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Abstract. Viscoelastic solids in Kelvin-Voigt rheology at small strains exhibiting also stress-driven Prandtl-Reuss perfect plasticity are considered quasistatic (i.e. inertia neglected) and coupled with heat-transfer equation through dissipative heat production by viscoplastic effects and through thermal expansion and corresponding adiabatic effects. Enthalpy transformation is used and existence of a weak solution is proved by an implicit suitably regularized time discretisation.

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1. **Introduction.** Plasticity represents both theoretically and industrially important problem. Isothermal models are typically considered *rate-independent*, which is a certain idealisation but well reflects typical hysteretic stress/strain response and simultaneously allows for efficient mathematical analysis. Some processes cannot be considered isothermal. Mathematics supported thermodynamically consistent anisothermal models has been developed rather recently because the relevant L^1-theory for the nonlinear heat equation was developed only in the 1990s in [4, 5], and the mathematical theory for rate-independent processes is even more recent, cf. [8,10,15,16,18,19], as well as the interpolation technique of the adiabatic-heat term in three-dimensional case [21], and the coupling with rate-independent processes with viscous/inertial effects [22] and thermal effects [23].

Conventional plasticity involves isotropic or/and kinematic hardening. Hardening has a regularizing effect and makes mathematics relatively simpler comparing to so-called *perfect plasticity* (=*no hardening*) where shear bands may develop and strains may spatially concentrate and special advanced mathematical tools are needed, in particular the so-called *bounded-deformation spaces* developed by P.-M. Suquet [24]. Perfect plasticity has mathematically been studied in the isothermal case in [1, 6, 8, 9, 11, 25]. The strain concentration typical in perfect plasticity leads to concentrated heat production and, in general, to awkward interactions of concentrating plastic-strain rate with thermal effects. It is, in general, difficult to devise a model that would cover this phenomenon and simultaneously be amenable to mathematical analysis. E.g. the model of plasticity with strain-viscosity cannot be successfully modified for perfect plasticity, cf. Remark 1 below. The key feature of modelling of thermodynamics of perfect plasticity is to involve stress viscosity and also stress-driven perfect plasticity rather than strain viscosity and strain-driven plasticity.

Needless to emphasize, the problem of thermodynamics of perfect plasticity combines a lot of difficult phenomena and this is why the various simplifications must be adopted. Here, in particular, we confine ourselves to *small strains* and *linearized additive plasticity*, and a *linear viscoelastic* response. On the other hand, we allow for a fully *rate-independent plastic* flow rule although, of course, the whole system is necessarily rate dependent due to the heat transfer, and here also due to considered viscous effects. We also neglect kinetic effects, cf. Remark 6 below.

The paper is organised as follows: The equations (or rather also inclusions) and the particular initial/boundary-value problem is formulated in Section 2 together with discussing its energetics and thermodynamics. The problem is then slightly transformed by re-scaling temperature (=a

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so-called enthalpy transformation) and shifting Dirichlet boundary conditions to make them temporarily constant, and a weak solution is defined and main existence result formulated in Section 3, while its proof is performed in Section 4.

2. **The model, its energetics and thermodynamics.** We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \le 3$. The state variables will be the *displacement* $u : \Omega \to \mathbb{R}^d$, the *plastic strain* $\pi : \Omega \to \mathbb{R}^{d\times d}_{\rm dev}$, and the *temperature* $\theta : \Omega \to \mathbb{R}$, where

$$\mathbb{R}^{d\times d}_{\rm dev} := \big\{A\in\mathbb{R}^{d\times d}_{\rm sym};\ {\rm tr}(A) = 0\big\} \quad \text{and} \quad \mathbb{R}^{d\times d}_{\rm sym} := \big\{A\in\mathbb{R}^{d\times d};\ A^\top = A\big\}. \tag{1}$$

The variables π play the role of *internal parameters*. We consider plastic response determined by a convex closed neighbourhood of the origin, say $S \subset \mathbb{R}^{d\times d}_{\rm dev}$, defining an *elasticity domain*, while its boundary is called the *yield surface* and has the meaning of the stress that triggers the evolution of plastic strains. Let δ_S denote its indicator function and δ_S^* the Fenchel-Legendre conjugate functional to δ_S with respect to the inner product $\sigma : e = \sum_{i,j=1}^d \sigma_{ij} e_{ij}$. We remark that the condition $0 \in {\rm int}(S)$ implies that δ_S^* is coercive. Usually, S is considered bounded, which implies δ_S^* to be everywhere finite.

Considering a Kelvin-Voigt-type viscous material, our model will consist of the *equilibrium equation* balancing viscous and elastic stresses,

$$\operatorname{div}\sigma = 0, \qquad \sigma = \sigma_{\rm vi} + \sigma_{\rm el}, \qquad \sigma_{\rm vi} = \mathbb{D}\dot\varepsilon, \qquad \sigma_{\rm el} = \mathbb{C}\varepsilon_{\rm el}, \tag{2a}$$

$$\varepsilon{+}\pi = e(u) := \frac12(\nabla u)^\top{+}\ \frac12\nabla u, \qquad \varepsilon = \varepsilon_{\rm el}{+}\mathfrak{e}, \qquad \mathfrak{e} = \mathbb{E}\theta, \tag{2b}$$

where $\dot\varepsilon = \frac{\partial}{\partial t}\varepsilon$, $\mathbb{D}$ the tensor determining the viscous-type response, $\mathbb{C}$ the tensor determining the elastic response, and $\mathbb{E}$ the thermal expansion tensor; cf. Figure 1. The *evolution of the internal*

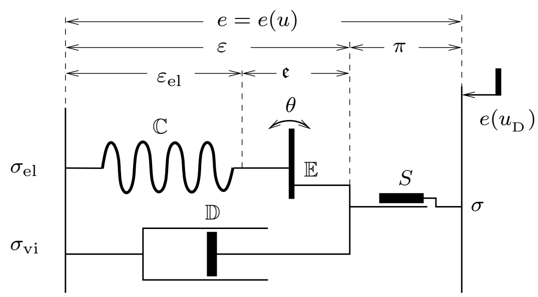

Fig. 1. Schematic rheological model used in (2a-c): thermally-expanding Kelvin-Voigt and perfectly-plastic elements in series.

parameters, i.e. here the plastic strain π, is governed by the inclusion

$$\partial\delta_S^*\big(\dot\pi\big) \ni \operatorname{dev}\sigma, \tag{2c}$$

where $\operatorname{dev}\sigma := \sigma - \sigma^{\rm s}$ is the deviatoric part of σ with $\sigma^{\rm s} = ({\rm tr}\,\sigma/d)\mathbb{I}$ the spherical part of σ. The *heat transfer/production* is governed by the equation

$$c_{\rm v}(\theta)\dot\theta - \operatorname{div}\big(\mathbb{K}(\theta)\nabla\theta\big) = \delta_S^*\big(\dot\pi\big) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \theta\mathbb{C}\mathbb{E}\dot\varepsilon \quad \text{ with } \dot\varepsilon = \mathbb{C}\big(e(\dot u) - \dot\pi\big), \tag{2d}$$

where $c_{\rm v} = c_{\rm v}(\theta)$ is the *heat capacity*, $\mathbb{K} = \mathbb{K}(\theta)$ is the *thermal conductivity* tensor, and "$:$" denotes the product of two $(d{\times}d)$-tensors. Note that we neglected inertial forces, cf. Remark 6 below, and also bulk forces to avoid the so-called safe-load condition usually considered in perfect plasticity, cf. [25, Sect.1.4] or [8, Formulae (2.17)-(2.18)] or [6, Def.6]. Note also that, in view of (2a,b), the stress σ is

$$\sigma = \mathbb{D}(e(\dot u){-}\dot\pi) + \mathbb{C}(e(u){-}\pi) - \mathbb{B}\theta, \qquad \text{where } \mathbb{B} = \mathbb{C}\mathbb{E}, \tag{3}$$

which is the Kelvin-Voigt rheological model in terms of ε, cf. again Figure 1.

Throughout this paper, we assume *isotropic material* with the symmetric positive definite fourth order tensors $\mathbb{C} = [\mathbb{C}_{ijkl}]$, $\mathbb{D} = [\mathbb{D}_{ijkl}]$, i.e.

$$\mathbb{C}_{ijkl} = \lambda_\mathrm{e}\delta_{ij}\delta_{kl} + \mu_\mathrm{e}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),$$
$$\mathbb{D}_{ijkl} = \lambda_\mathrm{v}\delta_{ij}\delta_{kl} + \mu_\mathrm{v}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \quad \text{with } \mu_\mathrm{e}, \mu_\mathrm{v} > 0,\ \lambda_\mathrm{e} > -2\mu_\mathrm{e}/d,\ \lambda_\mathrm{v} > -2\mu_\mathrm{v}/d, \tag{4a}$$
$$\mathbb{E}_{ij} = \alpha\delta_{ij}, \quad \text{hence } \ \mathbb{B}_{ij} = \alpha(d\lambda_\mathrm{e}{+}2\mu_\mathrm{e})\delta_{ij}, \tag{4b}$$

with δ denoting here the Kronecker symbol, λ's and μ's are the Lamé constants, α the thermal-expansion coefficient. Thus the elastic stress is $\mathbb{C}\varepsilon = \lambda_\mathrm{e}\mathrm{tr}(\varepsilon_\mathrm{el})\mathbb{I} + 2\mu_\mathrm{e}\varepsilon_\mathrm{el}$ with $\mathbb{I} = [\delta_{ij}]$ denoting the unit matrix, and corresponding energy is $\frac{1}{2}\mathbb{C}\varepsilon_\mathrm{el}{:}\varepsilon_\mathrm{el} = \frac{1}{2}\lambda_\mathrm{e}|\mathrm{tr}(\varepsilon_\mathrm{el})|^2 + \mu_\mathrm{e}|\varepsilon_\mathrm{el}|^2$ and, as a quadratic form of ε_el, it is positive definite, and similarly also the quadratic form $\dot\varepsilon \mapsto \frac{1}{2}\mathbb{D}\dot\varepsilon{:}\dot\varepsilon$ is positive definite. In fact, the essential condition we will rely on is

$$\mathrm{dev}\,\mathbb{B} = 0, \tag{5}$$

which is satisfied for the particular choice (4b).

Using the identity $[\partial\delta_S^*]^{-1} = \partial([\delta_S^*]^*) = \partial\delta_S^{**} = \partial\delta_S$, the inclusion (2c) can equivalently be written in a form which is more standard in engineering literature, namely

$$\dot\pi \in N_S(\mathrm{dev}\,\sigma) \tag{6}$$

where $N_S = \partial\delta_S$ is the normal cone to S. Note also that, due to (5), the driving stress σ in the flow rule (2c) (or now in (6) too) can effectively be replaced by $\sigma_\mathrm{vi} + \sigma_\mathrm{el}$ with exactly the same effect.

The above equations/inclusion (2) are to hold on the space/time domain $Q := (0,T)\times\Omega$ with $T > 0$ a fixed time horizon.

We consider an initial-boundary-value problem for the system (2). Hence, we take the initial conditions

$$u(0,\cdot) = u_0, \qquad \pi(0,\cdot) = \pi_0, \qquad \theta(0,\cdot) = \theta_0. \tag{7}$$

As we focus on processes in the bulk, we consider only the simplest boundary conditions, namely a prescribed normal stress and heat flux on $\Gamma := \partial\Omega$:

$$\sigma\nu = 0 \qquad \text{on } \Gamma_{\rm N}, \tag{8a}$$
$$u = u_{\rm D} \qquad \text{on } \Gamma_{\rm D}, \tag{8b}$$
$$\big(\mathbb{K}(\theta)\nabla\theta\big)\cdot\nu = f \qquad \text{on } \Gamma, \tag{8c}$$

where "$\cdot$" denotes the scalar product of two vectors and ν is the outward unit normal to Γ.

The energetics of the model (2) can be obtained by testing (2a) and (2c) respectively by the shifted velocity $\dot u{-}\dot u_{\rm D}$ (which has zero traces on $\Gamma_{\rm D}$) and by the plastic strain rate $\dot\pi$, which gives after using Green's formula for (2a) together with the boundary conditions (8) and eventually by summation the *mechanical energy balance*

$$\int_\Omega \delta_S^*(\dot\pi) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \frac{1}{2}\frac{\partial}{\partial t}\mathbb{C}\varepsilon{:}\varepsilon\,\mathrm{d}x = \int_\Omega \mathbb{B}\theta{:}\big(\dot\varepsilon - e(\dot u_{\rm D})\big) + \big(\mathbb{D}\dot\varepsilon{+}\mathbb{C}\varepsilon\big){:}e(\dot u_{\rm D})\,\mathrm{d}x. \tag{9}$$

Testing further (2d) by 1 and using Green's formula gives, when summing with (9), the *total energy balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon + C_\mathrm{v}(\theta)\,\mathrm{d}x}_{\substack{\text{stored and heat parts}\\ \text{of the internal energy}}} = \underbrace{\int_\Omega \sigma{:}e(\dot u_{\rm D})\,\mathrm{d}x}_{\substack{\text{power of the}\\ \text{external loading}}} + \underbrace{\int_\Gamma f\,\mathrm{d}S}_{\substack{\text{power of the}\\ \text{external heating}}} \tag{10}$$

where

$$C_\mathrm{v}(\theta) := \int_0^\theta c_\mathrm{v}(r)\,\mathrm{d}r\,; \tag{11}$$

thus C_v is a primitive function to c_v normalized such that $C_\mathrm{v}(0) = 0$.

