Thermodynamics of rate independent processes in viscous solids at small strains

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THERMODYNAMICS OF RATE INDEPENDENT PROCESSES IN VISCOUS SOLIDS AT SMALL STRAINS[1]

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Abstract: So-called generalized standard solids (of the Halphen-Nguyen type) involving also activated rate-independent processes such as plasticity, damage, or phase transformations, are described as a system of a momentum equilibrium equation and a variational inequality for inelastic evolution of internal-parameter variables. The stored energy is considered as temperature dependent and then the thermodynamically consistent system is completed with the heat-transfer equation. Existence of a suitably defined "energetic" solution is proved by a nontrivial combination of theory of rate-independent processes by Mielke at al. adapted for coupling with viscous/inertial effects and of sophisticated estimates by Boccardo and Gallouët of the temperature gradient of the heat equation with L^1-data. Illustrative examples are presented, too.

Keywords: generalized standard materials, heat equation, enthalpy transformation, doubly-nonlinear variational inequalities, energetic solution, plasticity, damage, magnetostriction, shape-memory alloys.

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1 Introduction, generalized standard materials

Theory of rate-independent processes based on the so-called energetic formulation by Mielke at al. [54–56] has been extensively developed and widely applied in [16, 20–22, 27, 39, 44, 47, 49, 50, 52, 71]. The rate-independent processes may involve plasticity, damage, or various phase transformations in ferroic materials. It is well known that coupling rate-independent processes with some others that are rate dependent brings, in general, serious difficulties, cf. e.g. [25,38,48]. In some cases when such processes are coupled rather indirectly, such combination is, however, well possible as shown in [64] for viscous and inertial effects; for some special cases we refer to [1, 19, 74]. The goal of this contribution is to expand this coupling also for thermal processes that are, of course, inevitably rate dependent.

After formulation of the problem here and in Section 2 in terms of displacements, internal parameters, and temperature, we reformulate the problem in terms of enthalpy in Section 3 to be better fitted with the semi-discretization method proposed in Section 4 where we show existence of the discrete solution, derive basic a-priori estimates and prove convergence to the special weak (so-called "energetic") solution of the continuous problem. Eventually, Section 5 presents various illustrative examples.

The thermodynamics will be governed, beside dissipation mechanisms and constitutive equations specified latter in (2.3) and (2.5), by the specific Helmholtz *free energy* $\psi : \mathbb{R}^{n\times n}_{\rm sym} \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \times \mathbb{R} \to [0,+\infty]$ with $\mathbb{R}^{n\times n}_{\rm sym} := \{A \in \mathbb{R}^{n\times n};\ A^\top = A\}$. Here ψ is a function of small-strain tensor e and the vector z of internal parameters, of its spatial gradient for which we will use the notation $Z \in \mathbb{R}^{m\times n}$ in the position of a variable in $\psi(e,z,Z,\theta)$, and of the temperature θ. Generalization and modifications for large-strains are outlined in Remark 4.7 below. We assume a partly linearized free energy in the form

$$\psi(e,z,Z,\theta) := \varphi(e,z,Z) + \theta\phi(e) - \phi_0(\theta). \tag{1.1}$$

This ansatz is to ensure that entropy separates thermal and mechanical variables, cf. (2.10) below, which facilitates the analysis of the coupled thermodynamical model, cf. also Remark 4.9 below.

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Using the ansatz of so-called *generalized standard solids* (due to Halphen and Nguyen [32]), we consider a momentum equilibrium equation involving a *viscous-like response* of the material in a *Kelvin-Voigt*-type *rheology* and inertia, combined with an inclusion for inelastic evolution of internal-parameter variables. We assume *small strains* and allow for a so-called *gradient theory* as far as the internal parameters are concerned. Altogether, when completed also by the heat-transfer equation, we then will deal with the following system

$$\varrho\frac{\partial^2 u}{\partial t^2}-\mathrm{div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big)+\sigma_{\mathrm{el}}\Big)=f,\qquad \sigma_{\mathrm{el}}=\varphi_e'\big(e(u),z,\nabla z\big)+\theta\phi'\big(e(u)\big),\tag{1.2a}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\sigma_{\mathrm{in}}\ni 0,\qquad\qquad \sigma_{\mathrm{in}}=\varphi_z'\big(e(u),z,\nabla z\big)-\mathrm{div}\,\varphi_Z'\big(e(u),z,\nabla z\big),\tag{1.2b}$$

$$c_{\mathrm{v}}\frac{\partial\theta}{\partial t}-\mathrm{div}\big(\mathbb{K}\nabla\theta\big)=\xi\Big(\frac{\partial z}{\partial t},e(\frac{\partial u}{\partial t})\Big)+\theta\phi'(e(u)):\frac{\partial e(u)}{\partial t},\tag{1.2c}$$

where $u:Q\to\mathbb{R}^n$ is a *displacement*, $z:Q\to\mathbb{R}^m$ a vector of certain *internal parameters*, $\theta:Q\to\mathbb{R}$ absolute *temperature* with $Q:=(0,T)\times\Omega$ with $T>0$ a fixed time horizon, Further, $\varrho>0$ is a constant mass density, $c_{\mathrm{v}}=c_{\mathrm{v}}(\theta)>0$ heat *capacity*, $\mathbb{K}=\mathbb{K}(e,z,\theta)$ heat *conductivity*, and $\zeta_2:\mathbb{R}^{n\times n}_{\mathrm{sym}}\to[0,+\infty)$ and $\zeta_1:\mathbb{R}^m\to[0,+\infty]$ are (pseudo)*potentials of dissipative forces*. From ψ, one derives the "elastic" stress σ_{el} and an "inelastic" driving force σ_{in} as said in (1.2a,b). Such z may involve plastic strain, hardening, damage, or volume fractions in various phase transformations, etc. We will assume each ζ_ℓ positively *homogeneous* of degree ℓ, i.e. for all v it holds $\zeta_\ell(rv)=r^\ell\zeta_\ell(v)$ with any $\ell=1,2$, $r\ge 0$. As to ζ_2, its homogeneity of degree 2 is just responsible for the viscous-like response. Elementary calculus shows the formula for the directional derivative $\zeta_\ell'(v)v$, namely $\zeta_\ell'(v)v=\lim_{\epsilon\to 0+}\frac{\zeta_\ell(v+\epsilon v)-\zeta_\ell(v)}{\epsilon}=\lim_{\epsilon\to 0+}\frac{(1+\epsilon)^\ell-1}{\epsilon}\zeta_\ell(v)=\ell\,\zeta_\ell(v)$. Then the *dissipation rate* $\xi(\dot z,\dot e)$ in (1.2c) is $\zeta_1(\dot z)+2\zeta_2(\dot e)$, cf. (2.4) below. We will confine ourselves to ζ_2' linear, hence ζ_2 quadratic. Then, without loss of generality, we may consider

$$\zeta_1(\dot z):=\delta_S^*(\dot z)\quad\text{ with } S\subset\mathbb{R}^m \text{ convex closed, and }\quad \zeta_2(\dot e):=\frac12\mathbb{D}\dot e{:}\dot e,\tag{1.3}$$

where δ_S^* is the Legendre-Fenchel conjugate function to the indicator function δ_S of S and $\mathbb{D}:\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}^{n\times n}_{\mathrm{sym}}$ is a 4th-order tensor (assumed positive definite and symmetric $\mathbb{D}_{ijkl}=\mathbb{D}_{jikl}=\mathbb{D}_{klij}$). This means $\delta_S^*(\dot z):=\sup_{z\in\mathbb{R}^m}\dot z\cdot z-\delta_S(z)=\sup_{z\in S}\dot z\cdot z$. Assuming S bounded (resp. containing 0 in its interior) makes ζ_1 bounded (resp. coercive). Also, $S=\partial\zeta_1(0)$. Nonsmoothness of ζ_1 at 0, which follows from its positive homogeneity of degree 1 (except the trivial case where ζ_1 is linear), may describe various *activated processes*, i.e. to trigger z evolving, the driving force $\varphi_z'(e(u),z,\nabla z)$ must exceed a certain activation threshold, namely the boundary of S.

2 Thermodynamics of generalized standard materials

We now justify thermodynamics of the model (1.2). Departing from the free energy ψ, we identify the partial derivatives with the elastic *stress* σ_{el}, the inelastic *driving stress* $\sigma_{\mathrm{in},0}$ and the "*hyper-stress*" $\sigma_{\mathrm{in},1}$, and the specific *entropy* by

$$\sigma_{\mathrm{el}}:=\psi_e',\qquad \sigma_{\mathrm{in},0}:=\psi_z',\qquad \sigma_{\mathrm{in},1}:=\psi_Z',\qquad s:=-\psi_\theta';\tag{2.1}$$

the last equation is the so-called Gibbs' relation. Then, as already introduced in (1.2b),

$$\sigma_{\mathrm{in}}:=\sigma_{\mathrm{in},0}-\mathrm{div}\sigma_{\mathrm{in},1}\tag{2.2}$$

so that the *total driving force* $\sigma_{\rm in}$ is the Gâteaux differential of the energy functional $z\mapsto\int_\Omega\varphi(e(u),z,\nabla z){\rm d}x$. Further, we define the specific *internal energy* ε by

$$\varepsilon:=\psi+\theta s. \tag{2.3}$$

Then the so-called *entropy equation*

$$\theta\frac{\partial s}{\partial t}+{\rm div}(j)=\xi:=\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+2\zeta_2\Big(e(\frac{\partial u}{\partial t})\Big) \tag{2.4}$$

balances the *heat flux* j and the rate of the heat production due to the *dissipation rate* $\xi\ge0$ (here due to the loss of mere mechanical energy but some additional sources might be considered too, see Remark 4.4 below).

The important fact is that the above procedure satisfies the 2nd thermodynamical law provided

$$j=j(e,z,\theta,\nabla\theta):=-\mathbb{K}(e,z,\theta)\nabla\theta \tag{2.5}$$

with the matrix of *heat-conduction* coefficients $\mathbb{K}(e,z,\theta)$ positive definite (the so-called *Fourier law* in the nonlinear anisotropic medium). Indeed, dividing (2.3) by θ and using Green's formula, the *Clausius-Duhem inequality* reads as:

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\int_\Omega s(t,x){\rm d}x=\int_\Omega\frac{\xi+{\rm div}(\mathbb{K}(e(u),z,\theta)\nabla\theta)}{\theta}{\rm d}x&=\int_\Omega\frac{1}{\theta}\Big(\xi-\mathbb{K}(e(u),z,\theta)\nabla\theta\cdot\nabla\frac{1}{\theta}\Big){\rm d}x\\&=\int_\Omega\frac{\xi}{\theta}+\frac{\mathbb{K}(e(u),z,\theta)\nabla\theta\cdot\nabla\theta}{\theta^2}{\rm d}x\ge0\end{aligned} \tag{2.6}$$

provided $\theta>0$ and provided the system is thermally isolated. Differentiating (2.3) in time and using the Gibb's relation (2.1) and the entropy equation (2.4) gives

$$\frac{{\rm d}\varepsilon}{{\rm d}t}=\frac{{\rm d}\psi}{{\rm d}t}+\frac{\rm d}{{\rm d}t}(\theta s)=\Big(\psi'_e{:}\frac{\partial e}{\partial t}+\psi'_z\cdot\frac{\partial z}{\partial t}+\psi'_Z:\frac{\partial\nabla z}{\partial t}+\underbrace{\psi'_\theta\frac{\partial\theta}{\partial t}\Big)+\Big(\frac{\partial\theta}{\partial t}s}_{=\,0\text{ due to (2.1)}}+\theta\frac{\partial s}{\partial t}\Big). \tag{2.7}$$

By Green's formula and by (2.4), we get

$$\frac{\rm d}{{\rm d}t}\int_\Omega\varepsilon\,{\rm d}x=\int_\Omega\sigma_{\rm el}{:}\frac{\partial e}{\partial t}+\sigma_{\rm in}\cdot\frac{\partial z}{\partial t}+\xi-{\rm div}j\,{\rm d}x; \tag{2.8}$$

in fact, (2.8) is to be understood formally in general since $\int_\Omega\sigma_{\rm in}\cdot\frac{\partial z}{\partial t}\,{\rm d}x$ rather means the duality $\langle\sigma_{\rm in},\frac{\partial z}{\partial t}\rangle$ with $\sigma_{\rm in}$ being the Gâteaux differential of $z\mapsto\int_\Omega\varphi(e(u),z,\nabla z){\rm d}x$. Testing (1.2a) by $\frac{\partial u}{\partial t}$, we get $\int_\Omega\frac{\varrho}{2}\frac{\partial}{\partial t}|\frac{\partial u}{\partial t}|^2+\sigma:e(\frac{\partial u}{\partial t})-f\cdot\frac{\partial u}{\partial t}{\rm d}x=0$ with $\sigma=\mathbb{D}e(\frac{\partial u}{\partial t})+\sigma_{\rm el}$. Testing (1.2b) by $\frac{\partial z}{\partial t}$, we get $\sigma_{\rm in}\frac{\partial z}{\partial t}+\zeta_1(\frac{\partial z}{\partial t})=0$. Using these identities for (2.7) integrated over Ω, we obtain

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\int_\Omega\underbrace{\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2+\varepsilon}_{\text{total energy}}\,{\rm d}x=\int_\Omega\frac{\varrho}{2}\frac{\partial}{\partial t}\Big|\frac{\partial u}{\partial t}\Big|^2+\sigma:\frac{\partial e}{\partial t}+\sigma_{\rm in}\frac{\partial z}{\partial t}&\\+\zeta_1\Big(\frac{\partial z}{\partial t}\Big)-{\rm div}(j)\,{\rm d}x=\underbrace{\int_\Omega f\cdot\frac{\partial u}{\partial t}{\rm d}x+\int_\Gamma j\,{\rm d}S}_{\text{power of external load and heat}}&.\end{aligned} \tag{2.9}$$

This reveals the total *energy balance* in terms of the sum of the kinetic and the internal energies.

Now, we confine ourselves to the special ansatz (1.1) and, from (2.1), we get the entropy

$$s = s(e,\theta) = \phi_0'(\theta) - \phi(e) \tag{2.10}$$

and, from (2.3), also the internal energy

$$\varepsilon(e,z,Z,\theta) := \psi(e,z,Z,\theta) + \theta s(e,z,\theta) \ = \varphi(e,z,\nabla z) + h(\theta), \tag{2.11}$$

where we denoted

$$h(\theta) = \theta\phi_0'(\theta) - \phi_0(\theta). \tag{2.12}$$

In some special cases (namely when Gibbs' and Helmholtz' free energies coincide with each other), $h(\theta)$ has the meaning of *enthalpy*, hence we dare call $h(\theta)$ (up to the mentioned tolerance) in this way. Substituting for s to (2.4), we obtain an equation for temperature, the so-called *heat-transfer equation*. Assuming the anisotropic nonlinear Fourier law (2.5), this heat equation results as

$$c_{\rm v}(\theta)\frac{\partial\theta}{\partial t} - {\rm div}\big(\mathbb{K}(e(u),z,\theta)\nabla\theta\big) = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + 2\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big) + \theta\phi'(e(u)){:}e\big(\frac{\partial u}{\partial t}\big) \tag{2.13}$$

with the dissipative heat ξ from (2.4) and with $c_{\rm v}$ *heat capacity* given by

$$c_{\rm v}(\theta) = \theta\phi_0''(\theta). \tag{2.14}$$

Altogether, counting also (1.3), we thus will treat the system

$$\varrho\frac{\partial^2 u}{\partial t^2} - {\rm div}\Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \varphi_e'\big(e(u),z,\nabla z\big) + \theta\phi'\big(e(u)\big)\Big) = f, \tag{2.15a}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'\big(e(u),z,\nabla z\big) - {\rm div}\varphi_Z'\big(e(u),z,\nabla z\big) \ni 0, \tag{2.15b}$$

$$c_{\rm v}(\theta)\frac{\partial\theta}{\partial t} - {\rm div}\big(\mathbb{K}(e(u),z,\theta)\nabla\theta\big) = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \mathbb{D}e\big(\frac{\partial u}{\partial t}\big){:}e\big(\frac{\partial u}{\partial t}\big) + \theta\phi'(e(u)){:}e\big(\frac{\partial u}{\partial t}\big). \tag{2.15c}$$

We now have naturally to prescribe the initial condition for displacement, velocity, the internal parameter, and temperature, i.e.

$$u(0,\cdot) = u_0, \qquad \frac{\partial u}{\partial t}(0,\cdot) = \dot u_0, \qquad z(0,\cdot) = z_0, \qquad \theta(0,\cdot) = \theta_0. \tag{2.16}$$

The problem is to be completed by boundary conditions. Let us consider $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ with Γ_0 and Γ_1 disjoint open sets and ${\rm meas}_{n-1}(\Gamma_2) = 0$, and denote $\Sigma_0 := (0,T)\times\Gamma_0$, $\Sigma_1 := (0,T)\times\Gamma_1$, and $\Sigma := (0,T)\times\partial\Omega$, and then (with a bit compromised generality) consider the boundary conditions

$$u\big|_{\Sigma_0} = 0, \qquad \Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \varphi_e'(e(u),z,\nabla z) + \theta\phi'(e(u))\Big)\Big|_{\Sigma_1}\cdot\nu = 0, \tag{2.17a}$$

$$\varphi_Z'(e(u),z,\nabla z)\big|_{\Sigma}\cdot\nu = 0, \tag{2.17b}$$

$$\mathbb{K}(e(u),z,\theta)\nabla\theta\big|_{\Sigma}\cdot\nu + b\big(\theta\big|_{\Sigma} - \theta_{\rm ext}\big) = 0, \tag{2.17c}$$

where ν denotes the outward normal to the boundary $\partial\Omega$ of Ω, $b = b(x)$ is a phenomenological coefficient of heat transfer through the boundary and $\theta_{\rm ext} = \theta_{\rm ext}(t,x)$ is the external temperature.

