$$

Weak solutions. Global existence

More properties and questions:

- Parabolic regularization? The regularized system

$$\begin{cases} (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla p = \mu \Delta \mathbf{u} + \rho \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad (\mathbf{x}, t) \in Q \\ \rho_t + \nabla \cdot (\rho \mathbf{u}) - \varepsilon \Delta \rho = 0, \quad (\mathbf{x}, t) \in Q \\ \mathbf{u} = 0, \quad \partial \rho \mathbf{u} / \partial \mathbf{n} = 0, \quad (\mathbf{x}, t) \in \Sigma \\ \rho|_{t=0} = \rho_0, \quad (\rho \mathbf{u})|_{t=0} = \rho_0 \mathbf{u}_0, \quad \mathbf{x} \in \Omega \end{cases}$$

and then try to take limits as $\varepsilon \to 0^+$
Are solutions obtained this way unique? (a challenging open problem)

Weak solutions. Global existence

More properties and questions:

- Solution in non-cylindrical domains?

$$\begin{cases} (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \mathbf{v}1_\omega, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega(t),\ t\in(0,T) \\ \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \mathbf{x}\in\Omega(t),\ t\in(0,T) \\ \mathbf{u} = 0, \quad \mathbf{x}\in\partial\Omega(t),\ t\in(0,T) \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x}\in\Omega(0) \end{cases}$$

Elliptic regularization? $Q = \{\,(\mathbf{x},t) : \mathbf{x}\in\Omega(t), \quad t\in(0,T)\,\}$; solve

$$\begin{cases} -\varepsilon\mathbf{u}_{tt} - \mu\Delta\mathbf{u} + (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \rho\mathbf{f} \\ -\varepsilon p_{tt} - \varepsilon\Delta p + \nabla\cdot\mathbf{u} = 0 \\ -\varepsilon\rho_{tt} - \varepsilon\Delta\rho + \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0 \\ +\ \text{appropriate boundary conditions for } \rho,\ \mathbf{u} \text{ and } p \text{ on } \partial Q \end{cases}$$

Then: what happens as $\varepsilon\to 0^+$?

A first step to solve free-boundary problems ...

THANK YOU VERY MUCH
SEE YOU TOMORROW!