One can derive *thermodynamics* of the above model by postulating the Helmholtz *free energy* ψ_0 as $\psi_0(\varepsilon_{\rm el},\theta) = \frac{1}{2}\mathbb{C}\varepsilon_{\rm el}{:}\varepsilon_{\rm el} - \phi(\theta)$ with $\varepsilon_{\rm el} = \varepsilon_{\rm el}(\varepsilon,\theta) = \varepsilon - \theta\mathbb{E}$. By substituting for $\varepsilon_{\rm el}$, we also denote

$$\psi(\varepsilon,\theta) = \psi_0(\varepsilon_{\rm el}(\varepsilon,\theta),\theta) = \frac{1}{2}\mathbb{C}(\varepsilon{-}\theta\mathbb{E}){:}(\varepsilon{-}\theta\mathbb{E}) - \phi(\theta). \tag{12}$$

Then *entropy* is given by

$$s = s(\varepsilon,\theta) := -\psi'_\theta(\varepsilon,\theta) = \phi'(\theta) - \mathbb{E}\mathbb{B}\theta + \mathbb{B}\varepsilon. \tag{13}$$

Note that the thermo-mechanically coupling terms in (12) are linear in terms of θ, so that the mechanical variables separate from temperature in (13) and thus

$$c_{\rm v} = c_{\rm v}(\varepsilon,\theta) := \theta\psi''_\theta(\varepsilon,\theta) = \theta\phi''(\theta) - \mathbb{E}\mathbb{B}\theta \tag{14}$$

in (2d) does not depend on these mechanical variables, which facilitates the analysis of the heat equation considerably.

We further introduce dissipative quantities involving also the variable $g\in\mathbb{R}^d$ as a place-holder for the quantity driving the heat transfer, i.e. here considered as $\nabla\theta$. Namely, we put

$$\zeta(\theta;\dot\pi,\dot\varepsilon,g) := \delta^*_S(\dot\pi) + \frac{1}{2}\mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \frac{1}{2}\mathbb{K}(\theta)g{\cdot}g \quad = \text{potential of dissipation}, \tag{15}$$

$$\xi(\dot\pi,\dot\varepsilon) = \big\langle\partial_{(\dot\pi,\dot\varepsilon)}\zeta(\theta;\dot\pi,\dot\varepsilon,g),(\dot\pi,\dot\varepsilon)\big\rangle = \delta^*_S(\dot\pi) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon \quad = \text{mechanical dissipation rate}, \tag{16}$$

$$\begin{aligned}\varrho(\theta;\dot\pi,\dot\varepsilon,g) &= \frac{\zeta(\theta;\dot\pi,\dot\varepsilon,g)}{\theta}\\ &= \frac{\delta^*_S(\dot\pi)}{\theta} + \frac{\mathbb{D}\dot\varepsilon{:}\dot\varepsilon}{2\theta} + \frac{\mathbb{K}(\theta)}{2\theta^2}g{\cdot}g \quad = \text{potential of entropy-production rate},\end{aligned} \tag{17}$$

$$\begin{aligned}\rho(\theta;\dot\pi,\dot\varepsilon,\theta,g) &= \big\langle\partial_{(\dot\pi,\dot\varepsilon,g)}\varrho(\theta;\dot\pi,\dot\varepsilon,g),(\dot\pi,\dot\varepsilon,g)\big\rangle = \frac{\xi(\dot\pi,\dot\varepsilon)}{\theta} + \frac{\mathbb{K}(\theta)g{\cdot}g}{\theta^2}\\ &= \frac{\delta^*_S(\dot\pi)}{\theta} + \frac{\mathbb{D}\dot\varepsilon{:}\dot\varepsilon}{\theta} + \frac{\mathbb{K}(\theta)}{\theta^2}g{\cdot}g \quad = \text{entropy production rate}.\end{aligned} \tag{18}$$

The equation (2d) itself can be derived from a so-called *entropy equation*

$$\theta\dot s = \xi(\dot\pi,\dot\varepsilon) + {\rm div}\, j \qquad \text{with} \qquad j = \zeta'_g(\theta;\nabla\theta) = \mathbb{K}(\theta)\nabla\theta, \tag{19}$$

which further takes, in general, the form of the *heat-transfer equation*

$$\theta\psi''_{\theta\theta}(\theta)\dot\theta - {\rm div}\big(\mathbb{K}(\theta)\nabla\theta\big) = \xi - \theta\psi''_{\theta\varepsilon}(\varepsilon,\theta){:}\dot\varepsilon. \tag{20}$$

Counting (12) and (14), we arrive to (2d).

At least formally, assuming positivity of temperature and $f\geq 0$ and positive-definiteness of $\mathbb{K}$, and realizing that always $\xi\geq 0$, from (19) we can see the *Clausius-Duhem inequality*

$$\frac{\rm d}{{\rm d}t}\int_\Omega s\,{\rm d}x = \int_\Omega {\rm div}\Big(\mathbb{K}\frac{\nabla\theta}{\theta}\Big) + \frac{\mathbb{K}\nabla\theta{\cdot}\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\,{\rm d}x = \int_\Omega \rho(\theta;\dot\pi,\dot\varepsilon,\theta,\nabla\theta)\,{\rm d}x + \int_\Gamma\frac{f}{\theta}\,{\rm d}S \geq 0. \tag{21}$$

Note also that the combination of (4a), (4b), and the orthogonality $\mathbb{B}{:}\pi = \alpha(d\lambda_{\rm e}{+}2\mu_{\rm e}){\rm tr}(\pi) = 0$, cf. (5), allows us to write the free energy from (12) in the more specific form as a function of (e,π,θ):

$$\hat\psi(e,\pi,\theta) := \psi(e{-}\pi,\theta) = \frac{\lambda_{\rm e}}{2}\big|{\rm tr}(e{-}\pi)\big|^2 + \mu_{\rm e}\big|e{-}\pi\big|^2 - \alpha\big(d\lambda_{\rm e}{+}2\mu_{\rm e}\big)\theta\,{\rm tr}(e) - \phi(\theta). \tag{22}$$

In terms of ε and θ, using (12) and (13), the *internal energy* given by $\psi{+}\theta s$ results to

$$\begin{aligned}\psi + \theta s &= \frac{1}{2}\mathbb{C}(\varepsilon{-}\theta\mathbb{E}){:}(\varepsilon{-}\theta\mathbb{E}) - \phi(\theta) + \theta\big(\phi'(\theta) - \mathbb{E}\mathbb{B}\theta + \mathbb{B}\varepsilon\big)\\ &= \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon - \phi(\theta) + \theta\phi'(\theta) - \frac{1}{2}\mathbb{E}\mathbb{B}\theta^2 = \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon + C_{\rm v}(\theta) - \phi(0),\end{aligned} \tag{23}$$

which is, up to a constant $\phi(0)$, the quantity occurring already in (10).

Remark 1 (Strain driven plasticity)**.** Some models, e.g. [2, 3, 13, 23], consider viscous behaviour imposed on $e(u)$ instead of $\varepsilon = e(u) - \pi$ and then naturally only $\sigma_{\rm el}$ as the driving stress for the plastic flow rule instead of $\sigma_{\rm el} + \sigma_{\rm vi}$ considered in (6). This is amenable to (relatively) simpler mathematical analysis but, on the other hand, does not allow for developing slip bands. More specifically, slip bands could develop only for infinitesimally slow processes which, however, suppress also thermo-coupling effects. Thus this alternative rheology smears out the most characteristic phenomenon for the perfect plasticity. Most engineering treatments do not focus on rigorous mathematical support and often do not consider any viscosity, hence the difference between these two models is smeared out.

Remark 2 (Doubly-nonlinear evolution)**.** The mechanical system (2a-c) possesses a simple structure as a doubly-nonlinear evolution in terms of $(u,\pi,\varepsilon,\theta)$ as

$$\partial\hat\zeta(\dot u,\dot\pi,\dot\varepsilon)+\partial_{(u,\pi,\varepsilon)}\hat\psi(u,\pi,\varepsilon,\theta)\ni 0 \tag{24}$$

with $\hat\zeta(\dot u,\dot\pi,\dot\varepsilon):=\zeta(\dot\pi,\dot\varepsilon)$ and with

$$\hat\psi(u,\pi,\varepsilon,\theta):=\begin{cases}\psi(\varepsilon,\theta) & \text{if } C(u,\pi,\varepsilon)=0,\\ \infty & \text{otherwise,}\end{cases} \tag{25}$$

where $C(u,\pi,\varepsilon):=e(u)-i_d\pi-\varepsilon$ is considered valued in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and i_d denotes merely the embedding $\mathbb{R}^{d\times d}_{\rm dev}\to\mathbb{R}^{d\times d}_{\rm sym}$. One can formally evaluate $\partial\hat\psi(u,\pi,\varepsilon,\theta)=\partial\psi(\varepsilon,\theta)+C^*\sigma$ where C^* is the adjoint operator to C, i.e. here $C^*\sigma=(\operatorname{div}\sigma,\,\sigma,\,\operatorname{dev}\sigma)$, and $\sigma\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ is the Lagrange multiplier to the holonomic constraint $C(u,\pi,\varepsilon)=0$. The particular three components of (24) can be identified successively with $\operatorname{div}\sigma=0$ in (2b), with (2c), and with (3).

Remark 3 (Entropy equation revisited)**.** The potential $\varrho(\theta;\dot\pi,\dot\varepsilon,\cdot)$ from (17) governs the entropy flux, let us denote it by i, and the entropy equation (19) can be thus written as balance of entropy:

$$\dot s=\rho(\theta;\dot\pi,\dot\varepsilon,\nabla\theta)+\operatorname{div} i \qquad \text{with}\quad i=\varrho_g'(\theta;\dot\pi,\dot\varepsilon,\nabla\theta)=\frac{j}{\theta} \tag{26}$$

with the heat flux j from (19) and the entropy production rate ρ from (18). Integrating it over Ω, we can again recover (21).

3. **Enthalpy transformation, weak solutions, main results.** It is desirable to allow for a certain growth of $c_{\rm v}(\cdot)$ if we have the viscosity in the form $\mathbb{D}e(\frac{\partial u}{\partial t})$ in order to be able to treat the adiabatic term, cf. [21]. On the other hand, the technique from [21] specifically relies on Galerkin's method and does not seem directly transferable if also the time discretisation is involved, which is in turn needed both for designing a fully discrete scheme and for efficient treatment of the rate-independent flow rule. The particular difficulty is in establishing the limit of a time-discretisation of the nonlinear term $c_{\rm v}(\theta)\dot\theta$. Therefore, we first write the original system (2) in terms of enthalpy instead of temperature, using so-called *enthalpy transformation*

$$w=C_{\rm v}(\theta) \tag{27}$$

with $C_{\rm v}(\theta):=\int_0^\theta c_{\rm v}(r)\,{\rm d}r$ defined already in (11). Further, we define

$$\Theta(w):=C_{\rm v}^{-1}(w),\qquad \mathscr{K}(w):=\frac{\mathbb{K}(\Theta(w))}{c_{\rm v}(\Theta(w))},\qquad \mathscr{B}(w):=\Theta(w)\mathbb{B}, \tag{28}$$

where $C_{\rm v}^{-1}$ here denotes the inverse function to $C_{\rm v}$; here we use a natural feature that $c_{\rm v}>0$ so $C_{\rm v}$ is increasing and its inverse does exist.