3 Enthalpy transformation, data qualification, and energetic solution

It is desirable to allow for a certain growth of $c_{\rm v}(\cdot)$ if we have the viscosity in the form $\mathbb{D}e(\frac{\partial u}{\partial t})$ in order to be able to treat the adiabatic term, cf. [63] and Remark 5.7 below. On the other hand, the technique from [63] specifically relies on Galerkin method and does not seem directly transferable to the Rothe method we use here which, in turn, seems better fitted to rate-independent part than the Galerkin method. The particular difficulty is in limiting a time-discretization of the nonlinear term $c_{\rm v}(\theta)\frac{\partial\theta}{\partial t}$. Therefore, we first write the original system (2.15) in terms of enthalpy instead of temperature, using so-called enthalpy transformation

$$w=h_0(\theta):=\int_0^\theta c_{\rm v}(r)\,{\rm d}r; \tag{3.1}$$

thus h_0 is a primitive function to $c_{\rm v}$ normalized such that $h_0(0)=0$. In view of (2.12) and (2.14), we have

$$h'(\theta)=(\theta\phi_0'(\theta)-\phi_0(\theta))'=\theta\phi_0''(\theta)+\phi_0'(\theta)-\phi_0'(\theta)=\theta\phi_0''(\theta)=c_{\rm v}(\theta)=h_0'(\theta), \tag{3.2}$$

hence h_0 differs from h just by a constant, namely $\phi_0(0)$. Further, we define

$$\mathscr{T}(w):=\begin{cases}h_0^{-1}(w) & \text{if } w\ge0,\\ 0 & \text{if } w<0,\end{cases}\qquad\qquad \mathscr{K}(e,z,w):=\frac{\mathbb{K}(e,z,\mathscr{T}(w))}{c_{\rm v}(\mathscr{T}(w))}, \tag{3.3}$$

where h_0^{-1} here denotes the inverse function to h. This transforms the system (2.15) into the form

$$\varrho\frac{\partial^2u}{\partial t^2}-{\rm div}\Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big)+\varphi_e'(e(u),z,\nabla z)+\mathscr{T}(w)\phi'(e(u))\Big)=f, \tag{3.4a}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\varphi_z'(e(u),z,\nabla z)-{\rm div}\varphi_Z'(e(u),z,\nabla z)\ni0, \tag{3.4b}$$

$$\frac{\partial w}{\partial t}-{\rm div}\big(\mathscr{K}(e(u),z,w)\nabla w\big)=\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\mathbb{D}e\big(\frac{\partial u}{\partial t}\big){:}e\big(\frac{\partial u}{\partial t}\big)+\mathscr{T}(w)\phi'(e(u)){:}e\big(\frac{\partial u}{\partial t}\big). \tag{3.4c}$$

We will call (3.4c) shortly the *enthalpy equation* rather than the heat-transfer equation in the enthalpy formulation.

Let us assume that the material described by (1.2) occupies a bounded Lipschitz domain Ω. The problem is to be completed by boundary conditions. Let us consider $\partial\Omega=\Gamma_0\cup\Gamma_1\cup\Gamma_2$ with Γ_0 and Γ_1 disjoint open sets and ${\rm meas}_{n-1}(\Gamma_2)=0$, and denote $\Sigma_0:=(0,T)\times\Gamma_0$, $\Sigma_1:=(0,T)\times\Gamma_1$, and $\Sigma:=(0,T)\times\partial\Omega$, and then consider the boundary conditions

$$u\big|_{\Sigma_0}=0,\qquad \Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big)+\varphi_e'(e(u),z,\nabla z)+\mathscr{T}(w)\phi'(e(u))\Big)\Big|_{\Sigma_1}\cdot\nu=0, \tag{3.5a}$$

$$\varphi_Z'(e(u),z,\nabla z)\big|_\Sigma\cdot\nu=0, \tag{3.5b}$$

$$\mathscr{K}(e(u),z,w)\nabla w\big|_\Sigma\cdot\nu+b\big(\mathscr{T}(w)\big|_\Sigma-\theta_{\rm ext}\big)=0, \tag{3.5c}$$

where ν denotes the outward normal to the boundary $\partial\Omega$ of Ω. In general, to have a-priori estimates, we will assume the coercivity of the specific stored and the dissipative energies:

$$\exists\, p>\max\Big(1,\frac{2n}{n{+}2}\Big),\quad q>\max\Big(1,\frac{2n}{n{+}4}\Big),\quad q_0>1,\quad c_0,c_1,c_2>0:$$

$$\forall e\in\mathbb{R}^{n\times n}_{\rm sym},\ z\in\mathbb{R}^m, Z\in\mathbb{R}^{m\times n}:\qquad \varphi(e,z,Z)\ge c_0|e|^p+c_0|z|^{q_0}+c_0|Z|^q, \tag{3.6a}$$

$$\forall\dot z\in\mathbb{R}^m:\qquad \zeta_1(\dot z)\ge c_1|\dot z|, \tag{3.6b}$$

$$\forall\dot e\in\mathbb{R}^{n\times n}_{\rm sym}:\qquad \zeta_2(\dot e)\ge c_2|\dot e|^2. \tag{3.6c}$$

In view of (1.3), the qualification (3.6b,c) means that S contains 0 in its interior and $\mathbb{D}$ is positive definite. Further, we will occasionally need φ'_e indepedent of Z and φ'_Z indepedent of e, which leads us to assume that

$$\varphi(e,z,Z) = \phi_1(e,z) + \phi_2(z,Z) \tag{3.7}$$

and we also qualify $\varphi(\cdot,z,Z)$ as smooth function with a "p-strongly monotone" gradient in the sense, with some $\alpha > 0$ (independent of z),

$$\forall e,\tilde e \in \mathbb{R}^{n\times n}_{\rm sym}: \quad \alpha\big(|e|^{p-2}e - |\tilde e|^{p-2}\tilde e\big){:}(e-\tilde e) \le \big(\varphi'_e(e,z) - \varphi'_e(\tilde e,z)\big){:}(e-\tilde e); \tag{3.8}$$

here we already used that φ'_e does not depend on Z because of (3.7). The qualification (3.8) will allow for proving strong convergence in terms of $e(u)$ in Step 3 of the proof of Proposition 4.3 which, in turn, seems invevitable starting step to prove further in Step 7 of that proof even better strong convegence in terms of $e(\frac{\partial u}{\partial t})$. An example $\varphi(e,z) = a(z)|e|^p$ with $p > 1$ satisfies (3.8) with $\alpha := p\inf_{\mathbb{R}^m} a(\cdot) > 0$. We will have also to assume

$$\exists \ell \ge 0: \quad (e,z,Z) \mapsto \varphi(e,z,Z) + \ell|e|^2 \quad \text{ is strictly convex.} \tag{3.9}$$

Let us comment that (3.9) seems essential for making running the implicit time discretization method, cf. (4.13)–(4.16), which, in turn, seems a very natural tool especially in the context of rate-independent processes, as already observed in [44, 55, 56], and a further limit passage would allow us to weaken this sometimes restricted structural assumption under the price, however, of further enlargment of the proofs. We will call φ satisfying (3.9) as strictly (e)-semiconvex; let us just remind the standard terminology calling φ semiconvex if $\varphi + \ell|\cdot|^2$ is convex (or equivalently strictly convex) for ℓ large enough. Note also that (e)-semiconvex functions must be convex in (z,Z). A nontrivial example for $n = 1 = m$ is $\varphi(e,z,Z) = ez + \epsilon z^2 + \varepsilon Z^2$ with $\epsilon,\varepsilon > 0$ which is nonconvex but strictly (e)-semiconvex, satisfying (3.9) for $\ell > \frac{1}{4\epsilon}$, because only in that case the Jacobian of the mapping from (3.9) is positive definite; note that this Jacobian is constant and equals $\begin{pmatrix} 2\ell & 1 & 0 \\ 1 & 2\epsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix}$. Also, the function $\varphi(e,z,Z) = ze^2 + \epsilon(e^6{+}z^2{+}Z^2)$ with $\epsilon > 0$ is strictly (e)-semiconvex on the domain $\{(e,z,Z);\ z \ge 0\}$.

It has been observed already e.g. in [51, 64] that continuity of ζ_1 or a certain quadratic structure of φ facilitates the limit passage in the rate-independent flow rule (3.4b). This is why we assume that one of the two cases holds:

$$\zeta_1 \text{ continuous, or} \tag{3.10a}$$

$q = 2$ in (3.6a), and $\varphi(e,\cdot,\cdot)$ quadratic/affine, and for some C:

$$\big|\varphi'_{(z,Z)}(e,z,Z)\big| \le C\big(1{+}|e|^{p(q^*-1)/q^*}\big) + C\big(1{+}|e|^{p(q^*-2)/(2q^*)}\big)\big(|z|{+}|Z|\big). \tag{3.10b}$$

where $q^* = nq/(n{-}q)$ if $q > n$ (or $q^* < +\infty$ for $q \ge n$) denotes the Sobolev critical exponent to q, here used for $q = 2$.

Moreover, for $p_1 \ge 0$, we need to assume

$$\phi \text{ convex}, \quad |\phi'(e)| \le C(1 + |e|^{p_1}), \tag{3.11a}$$

$$\exists C_1 \in \mathbb{R}\ \forall z \in \mathbb{R}^m: \quad \zeta_1(z) < +\infty \ \Rightarrow\ \zeta_1(z) \le C_1|z|. \tag{3.11b}$$

Other assumptions are on $c_{\rm v}$ and $\mathbb{K}$ and will facilitate interpolation of the adiabatic term (i.e. the last term in (3.4c)) similarly as in [63]. To be more specific, we require:

$$c_{\rm v} : [0,+\infty) \to \mathbb{R}^+ \ \text{ continuous,} \tag{3.12a}$$

$$\exists\, \omega_1 \ge \omega > 1,\ c_1 \ge c_0 > 0\ \forall \theta \in \mathbb{R}^+: \quad c_0(1{+}\theta)^{\omega-1} \le c_{\rm v}(\theta) \le c_1(1{+}\theta)^{\omega_1-1}, \tag{3.12b}$$

$$\mathscr{K} : \mathbb{R}^{n\times n} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{n\times n} \ \text{ bounded, continuous, and} \tag{3.12c}$$

$$\inf_{(e,z,w,v)\in\mathbb{R}^{n\times n}_{\rm sym}\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^n,\ |v|=1} \mathscr{K}(e,z,w)v:v > 0 \quad \text{ with } \mathscr{K} \text{ from (3.3);} \tag{3.12d}$$

For Proposition 4.2, we impose more restrictions on ω, namely we also need to assume the exponents p and p_1 and ω from (3.6a), (3.11a), (3.12b) to satisfy

$$\omega > \frac{2np_2}{(n{+}2)(p_2{-}2p_1)} \qquad \text{with} \qquad p_2 := \max(p,2) > 2p_1. \tag{3.13}$$

Furthermore, to have the accelaration $\frac{\partial^2}{\partial t^2}u$ controlled at least in some "dual" space, cf. (4.23) below, we need to assume $\big|\varphi_e'(e,z)\big| \le C(1+|e|^p+|z|^{q^*})$ with some C and with q^* the Sobolev exponent to q. Yet, due to Proposition 4.3, we will need even a stronger qualification of φ_e', namely

$$\big|\varphi_e'(e,z)\big| \le C\big(1+|e|^{p/2}+|z|^{q^*/2}\big), \tag{3.14a}$$

$$\varphi_e'(\cdot,z) \text{ be Lipschitz continuous uniformly with respect to } z; \tag{3.14b}$$

note that we used that φ_e' is independent of Z due to (3.7).

We consider evolution on the time interval $I := (0,T)$ with a fixed time horizon $T > 0$ and denote $Q := (0,T)\times\Omega$, $\Sigma := (0,T)\times\partial\Omega$, and $\bar I := [0,T]$. We will use a standard notation for function spaces, namely the space of the continuous $\mathbb{R}^k$-valued functions $C(\bar\Omega;\mathbb{R}^k)$, its dual $\mathscr{M}(\bar\Omega;\mathbb{R}^k)$ (i.e., up to an isometrical isomorphism, the space of Borel measures), the continuously differentiable functions $C^1(\bar\Omega;\mathbb{R}^k)$, the Lebesgue space $L^p(\Omega;\mathbb{R}^k)$, the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^k)$, and the Bochner space of X-valued Bochner measurable p-integrable functions $L^p(I;X)$. If $X=(X')^*$, the notation $L^\infty_{\mathrm{w}*}(I;X)$ stands for space of weakly* measurable essentially bounded functions $I\to X$; this space is dual to the space $L^1(I;X')$ and, in general, is not equal to $L^\infty(I;X)$. If X is separable reflexive, then $L^\infty(I;X)=L^\infty_{\mathrm{w}*}(I;X)$ by Pettis' theorem. For the Dirichlet boundary condition (2.17), we also introduce the Banach space

$$W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n) := \big\{v\in W^{1,p}(\Omega;\mathbb{R}^n);\ v|_{\Gamma_0}=0\big\}.$$

Moreover, we denote by $B(\bar I;X)$, $B_{\mathrm{w}*}(\bar I;X)$, $\mathrm{BV}(\bar I;X)$ or $C_{\mathrm{w}}(\bar I;X)$ Banach space of the functions $\bar I\to X$ that are bounded Bochner measurable, bounded weakly* measurable, have a bounded variation or are weakly continuous, respectively; note that all these functions are defined everywhere on $\bar I$. We will use the notation $q'=q/(q{-}1)$ for the conjugate exponent to q. Instead of $u(t,\cdot)$ or $z(t,\cdot)$ or $w(t,\cdot)$, we will mostly write briefly $u(t)$ or $z(t)$ or $w(t)$, respectively. As far as the data, we will assume

$$u_0\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n), \qquad \dot u_0\in L^2(\Omega;\mathbb{R}^n), \qquad \theta_0\in L^\omega(\Omega), \qquad \theta_0\ge 0, \tag{3.15a}$$

$$f\in L^1(I;L^2(\Omega;\mathbb{R}^n)), \quad \theta_{\mathrm{ext}}\in L^1(\Sigma), \qquad \theta_{\mathrm{ext}}\ge 0. \tag{3.15b}$$

Also, to have the energy balance, we will need the initial condition z_0 be "semi-stable" with respect to $u(0)=u_0$ in the sense

$$\forall v\in W^{1,q}(\Omega;\mathbb{R}^m):\quad \int_\Omega \varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(u_0),v,\nabla v\big)+\zeta_1\big(v{-}z_0\big)\,\mathrm{d}x. \tag{3.16}$$