We handle the Dirichlet condition by the additive shift, i.e. instead of the original u, we consider $u+u_{\rm D}$ with a suitable prolongation of $u_{\rm D}$ inside the domain; in particular, without any loss of generality, we will assume that

$$\big(\mathbb{D}e(\dot u_{\rm D})+\mathbb{C}e(u_{\rm D})\big)\nu=0 \qquad \text{on }\Gamma_{\rm N}, \tag{29}$$

which will simplify some formulae below. In terms of the enthalpy w and the shifted displacement, denoted again by u, the system (2) writes as

$$\operatorname{div}\sigma+f_{\rm D}=0, \qquad \sigma=\mathbb{D}\dot\varepsilon+\mathbb{C}\varepsilon-\mathscr{B}(w), \tag{30a}$$

$$\varepsilon=e(u)-\pi, \tag{30b}$$

$$f_{\rm D}=\operatorname{div}\sigma_{\rm D}, \tag{30c}$$

$$\partial\delta_S^*(\dot\pi)\ni\operatorname{dev}(\sigma+\sigma_{\rm D}), \qquad \sigma_{\rm D}=\mathbb{D}e(\dot u_{\rm D})+\mathbb{C}e(u_{\rm D}), \tag{30d}$$

$$\dot w-\operatorname{div}\big(\mathscr{K}(w)\nabla w\big)=\delta_S^*(\dot\pi)+\big(\mathbb{D}\dot\varepsilon+\mathbb{D}e(\dot u_{\rm D})+\mathscr{B}(w)\big){:}\big(\dot\varepsilon+e(\dot u_{\rm D})\big). \tag{30e}$$

We will call (30e) shortly the *enthalpy equation* rather than the heat-transfer equation in the enthalpy formulation. The boundary conditions (8) transform to

$$u = 0 \quad \text{on } \Gamma_{\mathrm{D}}, \qquad\qquad \sigma\nu = \sigma_{\mathrm{D}}\nu \quad \text{on } \Gamma_{\mathrm{N}}, \qquad\qquad \big(\mathscr{K}(w)\nabla w\big)\cdot\nu = f \quad \text{on } \Gamma, \tag{31a}$$

while the initial conditions (7) transform into

$$u(0,\cdot) = u_0, \qquad\qquad \pi(0,\cdot) = \pi_0, \qquad\qquad w(0,\cdot) = C_{\mathrm{v}}(\theta_0) \qquad \text{on } \Omega. \tag{31b}$$

The "shifted" system (30) exhibits naturally "shifted" energetics as it plays with a "shifted" displacement. The mechanical energy balance can be seen by testing (30a) by $\dot u$ and (30d) by $\dot\pi$, which modifies (9) as

$$\int_\Omega \delta_S^*(\dot\pi) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \frac12\frac{\partial}{\partial t}\mathbb{C}\varepsilon{:}\varepsilon \,\mathrm{d}x = \int_\Omega \big(\mathscr{B}(w){-}\mathbb{D}e(\dot u_{\mathrm{D}}){-}\mathbb{C}e(u_{\mathrm{D}})\big){:}\dot\varepsilon \,\mathrm{d}x. \tag{32}$$

Testing further (30e) by 1 and using Green's formula gives, when summing with (32) modifies the total energy balance (10) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac12\mathbb{C}\varepsilon{:}\varepsilon + w\,\mathrm{d}x = \int_\Omega &\big(\mathbb{D}e(\dot u_{\mathrm{D}}){+}\mathscr{B}(w)\big){:}e(\dot u_{\mathrm{D}})\\ &-\big(\mathbb{D}e(\dot u_{\mathrm{D}}){+}\mathbb{C}e(u_{\mathrm{D}})\big){:}\dot\varepsilon\,\mathrm{d}x + \int_\Gamma f\,\mathrm{d}S.\end{aligned} \tag{33}$$

This energy balance allows for obtaining a-priori estimates under some occasions. Assuming

$$\mathbb{B}{:}e(\dot u_{\mathrm{D}}) = 0, \tag{34}$$

integrating (33) over a time interval $[0,t]$ and using the by-part integration, one obtains

$$\begin{aligned}\int_\Omega \frac12\mathbb{C}\varepsilon(t){:}\varepsilon(t) + w(t)\,\mathrm{d}x = \int_0^t\!\bigg(\int_\Omega &\mathbb{D}e(\dot u_{\mathrm{D}}){:}e(\dot u_{\mathrm{D}})\\ -\big(\mathbb{D}e(\ddot u_{\mathrm{D}}){+}\mathbb{C}e(\dot u_{\mathrm{D}})\big){:}\varepsilon\,\mathrm{d}x &+ \int_\Gamma f\,\mathrm{d}S\bigg)\mathrm{d}t + \int_\Omega \frac12\mathbb{C}\varepsilon_0{:}\varepsilon_0 + C_{\mathrm{v}}(\theta_0)\\ -\big(\mathbb{D}e(\dot u_{\mathrm{D}}(t)){+}\mathbb{C}e(u_{\mathrm{D}}(t))\big){:}\varepsilon(t)&{+}\big(\mathbb{D}e(\dot u_{\mathrm{D}}(0)){+}\mathbb{C}e(u_{\mathrm{D}}(0))\big){:}\varepsilon_0\,\mathrm{d}x,\end{aligned} \tag{35}$$

where naturally $\varepsilon_0 = e(u_0){-}\pi_0$. Assuming u_{D} a "gentle" loading in the sense that, beside (34), also $u_{\mathrm{D}} \in W^{2,1}(I;W^{1,2}(\Omega;\mathbb{R}^d))$ and $f \in L^1(I;L^1(\Gamma))$ and also qualification of the initial conditions so that the last integral is finite, by the Hölder and the Young and the Gronwall inequalities, one can obtain the a-priori estimates of the type

$$\|\varepsilon\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}))} \quad \text{and} \quad \|w\|_{L^\infty(I;L^1(\Omega))} \qquad \text{bounded.} \tag{36}$$

This is an important departing ingredient for derivation of other apriori estimates, cf. Proposition 2 below.

We consider an evolution in the time interval $I := (0,T)$ with a fixed time horizon $T > 0$ and denote $Q := (0,T)\times\Omega$, $\Sigma := (0,T)\times\partial\Omega$, and $\bar I := [0,T]$. We will use a standard notation for function spaces, namely the space of the continuous $\mathbb{R}^k$-valued functions $C(\bar\Omega;\mathbb{R}^k)$, its dual $\mathscr{M}(\bar\Omega;\mathbb{R}^k)$ (i.e., up to an isometric isomorphism, the space of Borel measures), the continuously differentiable functions $C^1(\bar\Omega;\mathbb{R}^k)$, the Lebesgue space $L^p(\Omega;\mathbb{R}^k)$, the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^k)$, and the Bochner space of X-valued Bochner measurable p-integrable functions $L^p(I;X)$. If $X = (X')^*$, the notation $L^\infty_{\mathrm{w*}}(I;X)$ stands for space of weakly* measurable functions $I \to X$. Moreover, we denote by $B(\bar I;X)$, $B_{\mathrm{w*}}(\bar I;X)$, $\mathrm{BV}(\bar I;X)$ or $C_{\mathrm{w}}(\bar I;X)$ the Banach space of functions $\bar I \to X$ that are bounded Bochner measurable, bounded weakly* measurable, have a bounded variation or are weakly continuous, respectively; note that all these functions are defined everywhere on $\bar I$. We will use the notation $q' = q/(q{-}1)$ for the conjugate exponent to q.

In addition to (4a) which guarantees that $\mathbb{C}$ and $\mathbb{D}$ are positive definite, we will assume throughout this article that

$$c_{\rm v}:[0,+\infty)\to\mathbb{R}^+\ \ \text{continuous,} \tag{37a}$$

$$\exists\,\omega_1{\ge}\omega{>}\max\Big(1,\frac{2d}{d{+}2}\Big),\ c_1{\ge}c_0{>}0\ \forall\theta{\in}\mathbb{R}^+\!:\ \ c_0(1{+}\theta)^{\omega-1}\le c_{\rm v}(\theta)\le c_1(1{+}\theta)^{\omega_1-1}, \tag{37b}$$

$$\mathscr{K}{:}\mathbb{R}\to\mathbb{R}^{d\times d}\ \ \text{bounded, continuous, and}\quad \inf_{(w,\xi)\in\mathbb{R}\times\mathbb{R}^d,\ |\xi|=1}\mathscr{K}(w)\xi{\cdot}\xi>0; \tag{37c}$$

with $\mathscr{K}$ from (28). As far as the loading concerns, the basic natural qualification is:

$$u_{\rm D}\in W^{2,1}(I;W^{1,2}(\Omega;\mathbb{R}^d)),\quad \operatorname{div}\dot u_{\rm D}=0, \tag{38a}$$

$$f\in L^1(\Sigma),\quad f\ge0, \tag{38b}$$

$$u_0\in W^{1,1}(\Omega;\mathbb{R}^d) \tag{38c}$$

$$\pi_0\in L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}),\quad e(u_0){-}\pi_0\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}), \tag{38d}$$

$$\theta_0\in L^\omega(\Omega),\quad \theta_0\ge0, \tag{38e}$$

where we denoted $\Sigma:=I\times\Gamma$ in (38b). Note that $\operatorname{div}\dot u_{\rm D}=0$ ensures (34) when taking (4b) into account.

The following definition of a certain sort of a weak solution has been devised in [23], based on the concept of so-called energetic solution invented by Mielke et al. [10,15,18,19] for the theory of rate independent processes and adapted also for coupling with viscous/inertial effects in [22]. We refer to [23, Proposition 3.2] for justification (and not entirely obvious fact) that this definition is indeed selective in the sense that, under an additional absolute continuity of $\dot\pi$, it gives indeed a conventional notion of a weak solution. (For an isothermal situation, cf. also [22, Proposition 5.2].) It should be however emphasized that this additional regularity of $\dot\pi$ hardly can be expected due to the fully rate-independent flow rule, which just makes the devised concept properly fitted with this problem.

We will define the space of functions with bounded deformations and satisfying the Dirichlet boundary conditions (31a) by

$$\mathrm{BD}(\Omega;\mathbb{R}^d){:=}\big\{u{\in}L^1(\Omega;\mathbb{R}^d);\ \ e(u){\in}\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big\} \tag{39}$$

and moreover we define the space of admissible pairs (u,π) satisfying also the Dirichlet boundary conditions (31a) by

$$\begin{aligned}\mathfrak{Q}:=\big\{&(u,\pi)\in\mathrm{BD}(\Omega;\mathbb{R}^d){\times}\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev});\\ &e(u){-}\pi|_\Omega{\in}L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}),\ \ u\,{\rm Ⓢ}\,\nu{\rm d}S{+}\,\pi|_{\Gamma_{\rm D}}=0\ \text{ on }\Gamma_{\rm D}\big\},\end{aligned} \tag{40}$$

where $a\,{\rm Ⓢ}\,b$ means the symmetrized tensorial product $\frac12(a\otimes b+b\otimes a)$.

Definition 1. (Energetic solution.) Assuming (4a) and (38), we call a triple (u,π,w) with

$$u\in B(\bar I;\mathrm{BD}(\Omega;\mathbb{R}^d)), \tag{41a}$$

$$\pi\in B(\bar I;\mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})), \tag{41b}$$

$$\varepsilon=e(u)-\pi\in W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})), \tag{41c}$$

$$w\in L^1(I;W^{1,1}(\Omega)), \tag{41d}$$

$$\dot w\in\mathscr{M}(\bar I;W^{1+d,2}(\Omega)^*) \tag{41e}$$

an energetic solution to the problem (30)–(31) if the following five conditions hold:

(i) the weakly formulated momentum-equilibrium equation (30a) with (31a) holds, i.e. for all $v{\in}C^1(\bar Q;\mathbb{R}^d)$ such that $v|_{\Sigma_0}{=}0$,

$$\int_Q\big(\mathbb{D}\dot\varepsilon+\mathbb{C}\varepsilon-\mathscr{B}(w)\big){:}e(v)\,{\rm d}x{\rm d}t=-\int_Q\sigma_{\rm D}{:}e(v)\,{\rm d}x{\rm d}t, \tag{42a}$$

with $\sigma_{\rm D}$ from (30d).

(ii) the weakly formulated enthalpy equation (30e) with (52c) holds, i.e. for all $v{\in}C^1(\bar Q)$ with $v(T)=0$,

$$\int_Q \mathscr{K}(w)\nabla w{\cdot}\nabla v - w\dot v - \big(\mathscr{B}(w) + \mathbb{D}\dot\varepsilon + \mathbb{D}e(\dot u_{\rm D})\big){:}\big(\dot\varepsilon{+}e(\dot u_{\rm D})\big)v\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{\bar Q} v\,\mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_\Omega w_0 v(0)\,\mathrm{d}x + \int_\Sigma f v\,\mathrm{d}S\mathrm{d}t \tag{42b}$$

where $w_0 = C_{\rm v}(\theta_0)$ and $\mathfrak{h}_\pi$ is a measure (=heat produced by rate-independent dissipation) defined by prescribing its values for every closed set of the type $A := [t_1, t_2]{\times}B$ with B a Borel subset of $\bar\Omega$ by

$$\mathfrak{h}_\pi(A) := \mathrm{Var}_{\delta^*_S}\big(\pi|_B; t_1, t_2\big) \quad \text{with}$$
$$\mathrm{Var}_{\delta^*_S}(\pi; t_1, t_2) := \sup\sum_{i=1}^k \int_\Omega \delta^*_S\big(\pi(s_i,x){-}\pi(s_{i-1},x)\big)\,\mathrm{d}x \tag{42c}$$

where the supremum is taken over all partitions $t_1{\le}s_0{<}...{<}s_k{\le}t_2$, $k{\in}\mathbb{N}$,

(iii) the total energy equality holds, i.e.

$$\int_\Omega \frac12\mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_{\bar\Omega} w(T,\mathrm{d}x) = \int_\Omega\frac12\mathbb{C}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x + \int_\Omega C_{\rm v}(\theta_0)\,\mathrm{d}x$$
$$+ \int_\Sigma f\,\mathrm{d}S\mathrm{d}t + \int_Q \mathbb{D}e(\dot u_{\rm D}){:}e(\dot u_{\rm D}){+}\mathbb{C}e(u_{\rm D}){:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t, \tag{42d}$$

(iv) the "semistability" holds for any $\tilde u \in \mathrm{BD}(\Omega;\mathbb{R}^d)$ and $\tilde\pi \in \mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ such that $\tilde\varepsilon := e(\tilde u){-}\tilde\pi \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and for a.a. $t\in[0,T]$, i.e.

$$\int_\Omega\frac12\mathbb{C}\varepsilon(t){:}\varepsilon(t) + \mathfrak{s}(t){:}\varepsilon(t)\mathrm{d}x \le \int_\Omega\frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \mathfrak{s}(t){:}\tilde\varepsilon\mathrm{d}x + \int_{\bar\Omega}\delta^*_S(\cdot)\big[\tilde\pi{-}\pi(t)\big](\mathrm{d}x), \tag{42e}$$

with the "partial stress" $\mathfrak{s}(t) := \mathbb{D}\dot\varepsilon(t){-}\mathscr{B}(w(t)){+}\sigma_{\rm D}(t).$ (42f)

(v) the initial conditions $u(0) = u_0$ and $\pi(0) = \pi_0$ hold and if the boundary condition $u(t)\circledS\nu\mathrm{d}S{+}\pi(t)|_{\Gamma_{\rm D}} = 0$ holds on $\Gamma_{\rm D}$ for all t.

It should be emphasized that the energy balance (42d) arises from (33) but relies on the qualification of the loading regime (34) so that it does not contain the term $\mathscr{B}(\theta){:}e(\dot u_{\rm D})$. Also the semistability (42e) deserves some comments. By the degree-1 homogeneity of δ^*_S and (30d), one has $\mathrm{dev}(\sigma{+}\sigma_{\rm D}) \in \partial\delta^*_S(\dot\pi) \subset \delta^*_S(0)$ so that $0 = \delta^*_S(0) \le \delta^*_S(\tilde\pi) - \mathrm{dev}(\sigma{+}\sigma_{\rm D}){:}\tilde\pi$. Realizing that

$$(\sigma{+}\sigma_{\rm D}){:}\tilde\pi = \mathrm{dev}(\sigma{+}\sigma_{\rm D}){:}\mathrm{dev}\,\tilde\pi + (\sigma{+}\sigma_{\rm D})^{\rm s}{:}\tilde\pi^{\rm s} = \mathrm{dev}(\sigma{+}\sigma_{\rm D}){:}\tilde\pi \tag{43}$$

because the spherical part $\tilde\pi^{\rm s} = \frac1d(\mathrm{tr}\,\tilde\pi)\mathbb{I} = 0$ (so that $\mathrm{dev}\,\tilde\pi = \tilde\pi - \tilde\pi^{\rm s} = \tilde\pi$), and substituting $\tilde\pi{-}\pi(t)$ instead of $\tilde\pi$, we further obtain

$$-\big(\sigma(t){+}\sigma_{\rm D}(t)\big){:}\pi(t) \le \delta^*_S(\tilde\pi{-}\pi(t)) - (\sigma(t){+}\sigma_{\rm D}(t)){:}\tilde\pi. \tag{44}$$

Testing $\sigma(t) \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ by $\pi(t) \in \mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ is not convenient (even if $\mathrm{dev}\,\sigma \in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, cf. Remark 4 below) and thus we rather add to (44) integrated over Ω also the equilibrium equation (30a) tested by $\tilde u - u$, i.e. $\int_\Omega(\sigma(t) - \sigma_{\rm D}(t)){:}e(\tilde u(t){-}u(t))\mathrm{d}x = 0$. It gives

$$\int_\Omega \big(\sigma(t){+}\sigma_{\rm D}(t)\big){:}\varepsilon(t)\mathrm{d}x \le \int_\Omega \delta^*_S(\tilde\pi{-}\pi(t)) + \big(\sigma(t){+}\sigma_{\rm D}(t)\big){:}\tilde\varepsilon\mathrm{d}x. \tag{45}$$

Further we use the specific form (3) of the stress σ and the positive-definiteness of $\mathbb{C}$, implying $\mathbb{C}\varepsilon{:}(\varepsilon{-}\tilde\varepsilon) \ge \frac12\mathbb{C}\varepsilon{:}\varepsilon - \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon$. Thus we eventually arrive to (42e). Like in [22], an important fact is that such a semi-stability together with energy balance can, in regular cases, recover the flow rule (30d), so that Definition 1 does not loose selectivity. Perhaps even more important attribute of this semistability concept is that it still keeps the ability to serve satisfactorily for proving preservation of energy which, in turn, is intimately needed for accomplishing the limit passage in the heat equation, cf. (83) below.

Note also that (41e) makes values of $\vartheta(t)$ well defined in the sense of $W^{1+d,2}(\Omega)^*$ and (41d) further shows that even $\vartheta(t) \in \mathscr{M}(\bar\Omega)$, which has been exploited in (42d) for the time $t = T$. It should be emphasized that $t \mapsto \vartheta(t)$ cannot be expected to be continuous in any sense because, since δ^*_S is homogeneous degree-1, the measure $\mathfrak{h}_\pi$ may concentrate at particular time instances.

It is not surprising that we will be able to prove existence of energetic solution only for suitably qualified initial conditions (u_0, π_0, w_0) that can be interpreted as some sort of a "gentle" start. Standardly, we will assume that the triple (u_0, π_0, w_0) is semistable at $t = 0$, which is needed in (79) below. This needs here, however, a special care because $\dot\varepsilon(t)$ occurring in $\mathfrak{s}(t)$ is well defined only for a.a. t and not just for $t = 0$. One option how to overcome this trouble is to assume $\pi_0 = 0$ and to guarantee even $\pi(t,\cdot) = 0$ for all $t \in [0, t_0]$ with some $t_0 > 0$ because then the expression $\int_\Omega \mathfrak{s}(0){:}\tilde\varepsilon\,\mathrm{d}x$, needed only for $\tilde\varepsilon = \varepsilon(t) = e(u(t))$ for $t \in [0, t_0]$ in (79), is well defined from (42a), being equal $\int_\Omega \mathfrak{s}(0){:}\tilde\varepsilon\,\mathrm{d}x = \int_\Omega \mathfrak{s}(0){:}e(u(t))\,\mathrm{d}x = \int_\Omega(\mathbb{D}\dot\varepsilon(0){+}\mathbb{C}\varepsilon_0{-}\mathscr{B}(w_0)){:}e(u(t))\,\mathrm{d}x = -\int_\Omega \sigma_{\rm D}(0){:}e(u(t))\,\mathrm{d}x$. The other option of a "gentle" start, allowing for $\pi_0 \neq 0$, is to guarantee $e(u(t,\cdot)){-}\pi(t,\cdot) = 0$ for all $t \in [0, t_0]$. Then obviously $\dot\varepsilon(0) = 0$ since $\varepsilon(t,\cdot) = 0$ for all $t \in [0, t_0]$, and $\mathfrak{s}(0) = \sigma_{\rm D}(0){-}\mathscr{B}(w_0)$ is obviously well defined. We will consider this second option by assuming, for some $t_0 > 0$, that:

$$u_0 \in W^{1,\infty}(\Omega;\mathbb{R}^d), \qquad \pi_0 \in L^2(\Omega;\mathbb{R}^{d\times d}), \tag{46a}$$

$$w_0 \geq 0 \text{ constant on } \Omega, \quad \sigma_{\rm D}(t) \text{ constant on } [0, t_0], \quad f = 0 \text{ on } [0, t_0]{\times}\Gamma \tag{46b}$$

$$\begin{aligned}\int_\Omega \frac12\mathbb{C}\varepsilon_0{:}\varepsilon_0 + \tau|\pi_0|^2\mathrm{d}x \leq \int_\Omega \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \tau|\tilde\pi|^2 \\ + (\sigma_{\rm D}(0){-}\mathscr{B}(w_0)){:}(\tilde\varepsilon{-}\varepsilon_0)\,\mathrm{d}x + \int_{\bar\Omega}\delta_S^*(\cdot)\big[\tilde\pi{-}\pi_0\big](\mathrm{d}x),\end{aligned} \tag{46c}$$

with $\varepsilon_0 = e(u_0){-}\pi_0$ and $\tilde\varepsilon = e(\tilde u){-}\tilde\pi$ for all $\tilde u \in \mathrm{BD}(\Omega;\mathbb{R}^d)$ and $\tilde\pi \in \mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ such that $\tilde\varepsilon := e(\tilde u){-}\tilde\pi \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and for all $0 \leq \tau < t_0$. Note that, by (46a), we also strengthened (38c,d) and, by considering a possible small "isotropic-like hardening" τ, we strengthened the usual semistability (42e) at $t = 0$ with the intention rather to simplify the existence proof below.

The other, rather implicit and technical assumption formulated in [1] is that, for all $t \in [0, T]$ and all $(u, \pi) \in \mathfrak{Q}$ the following holds:

$$\left.\begin{array}{l}\text{If } E(t,u,\pi) \leq E(t,u{+}\hat u,\pi{+}\hat\pi) + R(\hat\pi) \text{ for all } (\hat u,\hat\pi) \in \mathfrak{Q}_0,\\ \text{then } E(t,u,\pi) \leq E(t,u{+}\tilde u,\pi{+}\tilde\pi) + R(\tilde\pi) \text{ for all } (\tilde u,\tilde\pi) \in \mathfrak{Q},\end{array}\right\} \tag{47}$$

where we abbreviated $E(t,u,\pi) := \int_\Omega(\frac12\mathbb{C}\varepsilon + \mathfrak{s}(t)){:}\varepsilon\,\mathrm{d}x$ with $\varepsilon = e(u){-}\pi$ and $R(\dot\pi) := \int_{\Omega\cup\Gamma_{\rm D}}\delta_S^*(\cdot)\mathrm{d}\dot\pi(x)$, and where $\mathfrak{Q}$ is from (40) and

$$\mathfrak{Q}_0 := \big\{(u,\pi){\in}W^{1,1}(\Omega;\mathbb{R}^d){\times}L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev});\ u{=}0 \text{ on } \Gamma_{\rm D},\ e(u){-}\pi{\in}L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big\}.$$

It was proved in [8] that the condition (47) is satisfied for a certain special $\mathbb{C}$ (in particular for $\mathbb{C}$ from (4)) and is Γ and the boundary of $\Gamma_{\rm D}$ is smooth. Anyhow, it was conjectured in [1] that (47) holds more generally.

Theorem 1. *Let (4), (37), (38a,b), (46), and (47) hold. Then the problem (30)–(31) has an energetic solution (u, π, w) in accord to Definition 1 satisfying, beside (41), also*

$$w \in L^r(I; W^{1,r}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap B_{\rm w*}(\bar I; \mathscr{M}(\bar\Omega)) \tag{48}$$

with any $1 \leq r < (d{+}2)/(d{+}1)$.

Scenario of the proof. We will use the implicit semi-discretisation in time (=the so-called Rothe method) suitably regularised, cf. (50)–(52) below, prove existence of such a discrete solution in Lemma 4.1, a-priori estimates in Propositions 1–2, and convergence in Proposition 3. An important fact is that the assumptions (54) used for the discrete scheme can always be satisfied without imposing any additional requirement on the original data. $\square$

4. **Time discretisation of the system** (30)–(31)**.** To prove stability of any discrete scheme needed for possible convergence, one must inevitably deal with executing fine a-priori estimates. This is not entirely easy in coupled systems with super-linear growth of nonmonotone terms, as it is typically the case of thermodynamically consistent continuum-mechanical problems. It has to be done essentially in two steps: first the physical internal energy is to be estimated uniformly in time using (and proving) also non-negativity of the enthalpy, and, from this, some additional finer estimates by using Gagliardo-Nirenberg interpolation several times. In this second step, one

estimates especially the gradient of enthalpy and also total dissipated energy, cf. also [21] and, for the plasticity, especially [23]. Further important phenomenon here is that, proving existence of a solution, we need to pass to the limit in the non-linear Nemytskiĭ operators induced by the dissipation heat ξ. Another peculiarity is that, due to degree-1 homogeneity of δ_S^*, the heat equation has its right-hand side not only in $L^1(Q)$ (as it would be in case of higher-degree homogeneity of dissipative-force potential) but even in measures. For this, the key trick is to recover the exact energy balance in the limit, cf. (83) below.