Definition 3.1 ***(Energetic solution.)*** *Assuming (3.7) and (3.15), we call a triple (u,z,w) with*

$$u\in C_{\mathrm{w}}(\bar I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{3.17a}$$

$$\frac{\partial u}{\partial t}\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))\cap\big(W^{1,2}(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)+W^{1,1}(I;L^2(\Omega;\mathbb{R}^n))\big), \tag{3.17b}$$

$$z\in B(\bar I;W^{1,q}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m)), \tag{3.17c}$$

$$w\in L^r(I;W^{1,r}(\Omega))\,\cap\,L^\infty(I;L^1(\Omega))\,\cap\,B_{\mathrm{w}*}(\bar I;\mathscr{M}(\bar\Omega)) \quad \textit{with any}\ \ 1\le r<\frac{n{+}2}{n{+}1}, \tag{3.17d}$$

$$\frac{\partial w}{\partial t}\in\mathscr{M}(\bar I;W^{1+n,2}(\Omega)^*) \tag{3.17e}$$

an energetic solution to (3.4) with the initial conditions (2.16) and the boundary conditions (3.5) if

(i) *the weakly formulated (3.4a) with (3.5a,b) holds, i.e. for all* $v\in C^1(\bar Q;\mathbb{R}^n)$ *such that* $v|_{\Sigma_0}=0$,

$$\int_\Omega \varrho\frac{\partial u}{\partial t}(T)\cdot v(T)\,\mathrm{d}x + \int_Q \Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \varphi_e'(e(u),z) + \mathscr{T}(w)\phi'(e(u))\Big) : e(v) \\ - \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_Q f\cdot v\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varrho\dot u_0\cdot v(0)\,\mathrm{d}x \qquad (3.18\mathrm{a})$$

(ii) *the weakly formulated enthalpy equation (3.4c) with (3.5c) holds, i.e. for all* $v\in C^1(\bar Q)$,

$$\int_{\bar\Omega} v(T)w(T,\mathrm{d}x) + \int_Q \mathscr{K}(e(u),z,w)\nabla w\cdot\nabla v - w\frac{\partial v}{\partial t} - \mathscr{T}(w)\phi'(e(u)) : e\big(\frac{\partial u}{\partial t}\big)v \\ - 2\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)v\,\mathrm{d}x\mathrm{d}t + \int_\Sigma b\mathscr{T}(w)v\,\mathrm{d}S\mathrm{d}t = \int_{\bar Q} v\,\mathfrak{h}_z(\mathrm{d}x\mathrm{d}t) + \int_\Omega w_0 v(0)\,\mathrm{d}x + \int_\Sigma b\theta_{\rm ext}v\,\mathrm{d}S\mathrm{d}t \qquad (3.18\mathrm{b})$$

where $w_0 = h_0(\theta_0)$ *and* $\mathfrak{h}_z\in\mathscr{M}(\bar Q)$ *is the measure (=heat produced by rate-independent dissipation) defined by prescribing its values for every closed set of the type* $A:=[t_1,t_2]\times B$ *with* B *a Borel subset of* Ω *by*

$$\mathfrak{h}_z(A) := \mathrm{Var}_S(z|_B;t_1,t_2) \quad \textit{with} \quad \mathrm{Var}_S(\tilde z;t_1,t_2) := \sup\sum_{i=1}^k\int_\Omega \delta_S^*\big(\tilde z(s_i,x){-}\tilde z(s_{i-1},x)\big)\,\mathrm{d}x$$

where the supremum is taken over all partitions of the type $t_1=s_0<...<s_k=t_2$, $k\in\mathbb{N}$,

(iii) *the total energy balance holds, i.e.*

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)),z(T),\nabla z(T))\,\mathrm{d}x + \int_{\bar\Omega} w(T,\mathrm{d}x) + \int_\Sigma b\mathscr{T}(w)\,\mathrm{d}S\mathrm{d}t \\ = \int_\Omega \frac{\varrho}{2}\big|\dot u_0\big|^2 + \varphi(e(u_0),z_0,\nabla z_0) + h_0(\theta_0)\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_\Sigma b\theta_{\rm ext}\mathrm{d}S\mathrm{d}t, \qquad (3.18\mathrm{c})$$

(iv) *the "semistability" holds for any* $v\in W^{1,q}(\Omega;\mathbb{R}^m)$ *and for a.a.* $t\in[0,T]$, *i.e.*

$$\int_\Omega \varphi(e(u(t)),z(t),\nabla z(t))\,\mathrm{d}x \le \int_\Omega \varphi(e(u(t)),v,\nabla v) + \zeta_1(v-z(t))\,\mathrm{d}x, \qquad (3.18\mathrm{d})$$

(v) *the initial conditions* $u(0)=u_0$ *and* $z(0)=z_0$ *hold.*

Note that (3.18c) is just (2.9) with ε from (2.11) when also (3.1)–(3.2) is taken into account. Note also that (3.17e) makes values of $w(t)$ well defined in the sense of $W^{1+n,2}(\Omega)^*$ and (3.17d) further shows that even $w(t)\in\mathscr{M}(\bar\Omega)$, which has been exploited in (3.18b,c) for the time $t=T$. It should be emphasized that $t\mapsto w(t)$ cannot be expect continuous in any sense because, since ζ_1 is homogeneous degree-1, the measure $\mathfrak{h}_z$ may concentrate at particular time instances. On the other hand, although (3.18b) itself could be used for $v(T)=0$ to eliminate $w(T)$, we actually need $w(T)$ for (3.18c). All these technicalities arise due to presense of rate-independent dissipation which may have tendency to concentrate heat production during jumping of the internal parameter z, in contrast to more conventional models with only rate-dependent terms like [63].

Since (3.4a,c) are standardly involved in (3.18a,b), the justification of Definition 3.1 needs to verify the inclusion (3.4b) in a weak sense, here

$$\int_Q \varphi_z'(e(u),z,\nabla z)\cdot\Big(v-\frac{\partial z}{\partial t}\Big)+\varphi_Z'(e(u),z,\nabla z):\nabla\Big(v-\frac{\partial z}{\partial t}\Big)+\zeta_1(v)\,\mathrm{d}x\mathrm{d}t\ge\int_Q\zeta_1\Big(\frac{\partial z}{\partial t}\Big)\mathrm{d}x\mathrm{d}t \tag{3.19}$$

for any $v\in C^1(\bar{Q};\mathbb{R}^m)$. Also note that the initial conditions $\frac{\partial u}{\partial t}(0)=\dot{u}_0$ and $w(0)=w_0=h_0(\theta_0)$, not explicitely required in Definition 3.1(v), are involved in (3.18a,b). Following [64, Sect.4], we can indeed prove (3.19) from (3.18):

Proposition 3.2 ***(Justification of energetic-solution concept.)*** *Let (3.6)–(3.16) hold. Any energetic solution with $\frac{\partial z}{\partial t}\in L^1(Q;\mathbb{R}^m)$ is also a weak solution in the sense that (3.18a,b), (3.19), and the initial conditions $u(0)=u_0$ and $z(0)=z_0$ hold.*

Sketch of proof. Using the definition $\mathrm{D}_z\Phi(e(u(t)),z(t),v)$ of the directional derivative of $\Phi(e(u(t)),\cdot):W^{1,q}(\Omega;\mathbb{R}^m)\to\mathbb{R}$ at $z(t)$ in the direction v with $\Phi(e,z):=\int_\Omega\varphi(e,z,\nabla z)\,\mathrm{d}x$, and using further the semi-stability (3.18d) of z at time t with respect to $z(t)+\varepsilon v$ and the degree-1 homogeneity of ζ_1, we obtain

$$\begin{aligned}\int_\Omega&\varphi_z'\big(e(u(t)),z(t),\nabla z(t)\big)\cdot v+\varphi_{\nabla z}'\big(e(u(t)),z(t),\nabla z(t)\big):\nabla v\,\mathrm{d}x\\&=\mathrm{D}_z\Phi(e(u(t)),z(t),v):=\lim_{\varepsilon\downarrow0}\frac{\Phi(e(u(t)),z(t){+}\varepsilon v))-\Phi(e(u(t)),z(t)}{\varepsilon}\\&=\lim_{\varepsilon\downarrow0}\int_\Omega\frac{\varphi(e(u(t)),z(t){+}\varepsilon v,\nabla z(t){+}\varepsilon\nabla v)-\varphi(e(u(t)),z(t),\nabla z(t))}{\varepsilon}\,\mathrm{d}x\\&\ge-\lim_{\varepsilon\downarrow0}\int_\Omega\frac{\zeta_1(z(t){+}\varepsilon v{-}z(t))}{\varepsilon}\,\mathrm{d}x=-\lim_{\varepsilon\downarrow0}\int_\Omega\zeta_1(v)\,\mathrm{d}x=-\int_\Omega\zeta_1(v)\,\mathrm{d}x.\end{aligned} \tag{3.20}$$

Then we test the force equilibrium (3.18a) by $\frac{\partial u}{\partial t}$. It is important that (3.18a) bears extension by continuity for the test functions $v\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^n))$ and that $\frac{\partial u^2}{\partial t^2}\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)+L^1(I;L^2(\Omega;\mathbb{R}^n))$ is in duality with $\frac{\partial u}{\partial t}\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^n))$, hence the by-part integration

$$\int_\Omega\varrho\frac{\partial u}{\partial t}(T)\cdot\frac{\partial u}{\partial t}(T)-\varrho\dot{u}_0\cdot\frac{\partial u}{\partial t}(0)\,\mathrm{d}x-\int_0^T\Big\langle\varrho\frac{\partial u}{\partial t},\frac{\partial^2u}{\partial t^2}\Big\rangle\,\mathrm{d}t=\frac{\varrho}{2}\int_\Omega\Big|\frac{\partial u}{\partial t}(T)\Big|^2-\big|\dot{u}_0\big|^2\,\mathrm{d}x \tag{3.21}$$

is legal as an equality. Thus we get the energy equality in the force equilibrium, cf. (4.67) below but as an equality. Moreover, $\mathfrak{h}_z=\zeta_1(\frac{\partial z}{\partial t})$ because now $\frac{\partial z}{\partial t}\in L^1(Q;\mathbb{R}^m)$; here also (3.11b) was used. Thus also $\frac{\partial w}{\partial t}\in L^1(I;W^{1+n,2}(\Omega)^*)$.

Further, we test (3.18b) by $v=1$; here it is important that 1 is in duality with $\frac{\partial w}{\partial t}\in L^1(I;W^{1+n,2}(\Omega)^*)$, hence we get also the energy equality in the thermal part.

Subtracting these two identities from (3.18c) gives

$$\int_Q\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\varphi_z'(e(u),z,\nabla z)\cdot\frac{\partial z}{\partial t}+\varphi_Z'(e(u),z,\nabla z):\nabla\frac{\partial z}{\partial t}\,\mathrm{d}x\mathrm{d}t\;=\;0. \tag{3.22}$$

Summing (3.22) with (3.20) integrated over $I=(0,T)$ just gives (3.19). $\square$

We can see that the concept of the weak solution as used in Proposition 3.2 requires additional qualification of $\varphi(e,\cdot,\cdot)$ and of $\frac{\partial z}{\partial t}$. In fact, $\frac{\partial z}{\partial t}$ could be eliminated from the left-hand side of (3.19) by substitution from (3.4a), cf. [64], but, more importantly, the concept of the weak solution does not wear enough information to track the energy balance. This is therefore particularly unsuitable in the context of thermodynamical evolution where we will ultimately exploit the concept of the energetic solution to execute Step 7 in the proof of Proposition 4.3. Let us sumarize the main analytical result in the following assertion:

Theorem 3.3 ***(Existence of energetic solutions.)*** *Let all the assumptions (3.6)–(3.16) hold. Then the initial-boundary-value problem for the system (3.4) with the initial conditions (2.16) and the boundary conditions (3.5) admits an energetic solution* (u,z,w) *in accord to Definition 3.1.*

4 Proof of existence of energetic solutions

An important phenomenon here is that, proving existence of a solution, we need to pass to the limit in the non-linear Nemytskiĭ operators induced by ζ_1 and ζ_2. Another peculiarity is that, due to degree-1 homogeneity of ζ_1, the heat equation has its right-hand side not only in $L^1(Q)$ (as it would be in case of a higher-degree homogeneity of dissipative-force potential) but even in measures.

The existence proof is therefore technically rather delicate. We will use a *fully implicit time-discretization* with a constant time step $\tau>0$ (assuming $K_\tau=T/\tau\in\mathbb{N}$) and a *regularization* of the force-equilibrium equation, leading to the following recursive increment formula

$$\begin{aligned}&\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}-\mathrm{div}\Big(\mathbb{D}e\Big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\varphi_e'(e(u_\tau^k),z_\tau^k)\\&\qquad\qquad\qquad+\mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k))+\tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\Big)=f_\tau^k,\end{aligned}\tag{4.1a}$$

$$\partial\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\varphi_z'(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k)-\mathrm{div}\varphi_Z'(z_\tau^k,\nabla z_\tau^k)\ni 0,\tag{4.1b}$$

$$\begin{aligned}&\frac{w_\tau^k-w_\tau^{k-1}}{\tau}-\mathrm{div}\big(\mathscr{K}(e(u_\tau^k),z_\tau^k,w_\tau^k)\nabla w_\tau^k\big)=\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)\\&\quad+\big(1-\sqrt{\tau}\big)\mathbb{D}e\Big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big):e\Big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k)):e\Big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big),\end{aligned}\tag{4.1c}$$

for $k=1,...,K_\tau=T/\tau$, starting for $k=1$ by using

$$u_\tau^0=u_{0,\tau},\qquad u_\tau^{-1}=u_{0,\tau}-\tau\dot{u}_0,\qquad z_\tau^0=z_0,\qquad w_\tau^0=w_0:=h_0(\theta_0).\tag{4.2}$$

Note that φ_e' does not depend on ∇z_τ^k due to (3.7). Of course, the system (4.1) is to be considered completed by the boundary conditions, i.e. here

$$u_\tau^k\big|_{\Gamma_0}=0,\tag{4.3a}$$

$$\Big(\mathbb{D}e\Big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\varphi_e'(e(u_\tau^k),z_\tau^k)+\mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k))+\tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\Big)\Big|_{\Gamma_1}\cdot\nu=0,\tag{4.3b}$$

$$\varphi_Z'(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k)\big|_\Gamma\cdot\nu=0,\tag{4.3c}$$

$$\mathscr{K}(e(u_\tau^k),z_\tau^k,w_\tau^k)\nabla w\big|_\Gamma\cdot\nu+b\mathscr{T}(w_\tau^k)\big|_\Gamma=b\theta_{\mathrm{ext},\tau}^k,\tag{4.3d}$$

where $\theta_{\mathrm{ext},\tau}^k$ is an approximation of θ_{ext} at time $t=k\tau$, similarly as f_τ^k in (4.1a) approximates f at $t=k\tau$. Note that, to compensate growth of the right-hand side terms in (4.1c), the equation (4.1a) involves a regularizing term $\tau\mathrm{div}(|e(u_\tau^k)|^{\gamma-2}e(u_\tau^k))$ which, later, will vanish when passing $\tau\downarrow 0$. For this reason we need also to regularize the initial condition u_0 by taking $u_{0,\tau}{\in}W^{1,\gamma}(\Omega;\mathbb{R}^n)$ in (4.2). We, however, did not regularize (4.1b) to avoid troubles with limit passage in semi-stability later.