Therefore, a design of a convergent numerical scheme is technically rather delicate. Following and modifying ideas from [23], we will use a *fully implicit time-discretisation* with a constant time-step $\tau > 0$, assuming $K_\tau = T/\tau \in \mathbb{N}$ and defining the *backward difference operator* by

$$\mathrm{D}_t\phi^k := \frac{\phi^k - \phi^{k-1}}{\tau} \tag{49}$$

for any sequence $\{\phi^k\}_{k\in\mathbb{N}\cup\{0\}}$, combined with a *regularization* of the momentum equation and of the flow rule, using as a parameter just the time-step $\tau > 0$. The term $\tau\pi_\tau^k$ in (50b) below is in a position of a kinematic hardening but with a vanishing coefficient like in [1] but here controlled directly by the time step τ. The purpose of these regularizing terms in (50a) and (50b) below is to compensate the superlinear growth of the right-hand-side terms in the heat equation if γ is chosen large enough. More specifically, we consider the following recursive increment formula

$$\mathrm{div}\big(\sigma_\tau^k + \sigma_{\mathrm{D},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_\tau^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_\tau^k\big) = 0, \tag{50a}$$

$$\partial\delta_S^*\big(\mathrm{D}_t\pi_\tau^k\big) + \tau\pi_\tau^k \ni \mathrm{dev}\big(\sigma_\tau^k + \sigma_{\mathrm{D},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_\tau^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_\tau^k\big), \tag{50b}$$

$$\mathrm{D}_t w_\tau^k - \mathrm{div}\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big) = \delta_S^*\big(\mathrm{D}_t\pi_\tau^k\big) + \big(\mathbb{D}\mathrm{D}_t\big(\varepsilon_\tau^k{+}e_{\mathrm{D},\tau}^k\big) + \mathscr{B}(w_\tau^k)\big){:}\mathrm{D}_t\big(\varepsilon_\tau^k{+}e_{\mathrm{D},\tau}^k\big), \tag{50c}$$

$$\text{with}\quad \varepsilon_\tau^k = e(u_\tau^k){-}\pi_\tau^k, \qquad \sigma_\tau^k = \mathbb{D}\mathrm{D}_t\varepsilon_\tau^k{+}\mathbb{C}\varepsilon_\tau^k{-}\mathscr{B}(w_\tau^k), \tag{50d}$$

for $k = 1, ..., K_\tau = T/\tau$, where

$$\sigma_{\mathrm{D},\tau}^k(t,x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}\sigma_{\mathrm{D}}(t,x)\,\mathrm{d}t \quad\text{and}\quad e_{\mathrm{D},\tau}^k(t,x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}e(u_{\mathrm{D}}(t,x))\,\mathrm{d}t, \tag{51}$$

with the corresponding boundary conditions

$$u_\tau^k = 0 \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{52a}$$

$$\big(\sigma_\tau^k + \sigma_{\mathrm{D},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_\tau^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_\tau^k\big)\nu = 0 \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{52b}$$

$$\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big)\cdot\nu = f_\tau^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}\tilde{f}_\tau(t,x)\,\mathrm{d}t \qquad \text{on } \Gamma, \tag{52c}$$

starting for $k = 1$ by using

$$u_\tau^0 = u_0, \qquad \pi_\tau^0 = \pi_0, \qquad w_\tau^0 = w_0. \tag{53}$$

Note that, in (51), we regularized the boundary flux f by $\tilde{f}_\tau$. We can and will assume:

$$\tilde{f}_\tau{\in}L^\infty(\Sigma), \qquad \tilde{f}_\tau \geq 0, \qquad \tilde{f}_\tau(t,\cdot) = 0 \quad \text{for } t \in [0,t_0], \tag{54a}$$

$$\lim_{\tau\downarrow 0}\sqrt{\tau}\big\|\tilde{f}_\tau\big\|_{L^2(I;L^{4/3}(\Gamma))} = 0 \quad\text{and}\quad \lim_{\tau\downarrow 0}\tilde{f}_\tau = f \text{ in } L^1(\Sigma) \tag{54b}$$

with t_0 occurring already in (46b).

Let us define the piecewise affine interpolants u_τ, π_τ, w_τ, σ_τ, and ε_τ by

$$\begin{aligned}\big[u_\tau,\pi_\tau,w_\tau,\sigma_\tau,\varepsilon_\tau\big](t) := {}& \frac{t-(k{-}1)\tau}{\tau}\big(u_\tau^k,\pi_\tau^k,w_\tau^k,\sigma_\tau,\varepsilon_\tau\big)\\ &+\frac{k\tau-t}{\tau}\big(u_\tau^{k-1},\pi_\tau^{k-1},w_\tau^{k-1},\sigma_\tau^{k-1},\varepsilon_\tau^{k-1}\big) \ \text{ for } \ t\in[(k{-}1)\tau,k\tau]\end{aligned} \tag{55}$$

with $k = 0, ..., K_\tau := T/\tau$. Besides, we define also the back-ward piecewise constant interpolants $\bar{u}_\tau$, $\bar{\pi}_\tau$, $\bar{w}_\tau$, $\bar{\sigma}_\tau$, and $\bar{\varepsilon}_\tau$ by

$$\big[\bar{u}_\tau,\bar{\pi}_\tau,\bar{w}_\tau,\bar{\sigma}_\tau,\bar{\varepsilon}_\tau\big](t) := \big(u_\tau^k,\pi_\tau^k,w_\tau^k,\sigma_\tau^k,\varepsilon_\tau^k\big) \qquad \text{for } \ (k{-}1)\tau < t \leq k\tau \tag{56}$$

with $k = 1, ..., K_\tau$. Similarly, we will later use u_τ, $\bar{u}_\tau$, etc. We will also use the notation $\bar{g}_\tau$ and $\bar{f}_\tau$ defined by $\bar{g}_\tau|_{((k-1)\tau,k\tau]} = g_\tau^k$ and $\bar{f}_\tau|_{((k-1)\tau,k\tau]} = f_\tau^k$ for $k = 1, ..., K_\tau$.

In terms of the interpolants, the scheme (50) can be written more lucidly as

$$\mathrm{div}\big(\bar\sigma_\tau+\overline\sigma_{\mathrm{D},\tau}+\tau\big|\dot\varepsilon_\tau\big|^{\gamma-2}\dot\varepsilon_\tau\big)=0, \tag{57a}$$

$$\partial\delta_S^*\big(\dot\pi_\tau\big)+\tau\bar\pi_\tau\ni\mathrm{dev}\big(\bar\sigma_\tau+\overline\sigma_{\mathrm{D},\tau}+\tau\big|\dot\varepsilon_\tau\big|^{\gamma-2}\dot\varepsilon_\tau\big), \tag{57b}$$

$$\dot w_\tau-\mathrm{div}\big(\mathscr{K}(\bar w_\tau)\nabla\bar w_\tau\big)=\delta_S^*\big(\dot\pi_\tau\big)+\big(\mathbb{D}\dot\varepsilon_\tau+\mathbb{D}e(\dot u_{\mathrm{D},\tau})+\mathscr{B}(\bar w_\tau)\big){:}\big(\dot\varepsilon_\tau+e(\dot u_{\mathrm{D},\tau})\big), \tag{57c}$$

$$\varepsilon_\tau=e(u_\tau)-\pi_\tau,\qquad \bar\sigma_\tau=\mathbb{D}\dot\varepsilon_\tau+\mathbb{C}\bar\varepsilon_\tau-\mathscr{B}(\bar w_\tau),\qquad \bar\varepsilon_\tau=e(\bar u_\tau)-\bar\pi_\tau. \tag{57d}$$

Lemma 4.1 (Existence and estimates of discrete solutions)**.** *Let (4), (37), (38), and (54) hold. Moreover, let*

$$\gamma>\max\Big(4,\frac{2\omega}{\omega{-}1}\Big). \tag{58}$$

Then there exists a weak solution $(u_\tau^k,\pi_\tau^k,w_\tau^k)\in W^{1,\gamma}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\times W^{1,2}(\Omega)$ *for the system (50). Moreover, there is some solution which additionally satisfies* $w_\tau^k\ge0$ *and also* $\mathrm{D}_t\varepsilon_\tau^1=0$ *for* $\tau>0$ *small enough.*

Note that, in (37b), we assume $\omega>1$ so that (58) allows for $\gamma<+\infty$.

Sketch of the proof of Lemma 4.1. We can see existence of a solution to (50) by a standard argument for coercive pseudomonotone set-valued operators; cf. e.g. [12] for a general concept or, here, [20, Sect.5.3] for inclusions with pseudomonotone operators whose set-valued part has a convex potential. The coercivity of the underlying operator can be shown by testing (50a), (50b), and (50c) by u_τ^k, π_τ^k, and $\delta_\tau w_\tau^k$, respectively. It is important that the right-hand sides of (50a,c) have the growth that can be dominated by the growth of the coercive terms in the left-hand sides if $\delta_\tau>0$ is taken sufficiently small; this is ensured by having taken γ large enough and by the assumption (37b) which ensures a sublinear growth of Θ and thus also of $\mathscr{B}$, namely

$$\big|\mathscr{B}(w)\big|\le|\mathbb{B}|\Big(\frac{\omega w}{c_0}{+}1\Big)^{1/\omega}-|\mathbb{B}|\le|\alpha|\big(d\lambda_{\mathrm{e}}{+}2\mu_{\mathrm{e}}\big)\Big(\frac{\omega w}{c_0}\Big)^{1/\omega} \tag{59}$$

because obviously $C_{\mathrm{v}}(\theta)\ge c_0((1{+}\theta)^\omega{-}1)/\omega$, cf. the definition (28) and (4b). Realize that the sum of the left-hand sides of (50) can be estimated (up to an additive constant) from below by

$$\tau|\varepsilon|^\gamma+\tau|\pi|^2+\delta_\tau|w|^2. \tag{60}$$

This indeed dominates the growth of the "right-hand-side terms" is of the type $|w|^{1/\omega}|\varepsilon|+|\pi|\,|w|+|\varepsilon|^2|w|+|w|^{1+1/\omega}|\varepsilon|$. More in detail, the heat-production δ_S^*-term can be estimated as

$$\delta_S^*\Big(\frac{\pi{-}\pi_\tau^{k-1}}{\tau}\Big)\delta_\tau w\le\frac1\tau K\delta_\tau\big|\pi{-}\pi_\tau^{k-1}\big|\,|w|\le\frac{K}{2\tau}\delta_\tau^{1/2}\big|\pi{-}\pi_\tau^{k-1}\big|^2+\frac{K}{2\tau}\delta_\tau^{3/2}|w|^2$$

with $K=\sup_{|\dot\pi|\le1}\delta_S^*(\dot\pi)$, and then absorbed in the left-hand side (60) if $\delta_\tau<4\min(\tau^4,\tau^2)/K^2$; note that $K<\infty$ because $\mathrm{int}(S)\ni0$. Similarly $|\varepsilon|^2|w|\le\delta|\varepsilon|^\gamma+\delta|w|^2+C_\delta$ with any $\delta>0$ and some C_δ; here $\gamma>4$ has been used. The last term can be estimated as $|w|^{1+1/\omega}|\varepsilon|\le\delta|\varepsilon|^\gamma+\delta^{-1/(\gamma-1)}|w|^{(1+1/\omega)\gamma/(\gamma-1)}\le\delta|\varepsilon|^\gamma+\delta|w|^2+C_\delta$ for arbitrary $\delta>0$ and some $C_\delta\in\mathbb{R}$; here the condition $\gamma>2\omega/(\omega{-}1)$ has originated.

To prove $w_\tau^1\ge0$, we test (50c) by $(w_\tau^1)^-$ (which belongs to $W^{1,2}(\Omega)$ and is therefore a legal test function) and use $f\ge0$ and $\theta_0\ge0$ (so that $w_\tau^0=w_0=C_{\mathrm{w}}^{-1}(\theta_0)\ge0$, too) and prove $(w_\tau^1)^-=0$ a.e. on Ω. Then we proceed recursively for $k=2,...,T/\tau$.

By using (46) with (54), we can see that $(u_\tau^k,\pi_\tau^k,w_\tau^k)=(u_0,\pi_0,w_0)$ is a solution to (50) for k and τ small enough. In particular, it holds for $k=1$ and thus $\mathrm{D}_t\varepsilon_\tau^1=0$ for $\tau>0$ small enough. $\square$

Note that (50c) has the right-hand side in $L^2(\Omega)$ since $\gamma\ge4$ and since $\pi_\tau^k-\pi_\tau^{k-1}$ is certainly in $L^2(\Omega;\mathbb{R}^{d\times d})$, hence the weak formulation of (50c) is understood standardly. In what follows, we will, of course, consider only the non-negative solutions w_τ^k whose existence we already proved.

Proposition 1 (A-priori estimates)**.** *Under the assumptions from Lemma 4.1, for some C and C_τ, it holds*

$$\big\|e(u_\tau){-}\pi_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C, \tag{61a}$$
$$\big\|\bar w_\tau\big\|_{L^\infty(I;L^1(\Omega))}\le C, \tag{61b}$$
$$\big\|e(\dot u_\tau){-}\dot\pi_\tau\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d}_{\rm sym})}\le C\tau^{-1/\gamma}, \tag{61c}$$
$$\big\|\bar\pi_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C\tau^{-1/2}. \tag{61d}$$

Proof. Taking $(u_\tau^k,\pi_\tau^k,w_\tau^k)$ solving (50), we can test (50a) and (50b) respectively by ${\rm D}_t u_\tau^k$ and ${\rm D}_t\pi_\tau^k$. Summing it up, we obtain

$$\int_\Omega \delta_S^*({\rm D}_t\pi_\tau^k)+\tau\pi_\tau^k{:}{\rm D}_t\pi_\tau^k \\ +\big(\mathbb{D}{\rm D}_t\varepsilon_\tau^k{+}\mathbb{C}\varepsilon_\tau^k{+}\sigma_{_{\rm D},\tau}^k{-}\mathscr{B}(w_\tau^k){+}\tau\big|{\rm D}_t\varepsilon_\tau^k\big|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}{\rm D}_t\varepsilon_\tau^k\,{\rm d}x=0. \tag{62}$$

By using the convexity of the underlying regularized stored energy

$$\Phi_\tau(u,\pi):=\int_\Omega\frac12\mathbb{C}(e(u){-}\pi){:}(e(u){-}\pi)+\frac\tau2|\pi|^2\,{\rm d}x, \tag{63}$$

and by summation over time steps, we obtain for any $k=1,...,K_\tau$, the following "discrete mechanical energy" balance:

$$\Phi_\tau\big(u_\tau^k,\pi_\tau^k\big)+\tau\sum_{l=1}^k\int_\Omega\delta_S^*\big({\rm D}_t\pi_\tau^l\big)+\mathbb{D}{\rm D}_t\varepsilon_\tau^l{:}{\rm D}_t\varepsilon_\tau^l+\tau\big|{\rm D}_t\varepsilon_\tau^l\big|^\gamma\,{\rm d}x \\ \le\Phi_\tau\big(u_0,\pi_0\big)+\tau\sum_{l=1}^k\int_\Omega\big(\mathscr{B}(w_\tau^l)+\sigma_{_{\rm D},\tau}^l\big){:}{\rm D}_t\varepsilon_\tau^l\,{\rm d}x, \tag{64}$$

cf. (32). Now, we still add (50c) tested by 1 and summed over time steps to (64); here it is important that the dissipative/adiabatic terms mutually cancel in the mechanical and the thermal parts. Thus, using also (38a) for $\mathscr{B}(w_\tau^l){:}e_{_{\rm D},\tau}^l=0$, we get the following "discrete total energy" balance:

$$\Phi_\tau\big(u_\tau^k,\pi_\tau^k\big)+\int_\Omega w_\tau^k\,{\rm d}x+\tau^2\sum_{l=1}^k\int_\Omega\big|{\rm D}_t\varepsilon_\tau^l\big|^\gamma\,{\rm d}x\le\Phi_\tau\big(u_0,\pi_0\big)+\int_\Omega w_0\,{\rm d}x \\ +\tau\sum_{l=1}^k\bigg(\int_\Omega\big(\mathbb{D}{\rm D}_te_{_{\rm D},\tau}^l{+}\mathbb{C}e_{_{\rm D},\tau}^l\big){:}{\rm D}_t\varepsilon_\tau^l+\mathbb{D}{\rm D}_te_{_{\rm D},\tau}^l{:}{\rm D}_te_{_{\rm D},\tau}^l\,{\rm d}x+\int_\Gamma f_\tau^l\,{\rm d}S\bigg), \tag{65}$$

cf. (33). Then we execute a discrete analog of (35), namely the by-part summations $\sum_{l=1}^k\mathbb{D}{\rm D}_te_{_{\rm D},\tau}^l{:}{\rm D}_t\varepsilon_\tau^l=\mathbb{D}{\rm D}_te_{_{\rm D},\tau}^k{:}\varepsilon_\tau^k-\sum_{l=2}^k\mathbb{D}{\rm D}_t^2e_{_{\rm D},\tau}^l{:}\varepsilon_\tau^{l-1}-\mathbb{D}{\rm D}_te_{_{\rm D},\tau}^1{:}\varepsilon_\tau^0$ and $\sum_{l=1}^k\mathbb{C}e_{_{\rm D},\tau}^l{:}{\rm D}_t\varepsilon_\tau^l=\mathbb{C}e_{_{\rm D},\tau}^k{:}\varepsilon_\tau^k-\sum_{l=2}^k\mathbb{C}{\rm D}_te_{_{\rm D},\tau}^l{:}\varepsilon_\tau^{l-1}-\mathbb{C}e_{_{\rm D},\tau}^1{:}\varepsilon_\tau^0$ and the discrete Gronwall inequality. From (65) thus one gets all the estimates (61). $\square$

Proposition 2 (Further a-priori estimates)**.** *Under the above assumptions, for some C and C_r, it holds*

$$\big\|e(u_\tau)\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C, \tag{66a}$$
$$\big\|{\rm div}\,u_\tau\big\|_{W^{1,2}(I;L^2(\Omega))}\le C, \tag{66b}$$
$$\big\|\pi_\tau\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C, \tag{66c}$$
$$\big\|e(u_\tau)-\pi_\tau\big\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C, \tag{66d}$$
$$\big\|\bar w_\tau\big\|_{L^r(I;W^{1,r}(\Omega))}\le C_r\quad\text{with any }1\le r<\tfrac{d+2}{d+1}, \tag{66e}$$
$$\big\|\dot w_\tau\big\|_{L^1(I;W^{1+d,2}(\Omega)^*)}\le C, \tag{66f}$$
$$\big\|\sigma_\tau\big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})}\le C. \tag{66g}$$

Sketch of the proof. We modify [23, proof of Proposition 4.2]; in contrast to [23], here we have the loading arising from shift-transformation of the Dirichlet conditions and especially the viscosity in terms of the elastic stress ε instead of $e(u)$, cf. again Figure 1.

Then one uses the L^1-theory for the evolutionary heat equation [4, 5] based on the test by $1-1/(1+\bar w_\tau)^\delta$, $\delta>0$, combined with the interpolation of the adiabatic term by using several-times Gagliardo-Nirenberg inequality as in [21, 23], which eventually allows us to bound the dissipation, yielding (66c) and (66d), and to bound the enthalpy gradient, yielding (66e). From (66c) and (66d) one gets also (66a).

As $\operatorname{tr}\dot\pi=0$, (66d) implies that $\operatorname{div}\dot u_\tau=\operatorname{tr}e(\dot u_\tau)=\operatorname{tr}(e(\dot u_\tau){-}\dot\pi)=\operatorname{tr}\dot\varepsilon_\tau$ bounded in $L^2(Q)$, i.e. (66b).

Testing (57c) by test functions from $L^\infty(I;W^{1+d,2}(\Omega))$ and using the already obtained estimates (61b) and (66d,e), one obtains (66f).

Eventually, interpolating (61b) and (66e) and using (37b), we can see that $\mathscr{B}(\bar w_\tau)$ is bounded in $L^2(Q)$ and then from (66d) we obtain (66g). $\square$

Proposition 3 (Convergence for $\tau{\downarrow}0$)**.** *Let $d\le 3$, let the assumptions (4), (37), (38), (47), (54), and (58) hold. Then there is a subsequence of $\{(u_\tau,\pi_\tau,w_\tau)\}_{\tau>0}$ weakly* converging to some (u,π,w) in the topologies indicated in (61a,b) and (66), and the stresses converges strongly, i.e.*

$$\sigma_\tau\to\sigma=\mathbb{D}(e(\dot u){-}\dot\pi){+}\mathbb{C}(e(u){-}\pi){-}\mathscr{B}(w)\qquad\text{strongly in }L^2(Q;\mathbb{R}^{d\times d}_{\rm sym}).\tag{67}$$

Moreover, any (u,π,w) obtained in this way is an energetic solution of the problem (30) with the initial/boundary conditions (31) according Definition 1.

The weak* topologies mentioned in Proposition 3 are meant, of course, in suitably extended spaces because (66a,c,f) involves L^1-spaces on which weak* topology is not defined at all. As to (66f), we consider $\mathscr{M}(\bar I;W^{1+d,2}(\Omega)^*)$ rather than $L^1(I;W^{1+d,2}(\Omega)^*)$, as used already in (41e). As to (66a,c), we enlarge $\mathscr{M}(\bar I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))$ and $\mathscr{M}(\bar I;L^1(\Omega))$ to the Borel measures $\mathscr{M}(\bar I{\times}\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ and $\mathscr{M}(\bar I{\times}\bar\Omega)$ so that the rate of plastic deformation $\dot\pi_\tau$ is a-priori bounded in $C(\bar I{\times}\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})^*$.

Proof of Proposition 3. First, by Banach's and Helly's selection principles, we select a weakly* convergent subsequence. In particular,

$$(\bar u_\tau,\bar\pi_\tau)\to(u,\pi)\qquad\text{weakly* in }L^\infty(I;{\rm BD}(\Omega;\mathbb{R}^d){\times}\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})),\tag{68a}$$
$$\bar\varepsilon_\tau=e(\bar u_\tau){-}\bar\pi_\tau\to\varepsilon=e(u){-}\pi\qquad\text{weakly* in }L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),\tag{68b}$$
$$\bar w_\tau\to w\qquad\text{strongly in }L^r(Q),\qquad 1\le r<\frac{d{+}2}{d},\tag{68c}$$
$$\varepsilon_\tau(t)\to\varepsilon(t)\qquad\text{weakly in }L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\qquad\forall t\in\bar I,\tag{68d}$$
$$\bar w_\tau(t)\to w(t)\qquad\text{weakly* in }\mathscr{M}(\bar\Omega)\qquad\forall t\in\bar I,\tag{68e}$$
$$\bar\pi_\tau(t)\to\pi(t)\qquad\text{weakly* in }\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})\qquad\forall t\in\bar I.\tag{68f}$$

The last four convergences are due to a-priori bounds uniform in time and a-priori bounds on time derivatives. Also

$$\bar u_\tau(t)\to u(t)\qquad\text{weakly* in }{\rm BD}(\Omega;\mathbb{R}^d)\qquad\forall t\in\bar I\tag{68g}$$

which is due to the bound (66a) and due to that limit of $\{\bar u_\tau(t)\}$ can be identified uniquely as $u(t)$ because its strain $e(u(t))=\mathbb{C}^{-1}\sigma(t)+\pi(t)|_\Omega\in\mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ has already been determined uniquely when choosing $\pi_\tau(t)\to\pi(t)$ weakly* in $\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})$ and because of the prescribed Dirichlet boundary conditions.

To pass to the limit in (57a) integrated over I to the weakly formulated momentum equation (42a) is simple because all terms are either linear, or enjoy compactness (which concerns $\mathscr{B}(w)$-term), or vanish due to the estimate (61g) since

$$\Big|\int_Q\tau|\dot\varepsilon_\tau|^{\gamma-2}\dot\varepsilon_\tau{:}e(v)\,{\rm d}x{\rm d}t\Big|\le\tau\big\|\dot\varepsilon_\tau\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}^{\gamma-1}\big\|e(v)\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}=\mathscr{O}(\tau^{1/\gamma})\to0\tag{69}$$

for v smooth.