As far as the (regularized) initial and boundary conditions and the loading are concerned, we assume

$$
\begin{aligned}
&u_{0,\tau}{\in}W^{1,\gamma}_{\Gamma_0}(\Omega;\mathbb{R}^n), \qquad \sup_{\tau>0}\int_\Omega\varphi\big(e(u_{0,\tau}),z_0,\nabla z_0\big)\,\mathrm{d}x<+\infty,\\
&\qquad\qquad \lim_{\tau\downarrow0}\sqrt[\gamma]{\tau}\big\|e(u_{0,\tau})\big\|_{L^\gamma(\Omega;\mathbb{R}^{n\times n})}=0, \qquad \lim_{\tau\downarrow0}u_{0,\tau}=u_0 \quad\text{in } W^{1,p}(\Omega;\mathbb{R}^n),
\end{aligned}\tag{4.4a}
$$

$$
\bar\theta_{\mathrm{ext},\tau}{\in}L^\infty(\Sigma), \qquad \bar\theta_{\mathrm{ext},\tau}\ge0, \qquad \lim_{\tau\downarrow0}\bar\theta_{\mathrm{ext},\tau}=\theta_{\mathrm{ext}}\ \text{in } L^1(\Sigma),\tag{4.4b}
$$

$$
\bar f_\tau{\in}L^\infty(I;L^2(\Omega;\mathbb{R}^n)), \quad \lim_{\tau\downarrow0}\bar f_\tau=f\ \text{in } L^1(I;L^2(\Omega;\mathbb{R}^n)), \quad \big\|\bar f_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))}\le\frac{K}{\sqrt{\tau}},\tag{4.4c}
$$

where $\bar\theta_{\mathrm{ext},\tau}|_{((k-1)\tau,k\tau]}=\theta^k_{\mathrm{ext},\tau}$ and $\bar f_\tau|_{((k-1)\tau,k\tau]}=f^k_\tau$ for $k=1,...,K_\tau$.

Lemma 4.1 *Let (3.6), (3.7), (3.10), (3.12), (3.15), and (4.4) hold, and let φ and ϕ be lower semicontinuous and satisfy (3.9) and (3.11). Moreover, let $\gamma\ge p$ be chosen so large that $\gamma>\max(\frac{2}{c(n)},(p_1{+}1)\frac{\omega}{\omega-1})$ and $\frac{1}{\omega}+\frac{p_1+1}{\gamma}<c(n)$ with $c(n)=1$ if $n\le2$ or $c(n)=\frac{n+2}{2n}$ if $n\ge3$ and let $\tau>0$ be sufficiently small, namely*

$$
\tau\le\Big(\frac{c_2}{\ell}\Big)^2\tag{4.5}
$$

with c_2 from (3.6c) and ℓ from (3.9) (or just $\tau\le T$ if φ is convex and thus $\ell=0$). Then there exists a weak solution $(u^k_\tau,z^k_\tau,w^k_\tau)\in W^{1,\gamma}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m)\times W^{1,2}(\Omega)$ to the boundary-value problem (4.1)–(4.3). Moreover, for any $k=1,...,K_\tau$, $w^k_\tau\ge0$ and the following "discrete mechanical energy" balance holds:

$$
\begin{aligned}
&\int_\Omega\frac{\varrho}{2}\Big|\frac{u^k_\tau{-}u^{k-1}_\tau}{\tau}\Big|^2+\varphi\big(e(u^k_\tau),z^k_\tau,\nabla z^k_\tau\big)+\frac{\tau}{\gamma}\big|e(u^k_\tau)\big|^\gamma\\
&\qquad+\tau\sum_{l=1}^k\bigg(\zeta_1\Big(\frac{z^l_\tau{-}z^{l-1}_\tau}{\tau}\Big)+2\big(1{-}\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{u^l_\tau{-}u^{l-1}_\tau}{\tau}\big)\Big)\bigg)\,\mathrm{d}x\\
&\qquad\le\int_\Omega\frac{\varrho}{2}\big|\dot u_0\big|^2+\varphi\big(e(u_{0,\tau}),z_0,\nabla z_0\big)+\frac{\tau}{\gamma}\big|e(u_{0,\tau})\big|^\gamma\\
&\qquad+\tau\sum_{l=1}^k\bigg(f^l_\tau{\cdot}\frac{u^l_\tau{-}u^{l-1}_\tau}{\tau}+\mathscr{T}(w^l_\tau)\phi'(e(u^l_\tau)):e\Big(\frac{u^l_\tau{-}u^{l-1}_\tau}{\tau}\Big)\bigg)\,\mathrm{d}x
\end{aligned}\tag{4.6}
$$

as well as the following "discrete total energy" balance holds:

$$
\begin{aligned}
&\int_\Omega\frac{\varrho}{2}\Big|\frac{u^k_\tau{-}u^{k-1}_\tau}{\tau}\Big|^2+\varphi\big(e(u^k_\tau),z^k_\tau,\nabla z^k_\tau\big)+w^k_\tau+\frac{\tau}{\gamma}\big|e(u^k_\tau)\big|^\gamma\,\mathrm{d}x+\tau\sum_{l=1}^k\int_\Gamma b\mathscr{T}(w^l_\tau)\mathrm{d}S\\
&\le\int_\Omega\frac{\varrho}{2}\big|\dot u_0\big|^2{+}\varphi\big(e(u_{0,\tau}),z_0,\nabla z_0\big){+}w_0{+}\tau\sum_{l=1}^k f^l_\tau{\cdot}\frac{u^l_\tau{-}u^{l-1}_\tau}{\tau}{+}\frac{\tau}{\gamma}\big|e(u_{0,\tau})\big|^\gamma\mathrm{d}x{+}\tau\sum_{l=1}^k\int_\Gamma b\theta^l_{\mathrm{ext},\tau}\mathrm{d}S,
\end{aligned}\tag{4.7}
$$

and also the "discrete semistability"

$$
\int_\Omega\varphi\big(e(u^k_\tau),z^k_\tau,\nabla z^k_\tau\big)\,\mathrm{d}x\le\int_\Omega\varphi\big(e(u^k_\tau),v,\nabla v\big)+\zeta_1(v-z^k_\tau)\,\mathrm{d}x\tag{4.8}
$$

holds for any $v\in W^{1,q}(\Omega;\mathbb{R}^m)$.

Proof. We can see existence of a conventional weak solution to (4.1) by standard methods for pseudomonotone set-valued operators induced by boundary-value problems for quasilinear elliptic equations. The coercivity of the underlying operator can be shown by testing the particular equations in (4.1) respectively by u_τ^k, z_τ^k, and $|w_\tau^k|^{\alpha-1}w_\tau^k$ with $0<\alpha<\min(q_0,\gamma/\max(2,(p_1{+}1)\omega/(\omega{-}1)))-1$ with q_0 from (3.6a). Here, for the a-priori estimate, we have used Hölder's inequality and the boundary conditions (4.3) for

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega|w_\tau^k|^{1+\alpha}\mathrm{d}x+\int_\Gamma b\mathscr{T}(w_\tau^k)|w_\tau^k|^{\alpha-1}w_\tau^k\mathrm{d}S\\&\le\int_\Omega\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)|w_\tau^k|^\alpha+2(1{-}\sqrt{\tau}\,)\zeta_2\Big(e\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big)\Big)|w_\tau^k|^\alpha\\&+\mathscr{T}(w_\tau^k)\big|\phi'(e(u_\tau^k))\big|\Big|e\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big)\Big||w_\tau^k|^\alpha+\frac{1}{\tau}|w_\tau^{k-1}|\,|w_\tau^k|^\alpha\,\mathrm{d}x+\int_\Gamma b\theta_{\mathrm{ext},\tau}^k|w_\tau^k|^\alpha\mathrm{d}S.\end{aligned}\tag{4.9}$$

The left-hand-side boundary term can be estimated from below by $\epsilon_{\omega_1,c_1}\int_\Gamma b|w_\tau^k|^{\alpha+1/\omega_1}\mathrm{d}x-C$ with $\epsilon_{\omega_1,c_1}>0$ depending on ω_1 and c_1 from (3.12b) and C large enough, which then can serve to handle the right-hand-side boundary term $\int_\Gamma b\theta_{\mathrm{ext},\tau}^k|w_\tau^k|^\alpha\mathrm{d}S$, using also (4.4b). The first right-hand-side term can be estimated as

$$\begin{aligned}\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)|w_\tau^k|^\alpha&\le C_\epsilon\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)^{1+\alpha}+\epsilon|w_\tau^k|^{1+\alpha}\\&\le C_\epsilon C_1^\alpha\Big|\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big|^{1+\alpha}+\epsilon|w_\tau^k|^{1+\alpha}\le C_{\epsilon,\tau}+\epsilon|z_\tau^k|^{q_0}+\epsilon|w_\tau^k|^{1+\alpha}\end{aligned}\tag{4.10}$$

with C_1 from (3.11b) and with C_ϵ and $C_{\epsilon,\tau}$ depending on ϵ and τ and, in the latter case, also α and C_1 and q_0 from (3.6a). Taking $\epsilon>0$ small enough, the last two terms can be absorbed in the left-hand sides of (4.9) and of (4.1b) tested by z_τ^k. The second right-hand-side term in (4.9) can be estimated similarly, using the sufficient growth of the regularing term on the left-hand sides in (4.1a) tested by u_τ^k; here we use that we can choose $\gamma>2(1{+}\alpha)$. The third right-hand-side term in (4.9) can be estimated

$$\begin{aligned}\mathscr{T}(w_\tau^k)\big|\phi'(e(u_\tau^k))\big|\Big|e\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big)\Big||w_\tau^k|^\alpha&\le C\big|w_\tau^k\big|^{1/\omega}\big(1+|e(u_\tau^k)|^{p_1}\big)\big(1+|e(u_\tau^k)|\big)\big|w_\tau^k\big|^\alpha\\&\le C\big(1+|e(u_\tau^k)|^{p_1+1}\big)^{\omega(1+\alpha)/(\omega-1)}+\epsilon\big|w_\tau^k\big|^{1+\alpha}\\&\le C+\epsilon\big|e(u_\tau^k)\big|^\gamma+\epsilon\big|w_\tau^k\big|^{1+\alpha}\end{aligned}\tag{4.11}$$

with C a generic constant; p_1 comes from (3.11a) and we used that $\gamma>(p_1{+}1)\omega(1{+}\alpha)/(\omega{-}1)$. Moreover, we used also (3.12b) which ensures $w=h_0(\theta)\ge\omega c_0(1{+}\theta)^\omega-\omega c_0$ so that

$$\theta=\mathscr{T}(w)\le\Big(\frac{w}{\omega c_0}{+}1\Big)^{1/\omega}-1\le\Big(\frac{w}{\omega c_0}\Big)^{1/\omega}.\tag{4.12}$$

Taking $\epsilon>0$ small enough, the last two terms in (4.11) can be absorbed in the left-hand sides of (4.9) and of (4.1a) tested by u_τ^k. Similarly and even more easily, using again (3.11a), one can estimate also the term $\mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k)):e(u_\tau^k)$ arising in (4.1a) when tested by u_τ^k. The remaining right-hand-side term in (4.9) can be estimated as $\frac{1}{\tau}|w_\tau^{k-1}|\,|w_\tau^k|^\alpha\le C|w_\tau^{k-1}|^{1+\alpha}+\epsilon|w_\tau^k|^{1+\alpha}$ and then again absorb the last term in the left-hand sides of (4.9). Altogether, we obtain an a-priori information of $(u_\tau^k,z_\tau^k,w_\tau^k)\in W^{1,\gamma}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m)\times L^{1+\alpha}(\Omega)$.

Now, we can see that the right-hand side of (4.1c) together with the regularized right-hand side $\theta^k_{\mathrm{ext},\tau}$ of the boundary conditions (4.3d) form a linear continuous functional on $W^{1,2}(\Omega)$, where we use that γ was chosen large enough. Indeed, as $q > \max(1, 2n/(n{+}4))$, cf. (3.6), we have $z^k_\tau, z^{k-1}_\tau \in W^{1,q}(\Omega;\mathbb{R}^m) \subset L^{1/c(n)+\epsilon}(\Omega)$, and thus the dissipative-heat term $\zeta_1(z^k_\tau{-}z^{k-1}_\tau)$ belongs to $L^{1/c(n)+\epsilon}(\Omega)$; here also (3.6) and (3.11b) have been used. As $\gamma > 2/c(n)$, the dissipative-heat term $\mathbb{D}e(u^k_\tau{-}u^{k-1}_\tau){:}(u^k_\tau{-}u^{k-1}_\tau)$ belongs to $L^{1/c(n)+\epsilon}(\Omega)$ for small (generic) $\epsilon > 0$. Also, due to (4.12), we have $\mathscr{T}(w^k_\tau) \in L^{1/\omega+\epsilon}(\Omega)$ and, due to (3.11a), we have $\phi'(e(u^k_\tau)) : (u^k_\tau{-}u^{k-1}_\tau) \in L^{\gamma/(p_1+1)}(\Omega)$, so that we have the adiabatic-heat term $\mathscr{T}(w^k_\tau)\phi'(e(u^k_\tau)) : (u^k_\tau{-}u^{k-1}_\tau) \in L^{1/c(n)+\epsilon}(\Omega)$ as we assumed $\frac{1}{\omega} + \frac{p_1+1}{\gamma} < c(n)$. The space $L^{1/c(n)+\epsilon}(\Omega)$ is naturally imbeded continuously into $W^{1,2}(\Omega)^*$. Hence by a bootstrap argument, testing the weak formulation of (4.1c) with the boundary conditions (4.3d) once again, now by w^k_τ, gives still the a-priori information about w^k_τ in $W^{1,2}(\Omega)$.

Since the regularization causes $w^k_\tau \in W^{1,2}(\Omega)$, we can use that $[w^k_\tau]^- \in W^{1,2}(\Omega)$ is a legal test function for (4.1c), which allows us to prove $w^k_\tau \geq 0$; here we use recursively that $w^{k-1}_\tau \geq 0$ starting from the initial condition $w^0_\tau = h_0(\theta_0) \geq 0$ and the property $[w^k_\tau]^- \mathscr{T}(w^k_\tau) = 0$ due to the definition (3.3), also by using $\theta^k_{\mathrm{ext},\tau} \geq 0$ assumed in (4.4b).

Let us now choose some $(u^k_\tau, z^k_\tau, w^k_\tau)$ solving (4.1). With these u^k_τ and w^k_τ given, let us still consider an auxiliary minimization problem, namely

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\int_\Omega \varrho\frac{u^k_\tau{-}2u^{k-1}_\tau{+}u^{k-2}_\tau}{\tau^2}\cdot u + \tau\zeta_1\Big(\frac{z{-}z^{k-1}_\tau}{\tau}\Big) + \frac{\tau}{\gamma}\big|e(u)\big|^\gamma \\ & \displaystyle\quad + \big(1{-}\sqrt{\tau}\,\big)\mathbb{D}e\big(\frac{u^k_\tau{-}u^{k-1}_\tau}{\tau}\big) : e(u) + \tau^{3/2}\zeta_2\Big(e\big(\frac{u{-}u^{k-1}_\tau}{\tau}\big)\Big) \\ & \displaystyle\quad + \varphi\big(e(u), z, \nabla z\big) + \mathscr{T}(w^k_\tau)\phi'(e(u^k_\tau)) : e(u) - f^k_\tau{\cdot}u\,\mathrm{d}x \\ \text{subject to} & (u,z) \in W^{1,\gamma}(\Omega;\mathbb{R}^n) \times W^{1,q}(\Omega;\mathbb{R}^m), \qquad u|_{\Gamma_0} = 0.\end{array}\right\} \tag{4.13}$$

Due to the assumed mode of convexity (3.9) and coercivity of φ and (3.6c), if τ is small as specified, (4.13) features a convex coercive functional and possesses therefore a solution which we denote by $(\tilde{u}^k_\tau, \tilde{z}^k_\tau)$. As this functional is even strictly convex, $(\tilde{u}^k_\tau, \tilde{z}^k_\tau)$ is determined uniquely as (u^k_τ, w^k_τ) is considered fixed. Writing optimality conditions for $(\tilde{u}^k_\tau, \tilde{z}^k_\tau)$ gives

$$\varrho\frac{u^k_\tau - 2u^{k-1}_\tau + u^{k-2}_\tau}{\tau^2} - \mathrm{div}\Big(\sqrt{\tau}\mathbb{D}e\big(\frac{\tilde{u}^k_\tau{-}u^{k-1}_\tau}{\tau}\big) + \varphi'_e\big(e(\tilde{u}^k_\tau), \tilde{z}^k_\tau\big) + \tau\big|e(\tilde{u}^k_\tau)\big|^{\gamma-2}e(\tilde{u}^k_\tau)\Big)$$

$$= f^k_\tau + \mathrm{div}\Big(\big(1{-}\sqrt{\tau}\,\big)\mathbb{D}e\big(\frac{u^k_\tau{-}u^{k-1}_\tau}{\tau}\big) + \mathscr{T}(w^k_\tau)\phi'(e(u^k_\tau))\Big), \tag{4.14a}$$

$$\partial\zeta_1\Big(\frac{\tilde{z}^k_\tau{-}z^{k-1}_\tau}{\tau}\Big) + \varphi'_z\big(e(\tilde{u}^k_\tau), \tilde{z}^k_\tau, \nabla\tilde{z}^k_\tau\big) - \mathrm{div}\varphi'_Z\big(e(\tilde{u}^k_\tau), \tilde{z}^k_\tau, \nabla\tilde{z}^k_\tau\big) \ni 0, \tag{4.14b}$$

with the boundary conditions (4.3a,c) with $(\tilde{u}^k_\tau, \tilde{z}^k_\tau)$ instead of (u^k_τ, z^k_τ) and

$$\Big(\sqrt{\tau}\mathbb{D}e\big(\frac{\tilde{u}^k_\tau{-}u^{k-1}_\tau}{\tau}\big) + \varphi'_e\big(e(\tilde{u}^k_\tau), \tilde{z}^k_\tau, \nabla\tilde{z}^k_\tau\big) + \tau\big|e(\tilde{u}^k_\tau)\big|^{\gamma-2}e(\tilde{u}^k_\tau)\Big)\Big|_{\Gamma_1}\cdot\nu$$

$$= \Big(\big(1{-}\sqrt{\tau}\,\big)\mathbb{D}e\big(\frac{u^k_\tau{-}u^{k-1}_\tau}{\tau}\big) + \mathscr{T}(w^k_\tau)\phi'(e(u^k_\tau))\Big)\Big|_{\Gamma_1}\cdot\nu \tag{4.15}$$

instead of (4.3b). Now it is important that the boundary-value problem (4.14)–(4.3a,c)–(4.15) represents the 1st-order sufficient optimality conditions for (4.13) if τ is small enough so that the functional in (4.13) is convex. Testing the difference of (4.1a) and (4.14a) by $u^k_\tau - \tilde{u}^k_\tau$ and the difference of (4.1b) and (4.14b) by $z^k_\tau - \tilde{z}^k_\tau$, in the sum we can see that $z_\tau^k=\tilde z_\tau^k$ and $u_\tau^k=\tilde u_\tau^k$ when taking into account the strict convexity of the underlying potential, namely

$$(u,z)\mapsto\int_\Omega\varphi(e(u),z,\nabla z)+\zeta_1(z{-}z_\tau^{k-1})+\frac{\tau\big|e(u)\big|^\gamma}{\gamma}+\frac{\zeta_2(e(u))}{\sqrt{\tau}}\,\mathrm{d}x \tag{4.16}$$

if $\tau>0$ is small as specified above in (4.5); note that here the assumed strict (e)-semiconvexity (3.9) is used. Then the functional in (4.13) must have a bigger or equal value on $(u_\tau^{k-1},z_\tau^{k-1})$ than on $(\tilde u_\tau^k,\tilde z_\tau^k)=(u_\tau^k,z_\tau^k)$, which gives

$$\begin{aligned}\int_\Omega&\frac{\varrho}{2}\Big|\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big|^2+\tau\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)+\tau\big(2{-}\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\big)\Big)\\&\qquad+\frac{\tau}{\gamma}\big|e(u_\tau^k)\big|^\gamma+\varphi\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)\,\mathrm{d}x\\&\qquad\le\int_\Omega\frac{\varrho}{2}\Big|\frac{u_\tau^{k-1}{-}u_\tau^{k-2}}{\tau}\Big|^2+\varphi\big(e(u_\tau^{k-1}),z_\tau^{k-1},\nabla z_\tau^{k-1}\big)+\tau f_\tau^k\cdot\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\\&\qquad+\frac{\tau\big|e(u_\tau^{k-1})\big|^\gamma}{\gamma}+\tau\mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k)):e\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x\end{aligned} \tag{4.17}$$

when employing also the algebraic inequality $(u_\tau^k{-}2u_\tau^{k-1}{+}u_\tau^{k-2})\cdot(u_\tau^k{-}u_\tau^{k-1})\ge\frac12|u_\tau^k{-}u_\tau^{k-1}|^2-\frac12|u_\tau^{k-1}{-}u_\tau^{k-2}|^2$. Summing it for $k=1,...,K_\tau$ just yields (4.6).

Now, to get (4.7), we still add (4.1c) tested by 1 to (4.17); here it is an important consequence of our carefully designed discretization (4.1) that the dissipative and adiabatic terms cancel.

As for (4.8), it suffices just to realize that (4.13) has a lower value on (u_τ^k,z_τ^k) than on (u_τ^k,v), which gives

$$\int_\Omega\varphi\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)+\tau\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x\le\int_\Omega\varphi\big(e(u_\tau^k),v,\nabla v\big)+\tau\zeta_1\Big(\frac{v{-}z_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x.$$

Then, one uses that ζ_1 is homogeneous degree-1 and thus satisfies the triangle inequality $\zeta_1(v{-}z_\tau^{k-1})\le\zeta_1(v{-}z_\tau^k)+\zeta_1(z_\tau^k{-}z_\tau^{k-1})$, which altogether gives

$$\begin{aligned}\int_\Omega\varphi\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)\,\mathrm{d}x&\le\int_\Omega\varphi\big(e(u_\tau^k),v,\nabla v\big)+\tau\zeta_1\Big(\frac{v{-}z_\tau^{k-1}}{\tau}\Big)-\tau\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x\\&\le\int_\Omega\varphi\big(e(u_\tau^k),v,\nabla v\big)+\tau\zeta_1\Big(\frac{v{-}z_\tau^k}{\tau}\Big)\,\mathrm{d}x,\end{aligned} \tag{4.18}$$

and thus (4.8) is proved. □

Let us define the piecewise affine interpolant (u_τ,z_τ,w_τ) by

$$u_\tau(t):=\frac{t-(k{-}1)\tau}{\tau}u_\tau^k+\frac{k\tau-t}{\tau}u_\tau^{k-1}\qquad\text{for }t\in[(k{-}1)\tau,k\tau], \tag{4.19}$$

and similarly $z_\tau(t)=\frac{t-(k-1)\tau}{\tau}z_\tau^k+\frac{k\tau-t}{\tau}z_\tau^{k-1}$ and $w_\tau(t)=\frac{t-(k-1)\tau}{\tau}w_\tau^k+\frac{k\tau-t}{\tau}w_\tau^{k-1}$ for $t\in[(k{-}1)\tau,k\tau]$ with $k=0,...,K_\tau:=T/\tau$. Also, we define the piecewise constant interpolant $(\bar u_\tau,\bar z_\tau,\bar w_\tau)$ by

$$\bar u_\tau(t):=u_\tau^k,\qquad\bar z_\tau(t):=z_\tau^k,\qquad\bar w_\tau(t):=w_\tau^k\qquad\text{for }t\in((k{-}1)\tau,k\tau], \tag{4.20}$$

for $k=0,...,K_\tau$. Eventually, we define $\bar f_\tau$ and $\bar\theta_{\mathrm{ext},\tau}$ by $\bar f_\tau|_{((k-1)\tau,k\tau]}:=f_\tau^k$ and recall that we already have defined $\bar\theta_{\mathrm{ext},\tau}|_{((k-1)\tau,k\tau]}:=\theta_{\mathrm{ext},\tau}^k$. Occasionally, we will use also a "retarded" piecewise constant interpolants $\underline{u}_\tau$, $\underline{z}_\tau$, and $\underline{w}_\tau$ defined by

$$\underline{u}_\tau(t):=u_\tau^{k-1},\qquad\underline{z}_\tau(t):=z_\tau^{k-1},\qquad\underline{w}_\tau(t):=w_\tau^{k-1}\qquad\text{for }t\in[(k{-}1)\tau,k\tau). \tag{4.21}$$

Proposition 4.2 ***(A-priori estimates for*** u_τ***,*** z_τ ***and*** w_τ***.)*** *Let, beside the assumptions from Lemma 4.1, the exponents* p *and* p_1 *and* ω *from (3.6a), (3.11a), (3.12b) satisfy (3.13), and let further (4.4) and (4.5) hold. Then it holds*

$$\big\|u_\tau\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^n))\,\cap\, L^\infty(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))\,\cap\, W^{1,2}(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))} \le C, \tag{4.22a}$$

$$\big\|\bar z_\tau\big\|_{L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))\,\cap\,\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))} \le C, \tag{4.22b}$$

$$\big\|\bar w_\tau\big\|_{L^\infty(I;L^1(\Omega))\,\cap\, L^r(I;W^{1,r}(\Omega))\,\cap\,\mathrm{BV}(\bar I;W^{1+n,2}(\Omega)^*)} \le C \quad \textit{with any } 1\le r<\tfrac{n+2}{n+1}, \tag{4.22c}$$

$$\big\|\bar u_\tau\big\|_{L^\infty(I;W^{1,\gamma}(\Omega;\mathbb{R}^n))} \le \frac{C}{\sqrt[\gamma]{\tau}}. \tag{4.22d}$$

Moreover, if also (3.14a) holds, then we also have the "dual" estimate of $\frac{\partial^2}{\partial t^2}u_\tau$ *as a measure, cf. (4.45) below, namely*

$$\Big\|\frac{\partial u_\tau}{\partial t}\Big\|_{\mathrm{BV}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)} \le C\,. \tag{4.23}$$

Proof. The first and second estimates in (4.22a), the first estimates in (4.22b,c), and (4.22d) follow quite directly from (4.7) by using the coercivity (3.6a) and by estimating

$$\tau\int_\Omega f_\tau^l\cdot\frac{u_\tau^l-u_\tau^{l-1}}{\tau}\,\mathrm{d}x \le \tau\big\|f_\tau^l\big\|_{L^2(\Omega;\mathbb{R}^n)}\Big(\frac{K\sqrt{T}}{\varrho}+\frac{\varrho}{4K\sqrt{T}}\Big\|\frac{u_\tau^l-u_\tau^{l-1}}{\tau}\Big\|^2_{L^2(\Omega;\mathbb{R}^n)}\Big) \tag{4.24}$$

and by using the discrete Gronwall's inequality which works here if the overall coefficient in front of $\frac{\varrho}{2}\|\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\|^2_{L^2(\Omega;\mathbb{R}^n)}$ in (4.7) is away from zero. This actually holds here since (4.4c) and $\tau\le T$ imply $\tau\big\|f_\tau^l\big\|_{L^2(\Omega;\mathbb{R}^n)}\frac{\varrho}{4K\sqrt{T}}\le\sqrt{\tau}\frac{\varrho}{4\sqrt{T}}\le\frac{\varrho}{4}$ so that the first term in (4.7) still dominates. Here we also have benefited from having already proved that $w_\tau^k\ge 0$.

Now we make estimation of ∇w_τ by exploiting the technique proposed by Boccardo and Gallouët [14, 15]. Simplified like in [26], we use the test of the enthalpy equation (4.1c) by $\chi(w_\tau^k)$ using an increasing nonlinear function $\chi:[0,+\infty)\to[0,1]$ defined by

$$\chi(w):=1-\frac{1}{(1{+}w)^\beta},\qquad \beta>0. \tag{4.25}$$

Let us abbreviate the right-hand side of (4.1c) by r_τ^k and then

$$\bar r_\tau=\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+2(1{-}\sqrt{\tau})\zeta_2\Big(e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big)+\mathscr{T}(\bar w_\tau)\phi'(e(\bar u_\tau)):e\Big(\frac{\partial u_\tau}{\partial t}\Big). \tag{4.26}$$

Note that we have already proved that $\bar r_\tau$ is in $L^1(Q)$, although we have not yet proved that it is bounded independently of τ. Now we execute the announced test of (4.1c) by $\chi(w_\tau^k)$, use the Green formula, and sum it for $k=1,...,K_\tau$. It is important here that $\chi(w_\tau^k)\in W^{1,2}(\Omega)$, hence it is a legal test function, because $0\le w_\tau^k\in W^{1,2}(\Omega)$ has already been proved and because χ is Lipschitz continuous on $[0,+\infty)$. Realizing that

$\chi'(w)=\beta/(1{+}w)^{1+\beta}$ and denoting by $\kappa_0>0$ the infimum in (3.12d), we get

$$\begin{aligned}\kappa_0\beta\int_Q\frac{|\nabla\bar w_\tau|^2}{(1+\bar w_\tau)^{1+\beta}}\,\mathrm{d}x\mathrm{d}t&=\kappa_0\int_Q\chi'(\bar w_\tau)|\nabla\bar w_\tau|^2\,\mathrm{d}x\mathrm{d}t\\&\le\int_Q\chi'(\bar w_\tau)\mathscr{K}(e(\bar u_\tau),\bar z_\tau,\bar w_\tau)\nabla\bar w_\tau\cdot\nabla\bar w_\tau\,\mathrm{d}x\mathrm{d}t\\&=\int_Q\mathscr{K}(e(\bar u_\tau),\bar z_\tau,\bar w_\tau)\nabla\bar w_\tau\cdot\nabla\chi(\bar w_\tau)\,\mathrm{d}x\mathrm{d}t\\&\le\int_Q\mathscr{K}(e(\bar u_\tau),\bar z_\tau,\bar w_\tau)\nabla\bar w_\tau\cdot\nabla\chi(\bar w_\tau)\,\mathrm{d}x\mathrm{d}t\\&\qquad+\int_\Omega\widehat\chi(w_\tau(T,\cdot))\,\mathrm{d}x+\int_\Sigma b\mathscr{T}(\bar w_\tau)\chi(w_\tau)\,\mathrm{d}S\mathrm{d}t\\&\le\int_\Omega\widehat\chi(w_0)\,\mathrm{d}x+\int_\Sigma b\bar\theta_{\mathrm{ext},\tau}\chi(\bar w_\tau)\,\mathrm{d}S\mathrm{d}t+\int_Q\bar r_\tau\chi(\bar w_\tau)\,\mathrm{d}x\mathrm{d}t\\&\le\big\|w_0\big\|_{L^1(\Omega)}+\big\|b\big\|_{L^\infty(\Gamma)}\big\|\bar\theta_{\mathrm{ext},\tau}\big\|_{L^1(\Sigma)}+\big\|\bar r_\tau\big\|_{L^1(Q)}\\&=:C_1+C_2\big\|\bar r_\tau\big\|_{L^1(Q)},\end{aligned}\tag{4.27}$$

where $\widehat\chi$ is the primitive function of χ such that $\widehat\chi(0)=0$. In (4.27), we used $\widehat\chi(w)\le w$ and also we used monotonicity of χ and hence convexity of $\widehat\chi$ so that the “discrete chain rule” holds:

$$\frac{\widehat\chi(w_\tau^k)-\widehat\chi(w_\tau^{k-1})}{\tau}\le\frac{w_\tau^k-w_\tau^{k-1}}{\tau}\chi(w_\tau^k).\tag{4.28}$$

Now we take $1\le r<2$. By Hölder's inequality and by (4.27),

$$\begin{aligned}\int_Q\big|\nabla\bar w_\tau\big|^r\mathrm{d}x\mathrm{d}t&=\int_Q\frac{|\nabla\bar w_\tau|^r}{(1+\bar w_\tau)^{(1+\beta)r/2}}(1+\bar w_\tau)^{(1+\beta)r/2}\mathrm{d}x\mathrm{d}t\\&\le\Big(\int_Q\frac{|\nabla\bar w_\tau|^2}{(1+\bar w_\tau)^{1+\beta}}\mathrm{d}x\mathrm{d}t\Big)^{r/2}\Big(\int_Q\big(1+\bar w_\tau\big)^{(1+\beta)r/(2-r)}\mathrm{d}x\mathrm{d}t\Big)^{(2-r)/2}\\&\le\Big(C_1+C_2\big\|\bar r_\tau\big\|_{L^1(Q)}\Big)^{r/2}\Big(\int_0^T\big\|1+\bar w_\tau(t,\cdot)\big\|_{L^{(1+\beta)r/(2-r)}(\Omega)}^{(1+\beta)r/(2-r)}\mathrm{d}t\Big)^{(2-r)/2}.\end{aligned}\tag{4.29}$$