We show a discrete analog of the semistability (42e). To this goal, let us realize that, at each time level k, $\pi_\tau^k \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ minimizes the functional $\tilde\pi \mapsto \int_\Omega \delta_S^*(\tilde\pi) - (\sigma_\tau^k{+}(\sigma_{\rm D})_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k){:}\tilde\pi + \frac{\tau}{2}|\tilde\pi|^2{\rm d}x$. This can be seen from (50b), which obviously represents just the 1st-order necessary and sufficient optimality condition for minimization of this convex functional. Therefore

$$
\begin{aligned}
&\int_\Omega \delta_S^*(\pi_\tau^k) - \big(\sigma_\tau^k{+}(\sigma_{\rm D})_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\pi_\tau^k + \frac{\tau}{2}|\pi_\tau^k|^2{\rm d}x\\
&\quad\le \int_\Omega \delta_S^*(\tilde\pi) - \big(\sigma_\tau^k{+}(\sigma_{\rm D})_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\tilde\pi + \frac{\tau}{2}|\tilde\pi|^2{\rm d}x. \qquad (70)
\end{aligned}
$$

Then we add the discrete equilibrium equation (50a) at time level k tested by $\tilde u{-}u_\tau^k$, i.e. $\int_\Omega(\sigma_\tau^k{+}(\sigma_{\rm D})_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k){:}e(\tilde u{-}u_\tau^k){\rm d}x = 0$. It gives

$$
\begin{aligned}
&\int_\Omega \delta_S^*(\pi_\tau^k) + \big(\sigma_\tau^k{+}[\sigma_{\rm D}]_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\varepsilon_\tau^k + \frac{\tau}{2}|\pi_\tau^k|^2{\rm d}x\\
&\qquad\le \int_\Omega \delta_S^*(\tilde\pi) + \big(\sigma_\tau^k{+}[\sigma_{\rm D}]_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\tilde\varepsilon + \frac{\tau}{2}|\tilde\pi|^2{\rm d}x. \qquad (71)
\end{aligned}
$$

Further we use the specific form (3) of the stress σ, the positive-definiteness of $\mathbb{C}$, implying $\mathbb{C}\varepsilon_\tau^k{:}(\varepsilon_\tau^k{-}\tilde\varepsilon) \ge \frac12\mathbb{C}\varepsilon_\tau^k{:}\varepsilon_\tau^k - \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon$, and the degree-1 homogeneity of δ_S^*, implying $\delta_S^*(\tilde\pi) - \delta_S^*(\pi_\tau^k) \le \delta_S^*(\tilde\pi{-}\pi_\tau^k)$. It gives

$$
\begin{aligned}
&\int_\Omega \frac12\mathbb{C}\varepsilon_\tau^k{:}\varepsilon_\tau^k + \big({\rm D}_t\mathbb{D}\varepsilon_\tau^k{-}\mathscr{B}(w_\tau^k){+}[\sigma_{\rm D}]_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\varepsilon_\tau^k + \frac{\tau}{2}|\pi_\tau^k|^2{\rm d}x\\
&\le \int_\Omega \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \big({\rm D}_t\mathbb{D}\varepsilon_\tau^k{-}\mathscr{B}(w_\tau^k){+}[\sigma_{\rm D}]_\tau^k{+}\tau|{\rm D}_t\varepsilon_\tau^k|^{\gamma-2}{\rm D}_t\varepsilon_\tau^k\big){:}\tilde\varepsilon + \frac{\tau}{2}|\tilde\pi|^2{+}\,\delta_S^*(\tilde\pi{-}\pi_\tau^k){\rm d}x.
\end{aligned}
$$

In terms of interpolants, it eventually gives

$$
\begin{aligned}
L_\tau(t) &:= \int_\Omega \Big[\frac12\mathbb{C}\bar\varepsilon_\tau{:}\bar\varepsilon_\tau + \big(\mathbb{D}\dot\varepsilon_\tau - \mathscr{B}(\bar w_\tau) + \overline{[\sigma_{\rm D}]}_\tau + \tau|\dot\varepsilon_\tau|^{\gamma-2}\dot\varepsilon_\tau\big){:}\bar\varepsilon_\tau + \frac{\tau}{2}|\bar\pi_\tau|^2\Big](t)\,{\rm d}x\\
&\le \int_\Omega \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \Big[\mathbb{D}\dot\varepsilon_\tau - \mathscr{B}(\bar w_\tau) + \overline{[\sigma_{\rm D}]}_\tau + \tau|\dot\varepsilon_\tau|^{\gamma-2}\dot\varepsilon_\tau\Big](t){:}\tilde\varepsilon\\
&\qquad\qquad + \frac{\tau}{2}|\tilde\pi|^2 + \delta_S^*\big(\tilde\pi{-}\bar\pi_\tau(t)\big){\rm d}x =: R_\tau(t) \qquad (72)
\end{aligned}
$$

for a.a. t. The limit passage in (72) can then be done by the classical binomial trick: having (68) and taking $(\tilde u,\tilde\pi)$ first so that $\tilde\pi{-}\pi \in L^2(Q;\mathbb{R}^{d\times d}_{\rm dev})$ and $\tilde\varepsilon{-}\varepsilon \in L^\gamma(Q;\mathbb{R}^{d\times d}_{\rm sym})$, we choose the so-called mutual recovery sequence (in the sense [17]) as

$$
\tilde u_\tau := \bar u_\tau + \tilde u - u, \qquad \tilde\pi_\tau := \bar\pi_\tau + \tilde\pi - \pi. \qquad (73)
$$

Substituting this choice to (72) gives, after using an elementary-school formula "$a^2{-}b^2 = (a{-}b)(a{+}b)$", that

$$
\begin{aligned}
0 \le R_\tau{-}L_\tau &= \int_\Omega \frac12\mathbb{C}(\bar\varepsilon_\tau{+}\tilde\varepsilon_\tau){:}(\tilde\varepsilon{-}\varepsilon) + \big(\mathbb{D}\dot\varepsilon_\tau{-}\mathscr{B}(\bar w_\tau){+}\overline{[\sigma_{\rm D}]}_\tau{+}\tau|\dot\varepsilon_\tau|^{\gamma-2}\dot\varepsilon_\tau\big){:}(\tilde\varepsilon{-}\varepsilon)\\
&\qquad\qquad + \frac{\tau}{2}(\bar\pi_\tau{+}\tilde\pi_\tau){:}(\tilde\pi{-}\pi) + \delta_S^*(\tilde\pi{-}\pi)\,{\rm d}x\\
&\rightharpoonup \int_\Omega \frac12\mathbb{C}(\varepsilon{+}\tilde\varepsilon){:}(\tilde\varepsilon{-}\varepsilon) + \big(\mathbb{D}\dot\varepsilon{-}\mathscr{B}(w){+}\sigma_{\rm D}\big){:}(\tilde\varepsilon{-}\varepsilon) + \delta_S^*(\tilde\pi{-}\pi){\rm d}x = R - L, \qquad (74)
\end{aligned}
$$

where we denoted

$$
R = R(t) := \int_\Omega \frac12\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \Big[\mathbb{D}\dot\varepsilon - \mathscr{B}(w) + \sigma_{\rm D}\Big](t){:}\tilde\varepsilon + \delta_S^*\big(\tilde\pi{-}\pi(t)\big)\,{\rm d}x, \qquad (75a)
$$

$$
L = L(t) := \int_\Omega \Big[\frac12\mathbb{C}\varepsilon{:}\varepsilon + \big(\mathbb{D}\dot\varepsilon - \mathscr{B}(w) + \sigma_{\rm D}\big){:}\varepsilon\Big](t)\,{\rm d}x. \qquad (75b)
$$

Let us realize that $R_\tau{-}L_\tau$ is bounded in $L^1(0,T)$ and its limit $R{-}L$ belongs even to $L^{2\gamma/(2+\gamma)}(0,T)$; in fact even higher integrability could be obtained by taking more special $\tilde u$ and $\tilde\pi$. The convergence in (74) is therefore weak in $L^1(0,T)$. The convergence in particular terms in (74) is obvious due to weak continuity or, as far as the regularizing terms concern, we have used

$$\int_Q\Big|\tau(\bar\pi_\tau+\tilde\pi_\tau){:}(\tilde\pi-\pi)\Big|\mathrm{d}x\mathrm{d}t \le \tau\big\|\bar\pi_\tau+\tilde\pi_\tau\big\|_{L^2(Q;\mathbb{R}^{d\times d})}\big\|\tilde\pi-\pi\big\|_{L^2(Q;\mathbb{R}^{d\times d})} \le \mathscr{O}(\tau^{1/2})\to 0$$

and, similarly as in (69),

$$\int_Q\Big|\tau\big|\dot\varepsilon_\tau\big|^{\gamma-2}\dot\varepsilon_\tau{:}(\tilde\varepsilon-\varepsilon)\Big|\mathrm{d}x\mathrm{d}t \le \tau\big\|\dot\varepsilon_\tau\big\|^{\gamma-1}_{L^\gamma(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})}\big\|\tilde\varepsilon-\varepsilon\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le \mathscr{O}(\tau^{1/\gamma})\to 0.$$

In particular, (74) shows that $L(t)\le R(t)$ for a.a. t, i.e. to (42e) written for $\tilde\varepsilon=\tilde\varepsilon(t)$ and $\tilde\pi=\tilde\pi(t)$.

Let us realize that we have thus shown (42e) for a special case that $\tilde\varepsilon-\varepsilon(t)\in L^\gamma(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\tilde\pi-\pi(t)\in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. Then we can extend the inequality $L(t)\le R(t)$ obtained in (74) for a general $(\tilde u,\tilde\pi)\in\mathfrak{Q}$ from (40); here (47) is used. Then $\int_\Omega\delta_S^*(\tilde\pi-\pi(t))\mathrm{d}x$ possibly converts to $\int_{\bar\Omega}\delta_S^*(\cdot)[\tilde\pi-\pi(t)](\mathrm{d}x)$; here we use in particular that any $\tilde\pi-\pi\in\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ can be weakly* (although not strongly!) approximated from $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ so that also the δ_S^*-variation converges.

The limit passage in the discrete energy inequality (65) written for $k=T/\tau$ is simply by weak lower semicontinuity or weak* continuity as far as $\int_\Omega w_\tau(T)\mathrm{d}x\to\int_{\bar\Omega}w(T,\mathrm{d}x)$, using (68e). Thus we get (42d) as an inequality.

It remains to prove (42d) as an equality and pass to the limit in the heat sources in the discrete heat equation, and then easily in the discrete heat equation itself. To this goal, we use the Riemann-sum trick applied to a Stieltjes-type integral with the fixed L^2-weight $\dot\varepsilon$ and the semistability (42e), which is the (here modified) technique developed in the theory of rate-independent processes [7, 15]. For any $\eta>0$, we consider a suitable partition $0=t_0^\eta<t_1^\eta<\ldots<t_{N_\eta}^\eta=T$ with $\max_{i=1,\ldots,N_\eta}(t_i^\eta-t_{i-1}^\eta)\le\eta$ so that the functions $\mathfrak{S}_1\in L^1([0,T])$ and $\mathfrak{S}_2\in L^2([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ defined by

$$\mathfrak{S}_1: t\mapsto\big\|\mathfrak{s}(t)\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}:[0,T]\to\mathbb{R}\qquad\text{and}\tag{76a}$$

$$\mathfrak{S}_2: t\mapsto\mathfrak{s}(t):[0,T]\to L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\tag{76b}$$

with $\mathfrak{s}(t)$ from (42f) with σ_{D} from (30d), can be approximated by their values on these partitions in the sense

$$\sum_{i=1}^{N_\eta}\int_{t_{i-1}^\eta}^{t_i^\eta}\big|\mathfrak{S}_1(t_{i-1}^\eta)-\mathfrak{S}_1(t)\big|\mathrm{d}t\to 0\qquad\text{and}\tag{77a}$$

$$\sum_{i=1}^{N_\eta}\int_{t_{i-1}^\eta}^{t_i^\eta}\big\|\mathfrak{S}_2(t_{i-1}^\eta)-\mathfrak{S}_2(t)\big\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}\mathrm{d}t\to 0;\tag{77b}$$

these properties just follows from [7, Lemma 4.12]. Let us define the piece-wise constant functions $\mathfrak{S}_{\ell,\eta}(t)=\mathfrak{S}_\ell(t_{i-1}^\eta)$ for $t\in(t_i^\eta,t_{i-1}^\eta)$ with $\ell=1,2$. Obviously, $\mathfrak{S}_{1,\eta}(t)=\|\mathfrak{S}_{2,\eta}(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}$ for a.a. t. In terms of these mappings, (77a) just says that $\mathfrak{S}_{1,\eta}\to\mathfrak{S}_1$ in $L^1(0,T)$ and (77b) says that $\mathfrak{S}_{2,\eta}\to\mathfrak{S}_2$ in $L^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. In particular, the sequence $\{\mathfrak{S}_{1,\eta}\}_{\eta>0}$ is bounded in $L^1(0,T)$, so that $\{\mathfrak{S}_{2,\eta}\}_{\eta>0}$ is bounded in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Therefore, there is a subsequence such that

$$\mathfrak{S}_{2,\eta}\to\mathfrak{S}_2\qquad\text{weakly in }L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}).\tag{78}$$