Then, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\big\|1{+}\bar w_\tau(t,\cdot)\big\|_{L^{(1+\beta)r/(2-r)}(\Omega)}&\le C_{\rm GN}\Big(\big\|1{+}\bar w_\tau(t,\cdot)\big\|_{L^1(\Omega)}+\big\|\nabla\bar w_\tau(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^d)}\Big)^\lambda\big\|1{+}\bar w_\tau(t,\cdot)\big\|_{L^1(\Omega)}^{1-\lambda}\\&\le C_{\rm GN}\big(|\Omega|{+}C_3\big)^{1-\lambda}\Big(|\Omega|+C_3+\big\|\nabla\bar w_\tau(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^d)}\Big)^\lambda\end{aligned}\tag{4.30}$$

for

$$\frac{2-r}{(1+\beta)r}\ge\lambda\Big(\frac1r-\frac1n\Big)+1-\lambda\qquad\text{with}\qquad 0<\lambda\le1.\tag{4.31}$$

We rise (4.30) to the power $(1{+}\beta)r/(2{-}r)$, use it in (4.29), and choose $\lambda:=$

$(2{-}r)/(1{+}\beta)$:

$$
\begin{aligned}
&\Big(\int_0^T \|1+\bar{w}_\tau(t,\cdot)\|_{L^{(1+\beta)r/(2-r)}(\Omega)}^{(1+\beta)r/(2-r)}\mathrm{d}t\Big)^{(2-r)/2}\\
&\qquad\le \Big(\int_0^T C_{\rm GN}^{\frac{(1+\beta)r}{2-r}}\big(|\Omega|{+}C_3\big)^{(1-\lambda)\frac{(1+\beta)r}{2-r}}\Big(|\Omega|{+}C_3{+}\|\nabla\bar{w}_\tau(t,\cdot)\|_{L^r(\Omega;\mathbb{R}^d)}\Big)^{\lambda\frac{(1+\beta)r}{2-r}}\mathrm{d}t\Big)^{\frac{2-r}{2}}\\
&\qquad\le \Big(\int_0^T C_{\rm GN}^{\frac{(1+\beta)r}{2-r}}\big(|\Omega|{+}C_3\big)^{(1-\lambda)\frac{(1+\beta)r}{2-r}}\Big(|\Omega|{+}C_3{+}\|\nabla\bar{w}_\tau(t,\cdot)\|_{L^r(\Omega;\mathbb{R}^d)}\Big)^{r}\mathrm{d}t\Big)^{\frac{2-r}{2}}\\
&\qquad= C_3+C_4\Big(\int_Q|\nabla\bar{w}_\tau|^r\mathrm{d}x\mathrm{d}t\Big)^{(2-r)/2}.
\end{aligned}\tag{4.32}
$$

By merging (4.29) with (4.32), one obtains the estimate $\|\nabla\bar{w}_\tau\|_{L^r(Q;\mathbb{R}^d)}^r/(1{+}\|\nabla\bar{w}_\tau\|_{L^r(Q;\mathbb{R}^d)}^{r(1-r/2)})\le C(1{+}\|\bar{r}_\tau\|_{L^1(Q)})^{r/2}$ with some C large enough, which further gives

$$
\big\|\nabla\bar{w}_\tau\big\|_{L^r(Q;\mathbb{R}^d)}^r-C_5\le\bigg(\frac{\|\nabla\bar{w}_\tau\|_{L^r(Q;\mathbb{R}^d)}^r}{1+\|\nabla\bar{w}_\tau\|_{L^r(Q;\mathbb{R}^d)}^{r(1-r/2)}}\bigg)^{2/r}\le C_6\big(1+\big\|\bar{r}_\tau\big\|_{L^1(Q)}\big)\tag{4.33}
$$

for C_5 and C_6 large enough. Substituting the above-mentioned choice of $\lambda:=(2-r)/(1{+}\beta)$ into (4.31), one gets after some algebra the conditions $r\le\frac{n+2-\beta n}{n+1}<\frac{n+2}{n+1}$, as indeed used in (4.22c); note that $0<\lambda<1$ needed in (4.31) is automatically ensured by $1\le r<2$ and $\beta>0$.

Further, we sum (4.17) for $k=1,...,K_\tau$, which gives, after forgetting the non-negative energy at time T, the estimate

$$
\begin{aligned}
\int_Q\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+(2{-}\sqrt{\tau})\zeta_2\Big(\frac{\partial e(u_\tau)}{\partial t}\Big)&\le\int_\Omega\varphi(e(u_{0,\tau}),z_0,\nabla z_0)+\frac{\tau}{\gamma}\big|e(u_{0,\tau})\big|^\gamma\,\mathrm{d}x\\
&\quad+\frac{\rho}{2}\big\|\dot{u}_0\big\|_{L^2(\Omega;\mathbb{R}^d)}^2+\int_Q\bar{f}_\tau{\cdot}\frac{\partial u_\tau}{\partial t}+\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)){:}\frac{\partial e(u_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{4.34}
$$

Now we substitute $\bar{r}_\tau$ from (4.26) into (4.33) and multiply by a sufficiently small weight, say $1/(2C_6)$, which gives

$$
\begin{aligned}
\frac{1}{2C_6}\big\|\nabla\bar{w}_\tau\big\|_{L^r(Q;\mathbb{R}^d)}^r\le\frac12+\frac{C_5}{2C_6}+\frac12\int_Q\bigg|\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+2(1{-}\sqrt{\tau}\,)\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)&\\
+\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)):e\Big(\frac{\partial u_\tau}{\partial t}\Big)\bigg|\mathrm{d}x\mathrm{d}t.&
\end{aligned}\tag{4.35}
$$

By this way, the dissipation terms $\int_Q\zeta_1(\frac{\partial z_\tau}{\partial t})\mathrm{d}x\mathrm{d}t$ and $\int_Q 2(1{-}\sqrt{\tau}\,)\zeta_2(e\big(\frac{\partial u_\tau}{\partial t}\big))\mathrm{d}x\mathrm{d}t$ contained in the right-hand side of (4.33) can be dominated by the corresponding left-hand-side terms in (4.17) when we sum (4.35) and (4.35). More specifically, we get

$$
\begin{aligned}
&\frac{c_1}{2}\Big\|\frac{\partial z_\tau}{\partial t}\Big\|_{L^1(Q;\mathbb{R}^m)}+c_2\Big\|\frac{\partial e(u_\tau)}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^{n\times n}_{\rm sym})}^2+\frac{1}{2C_6}\big\|\nabla\bar{w}_\tau\big\|_{L^r(Q;\mathbb{R}^n)}^r\\
&\qquad\le\int_Q\frac12\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+\zeta_2\Big(\frac{\partial e(u_\tau)}{\partial t}\Big)+\frac{1}{2C_6}\big|\nabla\bar{w}_\tau\big|^r\,\mathrm{d}x\mathrm{d}t\\
&\qquad\le\int_\Omega\varphi(e(u_{0,\tau}),z_0,\nabla z_0)+\frac{\tau}{\gamma}\big|e(u_{0,\tau})\big|^\gamma\,\mathrm{d}x+\frac{\rho}{2}\big\|\dot{u}_0\big\|_{L^2(\Omega;\mathbb{R}^d)}^2\\
&\qquad\quad+\int_Q\bar{f}_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t+\frac12+\frac{C_5}{2C_6}+\frac32\int_Q\Big|\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)):\frac{\partial e(u_\tau)}{\partial t}\Big|\,\mathrm{d}x\mathrm{d}t;
\end{aligned}\tag{4.36}
$$

note that c_1 and c_2 came from (3.6b,c). We estimate the last term in (4.36) by Hölder's and Young's inequalities as

$$\begin{aligned}
&\frac{3}{2}\int_Q\Big|\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)):\frac{\partial e(u_\tau)}{\partial t}\Big|\,\mathrm{d}x\mathrm{d}t\\
&\le C_{\delta_1}\big\|\mathscr{T}(\bar{w}_\tau)\big\|_{L^{p_3}(Q)}^{p_3}+\delta_1\big\|\phi'(e(\bar{u}_\tau))\big\|_{L^{p_2/p_1}(Q;\mathbb{R}^{n\times n})}^{p_2/p_1}+\delta_1\Big\|\frac{\partial e(u_\tau)}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^{n\times n})}^2\\
&\le \frac{C_{\delta_1}}{\sqrt[\omega]{\omega c_0}}\big\|\bar{w}_\tau\big\|_{L^{p_3/\omega}(Q)}^{p_3/\omega}+\delta_1\big\|\phi'(e(\bar{u}_\tau))\big\|_{L^{p_2/p_1}(Q;\mathbb{R}^{n\times n})}^{p_2/p_1}+\delta_1\Big\|\frac{\partial e(u_\tau)}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^{n\times n})}^2
\end{aligned}\tag{4.37}$$

with $p_1>0$, $p_2:=\max(p,2)$, $p_3:=2p_2/(p_2-2p_1)$ so that $\frac{1}{p_3}+\frac{p_1}{p_2}+\frac{1}{2}=1$, where C_{δ_1} depends on $\delta_1>0$. Note that p_3 is finite due to (3.13) and the constant c_0 comes from (3.12b); here we again used (4.12). For $p_1=0$, we have ϕ' bounded, cf. the assumption (3.11a), and we can simply forget the term $\delta_1\|\phi'(e(\bar{u}_\tau))\|_{L^{p_2/p_1}(Q;\mathbb{R}^{n\times n})}^{p_2/p_1}$ if taking C_{δ_1} large enough. If $\delta_1>0$ is small, the last term can be absorbed in the left-hand side of (4.36).

Next, we estimate the term $\delta_1\|\phi'(e(\bar{u}_\tau))\|_{L^{p_2/p_1}(Q;\mathbb{R}^{n\times n})}^{p_2/p_1}$ in (4.37) by $\delta_1\frac{p_2}{p_1}C^{p_2/p_1}(|Q|+\|e(\bar{u}_\tau)\|_{L^{p_2}(Q;\mathbb{R}^{n\times n})}^{p_2})$ with C here from (3.11a). Now, if $p\ge 2$, we have $p_2=p$ and thus we have the term $\|e(\bar{u}_\tau)\|_{L^{p_2}(Q;\mathbb{R}^{n\times n})}^{p_2}$ already estimated since $e(\bar{u}_\tau)$ has already been proved bounded in $L^\infty(I;L^p(\Omega;\mathbb{R}^{n\times n}))$. If $p<2$, then $p_2=2$ and we use

$$\big\|e(\bar{u}_\tau(t))\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}^2\le 2\big\|e(u_{0,\tau})\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}^2+2T\int_0^t\Big\|\frac{\partial e(u_\tau)}{\partial t}\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}^2\mathrm{d}t,\tag{4.38}$$

hence $\|e(\bar{u}_\tau)\|_{L^2(Q;\mathbb{R}^{n\times n})}^2\le 2T\|e(u_{0,\tau})\|_{L^2(\Omega;\mathbb{R}^{n\times n})}^2+2T^2\|\frac{\partial}{\partial t}e(u_\tau)\|_{L^2(Q;\mathbb{R}^{n\times n})}^2$. As it is pre-multiplied by δ_1 in (4.37), we can eventually absorb this term in the left-hand side of (4.36) if $\delta_1>0$ is taken small enough.

Further, using the already obtained estimate $\|\bar{w}_\tau\|_{L^\infty(I;L^1(\Omega))}\le C$ and Gagliardo-Nirenberg's inequality once more yields:

$$\begin{aligned}
\big\|\bar{w}_\tau(t,\cdot)\big\|_{L^{p_3/\omega}(\Omega)}&\le C_{\mathrm{GN},2}\big\|\bar{w}_\tau(t,\cdot)\big\|_{L^1(\Omega)}^{1-\mu}\Big(\big\|\bar{w}_\tau(t,\cdot)\big\|_{L^1(\Omega)}+\big\|\nabla\bar{w}_\tau(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^n)}\Big)^\mu\\
&\le C_{\mathrm{GN},2}C^{1-\mu}\Big(C+\big\|\nabla\bar{w}_\tau(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^n)}\Big)^\mu
\end{aligned}\tag{4.39}$$

for

$$\frac{\omega}{p_3}\ge\mu\Big(\frac{1}{r}-\frac{1}{n}\Big)+1-\mu.\tag{4.40}$$

Now, we rise (4.39) to the power p_3/ω and, assuming

$$\frac{\mu p_3}{\omega}<r,\tag{4.41}$$

we integrate it over $I=(0,T)$ and use Young's inequality

$$\big\|\bar{w}_\tau\big\|_{L^{p_3/\omega}(Q)}^{p_3/\omega}\le C_{\mathrm{GN},2}^{p_3/\omega}C^{\frac{1-\mu}{\omega}p_3}\int_0^T\Big(C+\big\|\nabla\bar{w}_\tau(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^n)}\Big)^{\frac{\mu}{\omega}p_3}\mathrm{d}t\le C_7+\delta_2\big\|\nabla\bar{w}_\tau\big\|_{L^r(Q;\mathbb{R}^n)}^r,\tag{4.42}$$

where C_7 depends here on $C_{\mathrm{GN},2}$, C, μ, p_3, ω, r, and δ_2. We further substitute it into (4.37), and then into (4.36). As we have δ_1 (and thus also C_{δ_1}) already fixed, we can now choose $\delta_2>0$ so small that we can absorb the right-hand-side term

$\delta_2 C_{\delta_1}(\omega c_0)^{-\omega}\|\nabla\bar w_\tau\|^r_{L^r(\Omega;\mathbb{R}^d)}$ in the left-hand side of (4.36). It eventually gives the the rest of (4.22a,b) and the second estimate in (4.22c). In particular, let us note that, although the left-hand side of (4.36) yields the L^1-estimate on $\frac{\partial}{\partial t}z_\tau$, we are able to formulate it as a BV-estimate for the piece-wise constant interpolant $\bar z_\tau$ in (4.22b) because of the identity $\|\bar z_\tau\|_{\mathscr{M}(\bar I;L^1(\Omega;\mathbb{R}^m))} = \|\frac{\partial}{\partial t}z_\tau\|_{L^1(Q;\mathbb{R}^m)}$.

Now, let us analyze the above conditions. Taking into account $r<\frac{n+2}{n+1}$, (4.40) and (4.41) imply respectively

$$\frac{\omega}{p_3} > 1-\mu\frac{2n+2}{n^2{+}2n} \qquad\text{and}\qquad \frac{\omega}{p_3} > \mu\frac{n+1}{n+2}. \tag{4.43}$$

The optimal value of μ makes both these lower bounds equal to each other, which takes place if $\mu = n/(n{+}1)$; note that $0<\mu<1$ is indeed satisfied, as desired for (4.39). In this way, (4.40) (or equally (4.41)) yields (3.13).

The “dual” estimate for $\frac{\partial w_\tau}{\partial t}$ follows, by using (4.1c) with (4.3d), from the already obtained estimates (4.22a-c) by

$$\Big\|\frac{\partial w_\tau}{\partial t}\Big\|_{L^1(I;W^{1+n,2}(\Omega)^*)} = \sup_{v\in L^\infty(I;W^{1+n,2}(\Omega))}\int_Q\frac{\partial w_\tau}{\partial t}v\,\mathrm{d}x\mathrm{d}t$$
$$= \sup_{v\in L^\infty(I;W^{1+n,2}(\Omega))}\int_Q \bar r_\tau{\cdot}v - \mathscr{K}(e(\bar u_\tau),\bar z_\tau,\bar w_\tau)\nabla\bar w_\tau{\cdot}\nabla v\,\mathrm{d}x\mathrm{d}t + \int_\Sigma b\big(\bar\theta_{\mathrm{ext},\tau}{-}\mathscr{T}(\bar w_\tau)\big)v\,\mathrm{d}S\mathrm{d}t. \tag{4.44}$$

Now we can estimate it by using ∇v bounded in $L^\infty(Q;\mathbb{R}^n)$ and the already proved (parts of) estimates (4.22a-c) and the fact that $\bar r_\tau$ is already proved bounded in $L^1(Q)$. Similarly as already used for $\bar z_\tau$, we have also here $\|\bar w_\tau\|_{\mathscr{M}(\bar I;W^{1+n,2}(\Omega)^*)} = \|\frac{\partial}{\partial t}w_\tau\|_{L^1(I;W^{1+n,2}(\Omega)^*)}$, which eventually gives the last BV-part in (4.22c).

Moreover, as for the estimate (4.23), let us realize that $\frac{\partial u_\tau}{\partial t}$ is piecewise constant in time, hence $\frac{\partial^2 u_\tau}{\partial t^2}$ is a measure on $\bar I=[0,T]$ supported just at the jumps of $\frac{\partial u_\tau}{\partial t}$, namely

$$\frac{\partial^2 u_\tau}{\partial t^2} = \sum_{k=1}^{K_\tau}\frac{u_\tau^k{-}2u_\tau^{k-1}{+}u_\tau^{k-2}}{\tau^2}\,\delta(\cdot-k\tau) \tag{4.45}$$

where here δ denotes the Dirac measure. Thus, we can estimate

$$\begin{aligned}\Big\|\frac{\partial^2 u_\tau}{\partial t^2}\Big\|_{\mathscr{M}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)} &\le \sum_{k=1}^{K_\tau}\Big\|\frac{u_\tau^k{-}2u_\tau^{k-1}{+}u_\tau^{k-2}}{\tau^2}\Big\|_{W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*}\\ &= \sum_{k=1}^{K_\tau}\sup_{\|v\|_{W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)}\le 1}\int_\Omega\frac{1}{\varrho}\Big(\mathbb{D}e\big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\big)+\varphi_e'(e(u_\tau^k),z_\tau^k)\\ &\qquad + \mathscr{T}(w_\tau^k)\phi'(e(u_\tau^k)) + \tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\Big) : e(v) - f_\tau^k\cdot v\,\mathrm{d}x\\ &= \sup_{\|v\|_{C(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n))}\le 1}\int_Q\frac{1}{\varrho}\Big(\mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big)+\varphi_e'(e(\bar u_\tau),\bar z_\tau)\\ &\qquad + \mathscr{T}(\bar w_\tau)\phi'(e(\bar u_\tau)) + \tau\big|e(\bar u_\tau)\big|^{\gamma-2}e(\bar u_\tau)\Big){:}e(v) - \bar f_\tau{\cdot}v\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.46}$$

With the same constants as in (4.37), we have also

$$\big\|\mathscr{T}(\bar w_\tau)\phi'(e(\bar u_\tau))\big\|^2_{L^2(Q;\mathbb{R}^{n\times n})} \le \frac{2C_{\delta_1}}{3\sqrt[\omega]{\omega c_0}}\big\|\bar w_\tau\big\|^{p_3/\omega}_{L^{p_3/\omega}(Q)} + \frac{2\delta_1}{3}\big\|\phi'(e(\bar u_\tau))\big\|^{p_2/p_1}_{L^{p_2/p_1}(Q;\mathbb{R}^{n\times n})}$$

and thus, by (4.22a,c), we have $\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau))$ estimated in $L^2(Q;\mathbb{R}^{n\times n})$. Furthermore, by (4.22a-c) and (3.14a), we have $\mathbb{D}e(\frac{\partial u_\tau}{\partial t})$ bounded in $L^2(Q;\mathbb{R}^{n\times n})$, $\varphi_e'(e(\bar{u}_\tau),\bar{z}_\tau)$ bounded in $L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))$, $\tau|e(\bar{u}_\tau)|^{\gamma-2}e(\bar{u}_\tau)$ bounded in $L^{\gamma/(\gamma-1)}(Q;\mathbb{R}^{n\times n})$ (even as $\mathscr{O}(\tau^{1/\gamma})$), and also $\bar{f}_\tau$ is bounded in $L^1(I;L^2(\Omega;\mathbb{R}^n))$ uniformly with respect to τ. Using Hölder's inequality for (4.46), we eventually prove (4.23). $\square$

Proposition 4.3 ***(Convergence for*** $\tau\downarrow 0$***.)*** *Let the assumptions of Lemma 4.1 together with (3.8) and (3.13) hold. Moreover, let also (3.14) and (3.16) hold. Then, in terms of subsequences,* $\{(u_\tau,z_\tau,w_\tau)\}_{\tau>0}$ *converges weakly* in the topologies indicated in (4.22a-c) and (4.23). Every limit* (u,z,w) *obtained by this way is an energetic solution in accord to Definition 3.1. In particular, such a solution does exist, as claimed in Theorem 3.3.*

Proof. For lucidity, let us divide the proof into seven steps.

Step 1: Discrete variant of (3.18). Testing (4.1a) by some v^k and using the discrete "by-part" summation

$$\sum_{k=1}^{K_\tau}(u^k-2u^{k-1}+u^{k-2})\cdot v^k = \big(u^{K_\tau}-u^{K_\tau-1}\big)\cdot v^{K_\tau} - (u^0-u^{-1})\cdot v^1 - \sum_{k=2}^{K_\tau}(u^{k-1}-u^{k-2})\cdot(v^k-v^{k-1}), \tag{4.47}$$

we obtain the discrete variant of (3.18a), namely

$$\int_Q\Big(\varphi_e'(e(\bar{u}_\tau),\bar{z}_\tau)+\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau))+\tau|e(\bar{u}_\tau)|^{\gamma-2}e(\bar{u}_\tau)+\mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big)\Big):e(\bar{v}_\tau)-\bar{f}_\tau\cdot\bar{v}_\tau\,\mathrm{d}x\mathrm{d}t$$
$$-\int_\tau^T\!\!\int_\Omega\varrho\frac{\partial u_\tau}{\partial t}(\cdot-\tau)\cdot\frac{\partial v_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t+\int_\Omega\varrho\frac{\partial u_\tau}{\partial t}(T)\cdot v_\tau(T)\,\mathrm{d}x=\int_\Omega\varrho\dot{u}_{0,\tau}\cdot v_\tau(\tau)\,\mathrm{d}x, \tag{4.48}$$

where $\bar{v}_\tau$ and v_τ denote respectively the piecewise constant and the piecewise affine interpolants of $\{v^k\}_{k=0}^{K_\tau}$ on the equidistant partition of $[0,T]$ with the time step τ.

Like before in (4.47) but now for scalar-valued v's, we use the discrete "by-part" summation

$$\sum_{k=1}^{K_\tau}(w^k-w^{k-1})v^k = w^{K_\tau}v^{K_\tau}-w^0v^1-\sum_{k=2}^{K_\tau}w^{k-1}(v^k-v^{k-1}), \tag{4.49}$$

and get the discrete analog of (3.18b) as

$$\int_\Omega w_\tau(T)v_\tau(T)\,\mathrm{d}x+\int_Q\mathscr{K}(e(\bar{u}_\tau),\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{w}_\tau\cdot\nabla\bar{v}_\tau-\bar{\xi}_\tau\bar{v}_\tau$$
$$-\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)):e\big(\frac{\partial u_\tau}{\partial t}\big)\bar{v}_\tau\,\mathrm{d}x\mathrm{d}t-\int_\tau^T\!\!\int_\Omega\underline{w}_\tau\frac{\partial v_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t=\int_\Omega w_0v_\tau(\tau)\,\mathrm{d}x \tag{4.50}$$

with $\bar{v}_\tau$ and v_τ denoting again respectively the piecewise constant and the piecewise affine interpolants of some $\{v^k\}_{k=0}^{K_\tau}$ on the equidistant partition of $[0,T]$ and with the dissipative heat

$$\bar{\xi}_\tau:=\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+2(1-\sqrt{\tau})\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big). \tag{4.51}$$

Moreover, the discrete analog of (3.18c) as an inequality can be got simply by writing (4.7) for $k = K_\tau$, which gives

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2 + \varphi\big(e(u_\tau(T)), z_\tau(T), \nabla z_\tau(T)\big) + w_\tau(T)\,\mathrm{d}x + \int_\Sigma b\mathscr{T}(\bar{w}_\tau)\,\mathrm{d}S\mathrm{d}t$$
$$\leq \int_Q \bar{f}_\tau \cdot \frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\varrho}{2}\big|\dot{u}_0\big|^2 + \varphi\big(e(u_{0,\tau}), z_0, \nabla z_0\big) + w_0 + \frac{\tau}{\gamma}|e(u_{0,\tau})|^\gamma\,\mathrm{d}x + \int_\Sigma b\bar{\theta}_{\mathrm{ext},\tau}\,\mathrm{d}S\mathrm{d}t \tag{4.52}$$

with $w_0 := h_0(\theta_0)$; note that we simply forgot the non-negative term $\tau|e(u_\tau^k)|^\gamma/\gamma$ in (4.7) which would otherwise occur on the left-hand side of (4.52) as $\tau|e(u_\tau(T))|^\gamma/\gamma$.

Eventually, summing (4.8) for $k = 1, ..., K_\tau$, we obtain the discrete variant of semistability (3.18d) integrated over I:

$$\int_Q \varphi\big(e(\bar{u}_\tau), \bar{z}_\tau, \bar{\nabla} z_\tau\big)\,\mathrm{d}x\mathrm{d}t \leq \int_Q \varphi\big(e(\bar{u}_\tau), v, \nabla v\big) + \zeta_1\big(v - \bar{z}_\tau\big)\,\mathrm{d}x\mathrm{d}t \tag{4.53}$$

for any $v \in L^\infty(I; W^{1,q}(\Omega;\mathbb{R}^m))$.

Step 2: Selection of subsequences. By the Banach selection principle, in view of the estimates (4.22a-c) and (4.23), we take a weakly* convergent subsequence and denote its limit by (u, z, w). More precisely, we should first embed $L^1(\Omega) \subset \mathscr{M}(\bar{\Omega})$ in (4.22c) because only then one can use the weakly* topology. Thus, in fact, $\bar{w}_\tau$ converges in $L^\infty_{\mathrm{w}*}(I;\mathscr{M}(\bar{\Omega})) \cap L^r(I;W^{1,r}(\Omega)) \cap \mathrm{BV}(\bar{I};W^{1+n,2}(\Omega)^*)$. Here, we used also that $L^\infty_{\mathrm{w}*}(I;\mathscr{M}(\bar{\Omega}))$ is the dual to the separable space $L^1(I;C(\bar{\Omega}))$. Since w is also in $L^r(I;L^r(\Omega)$, the mapping $t \mapsto w(t,\cdot) : I \to \mathscr{M}(\bar{\Omega})$ is a.e. valued in, say, $L^r(\Omega)$ and is Bochner measurable, thus even $w \in L^\infty(I;L^1(\Omega))$, as involved in (3.17d). On top of it, due to the BV-estimates in (4.22b,c), we can use the Helly selection principle generalized for functions valued in Banach spaces with a separable predual, cf. e.g. [44, 52], so that the subsequence can be considered also to have

$$z_\tau(t,\cdot) \to z(t,\cdot) \qquad \text{weakly in } \ W^{1,q}(\Omega;\mathbb{R}^m) \ \text{ and} \tag{4.54a}$$
$$w_\tau(t,\cdot) \to w(t,\cdot) \qquad \text{weakly* in } \ \mathscr{M}(\bar{\Omega}) \ \text{ for any } t \in \bar{I}. \tag{4.54b}$$

Thus, in particular, we have also $w \in B_{\mathrm{w}*}(\bar{I};\mathscr{M}(\bar{\Omega}))$, as involved in (3.17d).

Standardly, one can also show that the limits of converging subsequences $\{u_\tau\}_{\tau>0}$ and $\{\bar{u}_\tau\}_{\tau>0}$ are the same and that $\{\frac{\partial}{\partial t}u_\tau\}_{\tau>0}$ converges to $\frac{\partial}{\partial t}u$ weakly in the topology indicated in (4.22a) and also $\{\frac{\partial^2}{\partial t^2}u_\tau\}_{\tau>0}$ converges to $\frac{\partial^2}{\partial t^2}u$ weakly in the topology indicated in (4.23). Analogous facts are at disposal for z_τ and w_τ, too. In addition,

$$\bar{w}_\tau \to w \quad \text{strongly in } \ L^{(n+2)/n-\epsilon}(Q) \ \text{ with } \epsilon > 0 \tag{4.55}$$

by the Aubin-Lions theorem (generalized for time-derivatives as measures as in [62, Cor.7.9]) and interpolated (as in [62, Cor.7.8]) with the the first and the second part of estimate (4.22c). The mentioned interpolation is due to Gagliardo-Nirenberg inequality and, in fact, we already made it when proving boundedness of $\{\bar{w}_\tau\}_{\tau>0}$ in $L^{(n+2)/n-\epsilon}(Q)$; cf. (4.42) with $\frac{p_3}{\omega} < \frac{n+2}{\mu(n+1)}$ from (4.43) and realize the previous choice $\mu = \frac{n}{n+1}$.

Step 3: Strong convergence of $e(\bar{u}_\tau)$. Let us take v_τ and $\bar{v}_\tau$ respectively a piecewise affine approximation of u and the corresponding approximation piecewise constant in time on the partition of $[0, T]$ such that $v_\tau \to u$ strongly in $L^p(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)) \cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$ and $\bar{v}_\tau \to u$ strongly in $L^p(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$; such approximation is always possible since u lies in this space due to (4.22a). In addition, we can assume $\{e(\bar{v}_\tau)\}_{\tau>0} \subset L^\gamma(Q;\mathbb{R}^{n\times n})$, although we cannot assume this sequence bounded but only, say, $\|e(\bar v_\tau)\|_{L^\gamma(Q;\mathbb{R}^{n\times n})} = \mathscr{O}(\tau^{-1/(\gamma+1)})$. Using the p-strong monotonicity (3.8) of $\varphi'_e(\cdot,z)$ and the convexity (3.11a) of ϕ, we have p-strong monotonicity of $\psi'_e(\cdot,z,\theta) = \varphi'_e(\cdot,z) + \theta\phi'(\cdot)$ uniformly for any z and $\theta \ge 0$. Moreover, we use monotonicity of $e \mapsto \tau|e|^{\gamma-2}e$. Using further the identity (4.48) with $u_\tau{-}v_\tau$ and $\bar u_\tau{-}\bar v_\tau$ in place of v_τ and $\bar v_\tau$, respectively, we obtain

$$
\begin{aligned}
&\alpha\Big(\big\|e(\bar u_\tau)\big\|^{p-1}_{L^p(Q;\mathbb{R}^{n\times n})} - \big\|e(\bar v_\tau)\big\|^{p-1}_{L^p(Q;\mathbb{R}^{n\times n})}\Big)\Big(\big\|e(\bar u_\tau)\big\|_{L^p(Q;\mathbb{R}^{n\times n})} - \big\|e(\bar v_\tau)\big\|_{L^p(Q;\mathbb{R}^{n\times n})}\Big)\\
&\quad\le \int_Q \alpha\big(|e(\bar u_\tau)|^{p-2}e(\bar u_\tau) - |e(\bar v_\tau)|^{p-2}e(\bar v_\tau)\big){:}e(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_Q \big(\varphi'_e(e(\bar u_\tau),\bar z_\tau) + \mathscr{T}(\bar w_\tau)\phi'(e(\bar u_\tau)) + \tau|e(\bar u_\tau)|^{\gamma-2}e(\bar u_\tau)\\
&\qquad - \varphi'_e(e(\bar v_\tau),\bar z_\tau) - \mathscr{T}(\bar w_\tau)\phi'(e(\bar v_\tau)) - \tau|e(\bar v_\tau)|^{\gamma-2}e(\bar v_\tau)\big) : e(\bar u_\tau{-}\bar v_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\quad= \int_Q \bar f_\tau\cdot(\bar u_\tau{-}\bar v_\tau) - \mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar u_\tau{-}\bar v_\tau)\\
&\qquad - \big(\varphi'_e(e(\bar v_\tau),\bar z_\tau) + \mathscr{T}(\bar w_\tau)\phi'(e(\bar v_\tau)) + \tau|e(\bar v_\tau)|^{\gamma-2}e(\bar v_\tau)\big) : e(\bar u_\tau{-}\bar v_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \int_\tau^T\!\!\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(\cdot-\tau)\cdot\frac{\partial(u_\tau{-}v_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\qquad - \int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(T)\cdot[u_\tau{-}v_\tau](T) - \varrho\dot u_0\cdot\big[u_\tau - v_\tau\big](\tau)\,\mathrm{d}x \to 0, \qquad (4.56)
\end{aligned}
$$

where we are still to prove the last convergence. In fact, it suffices to prove that the limit superior is nonpositive. Obviously, $\int_Q \bar f_\tau\cdot(\bar u_\tau{-}\bar v_\tau)\,\mathrm{d}x\mathrm{d}t \to 0$. As for the $\mathbb{D}$-term, we have

$$
\begin{aligned}
\limsup_{\tau\downarrow0}\int_Q -\mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar u_\tau{-}\bar v_\tau)\,\mathrm{d}x\mathrm{d}t &\le \lim_{\tau\downarrow0}\int_\Omega \frac{1}{2}\mathbb{D}e(u_{0,\tau}) : e(u_{0,\tau})\,\mathrm{d}x\\
-\liminf_{\tau\downarrow0}\int_\Omega \frac{1}{2}\mathbb{D}e(u_\tau(T)) : e(u_\tau(T))\,\mathrm{d}x &- \lim_{\tau\downarrow0}\int_Q \mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar v_\tau)\,\mathrm{d}x\mathrm{d}t\\
\le \frac{1}{2}\int_\Omega \mathbb{D}e(u_0) : e(u_0) - \mathbb{D}e(u(T)) : e(u(T))\,\mathrm{d}x &- \int_Q \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e(u)\,\mathrm{d}x\mathrm{d}t = 0. \qquad (4.57)
\end{aligned}
$$

The first inequality in (4.57) used $\mathbb{D}e(u^k{-}u^{k-1}) : e(u^k) \ge \frac{1}{2}\mathbb{D}e(u^k) : e(u^k) - \frac{1}{2}\mathbb{D}e(u^{k-1}) : e(u^{k-1})$ so that $\int_Q \mathbb{D}e(\frac{\partial u_\tau}{\partial t}) : e(\bar u_\tau)\,\mathrm{d}x\mathrm{d}t \ge \frac{1}{2}\int_\Omega \mathbb{D}e(u_\tau(T)) : e(u_\tau(T)) - \mathbb{D}e(u_{0,\tau}) : e(u_{0,\tau})\,\mathrm{d}x$; note that the last difference is indeed in $L^1(\Omega)$ although the particular terms need not be if $p < 2$.