It is well known from theory of Lebesgue integral that this holds for a.a. partitions, and therefore we can also choose these partitions so that also the semistability (42e) holds at all these points, namely $\{t_i^\eta;\ i=1,\ldots,N_\eta-1,\ \eta>0\}$ where the parameter η is assumed to range only countable values accumulating at 0, of course. Note that we do not need the semistability at any a-priori determined time, in particular not at $t_{N_\eta}^\eta=T$, except that we will need it at $t=0$. For this, we use the gentle-start assumption, in particular the semistability (46) for $\tau=0$ together with the proved fact that the solution is constant on $[0,t_0)$ so that certainly $\mathbb{D}\dot\varepsilon(0)=0$ which converts (46) for $\tau=0$ to the desired semistability (42e) at $t=0$ when taken into account that $\varepsilon(0)=\varepsilon_0$ and

$w(0)=w_0$. Then we take (42e) at t_{i-1}^η tested by the limit (u,π) at t_i^η, and sum it up, i.e.

$$
\begin{aligned}
0 \le & \int_\Omega \frac12 \mathbb{C}\varepsilon(t_{N_\eta}^\eta){:}\varepsilon(t_{N_\eta}^\eta) - \frac12 \mathbb{C}\varepsilon(t_0^\eta){:}\varepsilon(t_0^\eta)\mathrm{d}x \\
& + \sum_{i=1}^{N_\eta}\int_\Omega \mathfrak{s}(t_{i-1}^\eta){:}\big(\varepsilon(t_i^\eta){-}\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x + \sum_{i=1}^{N_\eta}\int_{\bar\Omega} \delta_S^*(\cdot)\big[\pi(t_i^\eta)-\pi(t_{i-1}^\eta))\big](\mathrm{d}x)
\end{aligned} \tag{79}
$$

with $\mathfrak{s}$ from (42f). Then we use

$$
\begin{aligned}
\sum_{i=1}^{N_\eta}\int_\Omega \mathfrak{s}(t_{i-1}^\eta){:}\big(\varepsilon(t_i^\eta){-}\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x &= \int_Q \mathfrak{S}_{2,\eta}{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t \\
\to \int_Q \mathfrak{S}_2{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_Q \mathfrak{s}{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t &= \int_Q \big(\mathbb{D}\dot\varepsilon{-}\mathscr{B}(w){+}\sigma_{\rm D}\big){:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{80}
$$

where we used (78). By definition of the variation, cf. (42c), we have

$$
\sum_{i=1}^{N_\eta}\int_{\bar\Omega} \delta_S^*(\cdot)\big[\pi(t_i^\eta)-\pi(t_{i-1}^\eta)\big](\mathrm{d}x) \le \mathrm{Var}_{\delta_S^*}(\pi;0,T). \tag{81}
$$

Altogether, realizing that $t_{N_\eta}^\eta=T$ and $t_0^\eta=0$ are fixed, we can make the limit passage in (79), obtaining thus the resting energy inequality

$$
\begin{aligned}
\int_\Omega \frac12 \mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x - \int_\Omega \frac12 \mathbb{C}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x + \mathrm{Var}_{\delta_S^*}(\pi;0,T)& \\
+ \int_Q \big(\mathbb{D}\dot\varepsilon{-}\mathscr{B}(w){+}\sigma_{\rm D}\big){:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t \ge 0.&
\end{aligned} \tag{82}
$$

Thus the equality (42d) has been proved.

Now, referring to the measure $\mathfrak{h}_\pi$ corresponding to $\delta_S^*(\dot\pi)$ defined in (42c), then like in [23] we have

$$
\begin{aligned}
\int_{\bar Q} \mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_Q \mathbb{D}\dot\varepsilon{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t &= \mathrm{Var}_{\delta_S^*}(\pi;0,T) + \int_Q \mathbb{D}\dot\varepsilon{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t \\
&\le \liminf_{\tau\downarrow 0}\int_Q \delta_S^*\big(\dot\pi_\tau\big) + \mathbb{D}\dot\varepsilon_\tau{:}\dot\varepsilon_\tau\,\mathrm{d}x\mathrm{d}t \le \limsup_{\tau\downarrow 0}\int_Q \delta_S^*\big(\dot\pi_\tau\big) + \mathbb{D}\dot\varepsilon_\tau{:}\dot\varepsilon_\tau\,\mathrm{d}x\mathrm{d}t \\
&\le \limsup_{\tau\downarrow 0}\Big(\Phi_\tau(u_0,\pi_0) - \Phi_\tau\big(u_\tau(T),\pi_\tau(T)\big) + \int_Q \big(\mathscr{B}(\bar w_\tau){-}\overline{\sigma}_{{\rm D},\tau}\big){:}\dot\varepsilon_\tau\,\mathrm{d}x\mathrm{d}t\Big) \\
&\le \int_\Omega \frac12 \mathbb{C}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x - \int_\Omega \frac12 \mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \big(\mathscr{B}(w){-}\sigma_{\rm D}\big){:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t \\
&\le \mathrm{Var}_{\delta_S^*}(\pi;0,T) + \int_Q \mathbb{D}\dot\varepsilon{:}\dot\varepsilon\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{83}
$$

The inequalities in (83) are based successively on the lower weak* semicontinuity, on general comparison "liminf$\,\le\,$limsup", on the discrete mechanical-energy inequality (64) for $k=K_\tau$ with simply "forgetting" the term $\tau|\dot\varepsilon_\tau^k|^\gamma$, on the upper weak* semicontinuity and the obvious estimate $\Phi_\tau(u,\pi)\ge\int_\Omega\frac12\mathbb{C}\varepsilon{:}\varepsilon\mathrm{d}x$ (cf. (63)) and on the convergence

$$
\mathscr{B}(\bar w_\tau){:}\dot\varepsilon_\tau \to \mathscr{B}(w){:}\dot\varepsilon \qquad \text{weakly in } L^1(Q) \tag{84}
$$

and finally on (82). Thus we have equality in the above chain of inequalities (83). This allows us to say that $\delta_S^*(\dot\pi_\tau)\to\mathfrak{h}_\pi$ weakly* as measures on $\bar Q$ and $\mathbb{D}\dot\varepsilon_\tau{:}\dot\varepsilon_\tau\to\mathbb{D}\dot\varepsilon{:}\dot\varepsilon$ even strongly in $L^1(Q)$. In particular, we obtain also the strong convergence (67) of σ_τ. This allows for the limit passage in the enthalpy equation.

Having (42b) already at disposal, we also obtain (41) and we can test (42b) by $v:=1$ which is obviously in duality with $\dot w\in\mathscr{M}(I;W^{1+d}(\Omega)^*)$, and adding it to (82), we obtain (42d) with "$\ge$". As the opposite inequality has already been discussed, altogether we proved the total energy equality (42d). $\square$

Remark 4 (More estimate of stress)**.** From (2c), one can see that $\operatorname{dev}\sigma \in L^\infty(Q;\mathbb{R}^{d\times d}_{\rm sym})$ although the full stress σ is only in $L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})$ and although the regularization in (50b) causes that $\operatorname{dev}\sigma_\tau$ is not in $L^\infty(Q;\mathbb{R}^{d\times d}_{\rm sym})$ in general.

Remark 5 (Uniqueness)**.** Even in the isothermal case, the uniqueness is difficult. In general, it can be proved at most for the stress only, cf. [14, Sect.4.2.3] or [8, Thm.5.9]. In particular, plastic strain (and thus also the heat production) is not expected to be unique even in such isothermal case. It indicates that, in the scrutinised anisothermal case, the uniqueness can hardly be expected, not only because of the coupling terms (which might possibly still be handled at least for small data) but mainly because of the mentioned nonuniqueness of the heat production in perfect plasticity itself.

Remark 6 (Inertial effects)**.** Sometimes, contribution of kinetic energy and occurrence of elastic waves cannot be neglected. The equilibrium equation in (2a) should then augment to $\varrho\ddot u-\operatorname{div}\sigma=0$ with $\varrho>0$ being the mass density. The shifted equation (30a) would then augment as $\varrho\ddot u = \operatorname{div}\sigma + f_{\rm D}$ with $f_{\rm D} = \operatorname{div}\sigma_{\rm D} - \varrho\ddot u_{\rm D}$ and an additional initial condition $\dot u(0)=\dot u_0 \in L^2(\Omega;\mathbb{R}^d)$ should be prescribed. The discrete scheme (50) must be augmented by the term $\varrho{\rm D}_t^2 u_\tau^k$ and started for $k=1$ by putting $u_\tau^{-1}=u_0-\tau\dot u_0$. The proper space where (discrete) velocity $\dot u_\tau$ lives is then

$$\mathscr{V} := \Big\{v\in L^\infty(I;L^2(\Omega;\mathbb{R}^d));\quad e(v)\in L^1(Q;\mathbb{R}^{d\times d}_{\rm sym}),\quad \operatorname{div} v\in L^2(Q)\Big\}, \tag{85}$$

instead of $\{v\in L^1(Q;\mathbb{R}^d);\ e(v)\in L^1(Q;\mathbb{R}^{d\times d}_{\rm sym}),\ \operatorname{div} v\in L^2(Q)\}$, and the a-priori estimates (61) and (66) then augment by

$$\big\|u_\tau\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \le C. \tag{86}$$

Moreover, for the limit we have the information

$$\varrho\ddot u \in \mathscr{V}^*. \tag{87}$$

which follows from the estimate

$$\begin{aligned}\big\langle\varrho\ddot u, v\big\rangle_{\mathscr{V}^*\times\mathscr{V}} &= \int_Q f_{\rm D}{\cdot}v - \sigma{:}e(v)\,{\rm d}x{\rm d}t\\ &= \int_Q f_{\rm D}{\cdot}v - \operatorname{dev}\sigma{:}\operatorname{dev}e(v) - \operatorname{tr}(\sigma^{\rm s})\operatorname{div} v\,{\rm d}x{\rm d}t\\ &\le \big\|f_{\rm D}\big\|_{L^1(I;L^2(\Omega;\mathbb{R}^d))}\big\|v\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))}\\ &\quad+\big\|\operatorname{dev}\sigma\big\|_{L^\infty(Q;\mathbb{R}^{d\times d}_{\rm sym})}\big\|e(v)\big\|_{L^1(Q;\mathbb{R}^{d\times d}_{\rm sym})}\\ &\quad+\big\|\sigma\big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})}\big\|\operatorname{div} v\big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})}\end{aligned} \tag{88}$$

together with the estimate (66g). In the weak formulation, the momentum equilibrium (42a) has to be augmented as

$$\begin{aligned}\int_\Omega \varrho\frac{\partial u}{\partial t}(T){\cdot}v(T)\,{\rm d}x + \int_Q \big(\mathbb{D}\dot\varepsilon + \mathbb{C}\varepsilon - \mathscr{B}(w)\big){:}e(v) - \varrho\frac{\partial u}{\partial t}{\cdot}\frac{\partial v}{\partial t}\,{\rm d}x{\rm d}t\\ = \int_\Omega \varrho\dot u_0{\cdot}v(0)\,{\rm d}x - \int_Q \varrho\ddot u_{\rm D}{\cdot}v + \sigma_{\rm D}{:}e(v)\,{\rm d}x{\rm d}t.\end{aligned} \tag{89}$$

Also the energy balance (42d) augments by $T_{\rm kin}(\dot u(T))$ and $T_{\rm kin}(\dot u_0)$ with the kinetic energy $T_{\rm kin}(\cdot) := \int_\Omega \varrho|\cdot|^2{\rm d}x$. There is however a serious problem with preservation of energy and thus also with the passage to the limit when $\tau\to 0$. This is related with a need of the by-part integration formula of the type $\frac12\|u(T)\|^2-\frac12\|u(0)\|^2=\int_0^T\langle\ddot u,\dot u\rangle\,{\rm d}t$. Such formula is however not clear, mainly because of $\mathscr{V}$ is not reflexive. The limit velocity $\dot u$ is thus expected to live rather in an extension of $\mathscr{V}$ allowing for $e(\dot u)\in\mathscr{M}(\bar Q;\mathbb{R}^{d\times d}_{\rm sym})$, but then the proof of the mentioned by-part integration formula based standardly on a strong convergence of mollifiers fails in the space of measures. It is not clear whether it is "only" a mathematical difficulty or whether it is related with some physical phenomenon of dissipation of energy during impacts of elastic waves on shear bands, similarly like it may possibly happen during impacts on a unilateral boundary contact (which remains for a long time an open difficult problem).

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