We use the Aubin-Lions theorem (again generalized as [62, Cor.7.9]) so that

$$
\bar z_\tau \to z \ \text{ strongly in } \ L^{q^*-\epsilon}(Q;\mathbb{R}^m) \ \text{ with } \ \epsilon > 0; \qquad (4.58)
$$

in fact, this convergence does not exploit any interpolation (unlike (4.55) before) and holds even in a bit smaller space $L^{1/\epsilon}(I;L^{q^*-\epsilon}(\Omega;\mathbb{R}^m))$. This gives $\varphi'_e(e(\bar v_\tau),\bar z_\tau)) : e(\bar u_\tau - v_\tau) \to \varphi'_e(e(u),z)) : e(u-u) = 0$ weakly in $L^1(Q)$; here the growth (3.14a) of φ'_e has been used. In (4.56), we also used that $\mathscr{T}(\bar w_\tau)\phi'(e(\bar v_\tau)) : e(\bar u_\tau{-}\bar v_\tau)$ converges to 0 weakly in $L^1(Q)$ since (4.55). For both terms, we used also that $e(\bar v_\tau) \to e(u)$ by assumption.

Also we use $\frac{\partial u_\tau}{\partial t} \to \frac{\partial u}{\partial t}$ strongly in $L^2(Q;\mathbb{R}^n)$, which can be proved by Aubin-Lions theorem (again generalized as [62, Cor.7.9]) based on the estimate of $\frac{\partial u_\tau}{\partial t}$ in $L^2(I;W^{1,2}(\Omega;\mathbb{R}^n)) \cap \mathrm{BV}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)$ from (4.22a) and (4.23). Also, $\frac{\partial u_\tau}{\partial t}(\cdot-\tau) \to \frac{\partial u}{\partial t}$ weakly in $L^2(Q;\mathbb{R}^n)$ due to the a-priori estimate (4.22a) and

$$
\Big\|\frac{\partial u_\tau}{\partial t}(\cdot-\tau) - \frac{\partial u_\tau}{\partial t}\Big\|_{\mathscr{M}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)} \le \tau\Big\|\frac{\partial^2 u_\tau}{\partial t^2}\Big\|_{\mathscr{M}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)} \to 0, \qquad (4.59)
$$

where $\mathscr{M}(\bar{I};X)$ denotes the space of X-valued measures on $\bar{I} = [0,T]$. Thus $\int_\tau^T\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(\cdot - \tau)\cdot\frac{\partial(u_\tau - v_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t \to \int_Q \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial(u-u)}{\partial t}\,\mathrm{d}x\mathrm{d}t = 0$.

Also, we can limit our regularizing term $\int_Q \tau|e(\bar{v}_\tau)|^{\gamma-2}e(\bar{v}_\tau) : e(\bar{u}_\tau - \bar{v}_\tau)\,\mathrm{d}x\mathrm{d}t$ by using (4.22d) and our assumption $\|e(\bar{v}_\tau)\|_{L^\gamma(Q;\mathbb{R}^{n\times n})} = \mathscr{O}(\tau^{-1/(\gamma+1)})$ so that $|\int_Q \tau|e(\bar{v}_\tau)|^{\gamma-2}e(\bar{v}_\tau) : e(\bar{u}_\tau - \bar{v}_\tau)\,\mathrm{d}x\mathrm{d}t| \le \tau\|e(\bar{v}_\tau)\|^{\gamma-1}_{L^\gamma(Q;\mathbb{R}^{n\times n})}\|e(\bar{v}_\tau - \bar{v}_\tau)\|_{L^\gamma(Q;\mathbb{R}^{n\times n})} = \mathscr{O}(\tau^{1-1/(\gamma+1)-1/\gamma}) \to 0$.

Since

$$\frac{\partial u_\tau}{\partial t}(T) = \dot{u}_0 + \int_0^T \frac{\partial^2 u_\tau}{\partial t^2}\,\mathrm{d}t \;\to\; \dot{u}_0 + \int_0^T \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}t = \frac{\partial u}{\partial t}(T) \;\text{ weakly in }\; W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*, \tag{4.60}$$

by the a-priori estimate of $\frac{\partial u_\tau}{\partial t}(T)$ in $L^2(\Omega;\mathbb{R}^n)$, we have $\frac{\partial u_\tau}{\partial t}(T) \to \frac{\partial u}{\partial t}(T)$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Further we use also $u_\tau(T) \to u(T)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ and therefore strongly in $L^2(\Omega;\mathbb{R}^n)$ (here $p > 2n/(n+2)$ from (3.6a) is used to ensure $W^{1,p}(\Omega) \Subset L^2(\Omega)$), so that we have $\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(T)\cdot[u_\tau - v_\tau](T)\,\mathrm{d}x \to \int_\Omega \varrho\frac{\partial u}{\partial t}(T)\cdot[u-u](T)\,\mathrm{d}x = 0$.

Eventually, for limiting the last term in (4.56) we use $u_\tau(\tau) = u_\tau^1 \to u_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$ and $v_\tau(\tau) \to u_0$ in $L^2(\Omega;\mathbb{R}^n)$.

Altogether, from (4.56), we get $\|e(\bar{u}_\tau)\|_{L^p(Q;\mathbb{R}^{n\times n})} \to \|e(u)\|_{L^p(Q;\mathbb{R}^{n\times n})}$. As we already know that $e(\bar{u}_\tau) \to e(u)$ weakly in $L^p(Q;\mathbb{R}^{n\times n})$, by the well-known fact that $L^p(Q;\mathbb{R}^{n\times n})$ is a uniformly convex space, we obtain $e(\bar{u}_\tau) \to e(u)$ strongly in $L^p(Q;\mathbb{R}^{n\times n})$.

Step 4: Limit passage in discrete momentum balance (4.48). We use the strong convergence of $e(\bar{u}_\tau)$ in $L^p(Q;\mathbb{R}^{n\times n})$ already proved is Step 3, and also (4.55) and (4.58). Also, employing (4.22d), we use

$$\begin{aligned}\left|\int_Q \tau\big|e(\bar{u}_\tau)\big|^{\gamma-2}e(\bar{u}_\tau) : e(v)\,\mathrm{d}x\mathrm{d}t\right| &\le \tau\big\|e(\bar{u}_\tau)\big\|^{\gamma-1}_{L^\gamma(Q;\mathbb{R}^{n\times n})}\big\|e(v)\big\|_{L^\gamma(Q;\mathbb{R}^{n\times n})}\\ &\le \tau\bigg(\frac{C}{\sqrt[\gamma]{\tau}}\bigg)^{\gamma-1}\big\|e(v)\big\|_{L^\gamma(Q;\mathbb{R}^{n\times n})} = \mathscr{O}\big(\sqrt[\gamma]{\tau}\big) \to 0.\end{aligned} \tag{4.61}$$

Also, $\frac{\partial}{\partial t}u_\tau(\cdot - \tau) \to \dot{u}_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$ because, like in (4.60), $\frac{\partial}{\partial t}u_\tau(\cdot - \tau) = \dot{u}_0 + \int_0^\tau \frac{\partial^2 u_\tau}{\partial t^2}\,\mathrm{d}t \to \dot{u}_0 + \int_0^0 \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}t = \dot{u}_0$. Then the limit passage in (4.48) is easy and we thus obtain (3.18a).

Step 5: Limit passage in discrete semistability (4.53). This must be excecuted case by case to obtain the "integrated" semistability

$$\int_Q \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\mathrm{d}t \le \int_Q \varphi\big(e(u), v, \nabla v\big) + \zeta_1\big(v - z\big)\,\mathrm{d}x\mathrm{d}t \tag{4.62}$$

for any $v \in L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))$. Having obtained (4.62), we would like to see (3.18d) for a.a. $t \in I$. Like in [64, Proof of Proposition 5.2], assuming it would not be true, we could find $\varepsilon > 0$ and $J \subset I$ with a positive measure such that

$$\begin{aligned}\forall t \in J\;\;\exists v \in W^{1,q}(\Omega;\mathbb{R}^m):\qquad &\int_\Omega \varphi\big(e(u(t)), v, \nabla v\big) + \zeta_1\big(v - z(t)\big)\,\mathrm{d}x + \varepsilon\\ &\le \int_\Omega \varphi\big(e(u(t)), z(t), \nabla z(t)\big)\,\mathrm{d}x =: E(t).\end{aligned} \tag{4.63}$$

Let $M(t)$ denote the set of all v satisfying the inequality in (4.63). Each $M(t)$ is nonempty and closed, and the set-valued mapping $t \mapsto M(t) : I \rightrightarrows W^{1,q}(\Omega;\mathbb{R}^m)$ is measurable and bounded; note that the boundedness of $M(\cdot)$ follows from the coercivity (3.6a,b) and from the boundedness of $E(\cdot)$ which is guaranteed by the estimate $E(t) \leq \|f\|_{L^1(I;L^2(\Omega;\mathbb{R}^n))}\|\frac{\partial u}{\partial t}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))} + \sup_{\tau>0}\int_\Omega \frac{\varrho}{2}\big|\dot{u}_0\big|^2 + \varphi\big(e(u_{0,\tau}), z_0, \nabla z_0\big) + w_0\,\mathrm{d}x + \int_\Sigma b\bar{\theta}_{\mathrm{ext},\tau}\,\mathrm{d}S\mathrm{d}t$, cf. (4.52). Then it is well-known that there is a measurable selection of M, let us denote it as $\tilde{v}$. Considering $v \in L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))$ as $v(t) = \tilde{v}(t)$ for $t \in J$ and $v(t) = z(t)$ for $t \in I \setminus J$, we obtain

$$\begin{aligned}\int_Q \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\mathrm{d}t &= \int_J\int_\Omega \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\mathrm{d}t + \int_{I\setminus J}\int_\Omega \varphi\big(e(u), v, \nabla v\big)\,\mathrm{d}x\mathrm{d}t\\ &\geq \int_J\bigg(\int_\Omega \varphi\big(e(u), v, \nabla v\big) + \zeta_1(v{-}z)\,\mathrm{d}x + \varepsilon\bigg)\mathrm{d}t + \int_{I\setminus J}\int_\Omega \varphi\big(e(u), v, \nabla v\big)\,\mathrm{d}x\mathrm{d}t\\ &= \int_Q \varphi\big(e(u), v, \nabla v\big) + \zeta_1(v{-}z)\,\mathrm{d}x\mathrm{d}t + \varepsilon\,\mathrm{meas}(J),\end{aligned}\tag{4.64}$$

which would contradict (4.62) since $\varepsilon\,\mathrm{meas}(J) > 0$.

To obtain (4.62), we may use methods as in the isothermal case [64] because, in particular, ϕ is not involved in the discrete semistability (as ϕ does not depend on z).

Step 5a. Let us begin with the case (3.10a), which allows for the limit passage in (4.53) to get (4.62) simply by continuity as far as ϕ_1 and ζ_1 concerns and by weak lower semicontinuity as far as $\int_Q \phi_2(\bar{z}_\tau, \nabla\bar{z}_\tau)\,\mathrm{d}x\mathrm{d}t$ concerns. Here we also use (4.58).

Step 5b. In case (3.10b), the limit passage in (4.53) can rely on the binomial formula for the functional $\Phi(e,\cdot)$ from the proof of Proposition 3.2 which is now assumed quadratic/affine, so that:

$$\Phi(e, z) - \Phi(e, \tilde{z}) = \Big\langle \Phi'_z(e,\cdot) + \frac{1}{2}\big[\Phi''_{zz}(e)\big](z{+}\tilde{z})\,,(z{-}\tilde{z})\Big\rangle.\tag{4.65}$$

Thus, for any test function $\tilde{v} \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^m))$, we can use (4.8) with $v := v_\tau = \tilde{v} - z + \bar{z}_\tau$ in place of v, and use the binomial formula (4.65) with $\bar{z}_\tau$ instead of z and also use the assumed form $\varphi(e, z, Z) = (A(e)(z, Z) + b(e)) : (z, Z)$ with a matrix $A(e)$ and a vector $b(e)$, i.e.

$$\begin{aligned}&\int_Q \varphi\big(e(\bar{u}_\tau), \bar{z}_\tau, \nabla\bar{z}_\tau\big)\,\mathrm{d}x\mathrm{d}t - \int_Q \varphi\big(e(\bar{u}_\tau), v_\tau, \nabla v_\tau\big)\,\mathrm{d}x\mathrm{d}t\\ &\qquad = \int_Q \Big(A\big(e(\bar{u}_\tau)\big)\big(\bar{z}_\tau{+}v_\tau, \nabla(\bar{z}_\tau{+}v_\tau)\big) + b\big(e(\bar{u}_\tau)\big)\Big) : \big(\bar{z}_\tau{-}v_\tau, \nabla(\bar{z}_\tau{-}v_\tau)\big)\,\mathrm{d}x\mathrm{d}t\\ &\qquad = \int_Q \Big(A\big(e(\bar{u}_\tau)\big)\big(\bar{z}_\tau{+}v_\tau, \nabla(\bar{z}_\tau{+}v_\tau)\big) + b\big(e(\bar{u}_\tau)\big)\Big) : \big(z{-}\tilde{v}, \nabla(z{-}\tilde{v})\big).\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.66}$$

This then converges to $\int_Q (A(e(u))(z{+}v, \nabla(z{+}v)) + b(e(u))) : (z{-}\tilde{v}, \nabla(z{-}\tilde{v}))\,\mathrm{d}x\mathrm{d}t$ which equals $\int_Q \varphi(e(u), z, \nabla z)\,\mathrm{d}x\mathrm{d}t - \int_Q \varphi(e(u), v, \nabla v)\,\mathrm{d}x\mathrm{d}t$ by (4.65). Here we used the strong convergence $e(\bar{u}_\tau) \to e(u)$ from Step 3 and the growth conditions for $\varphi'_{(z,Z)}$ in (3.10b) to guarantee continuity of the Nemytskiĭ mapping induced by this integrand. Moreover, we have simply $\zeta_1(v_\tau{-}\bar{z}_\tau) = \zeta_1(\tilde{v}{-}z)$, so that the limit passage from (4.53) to (4.62) is proved in case (3.10b).

Step 6: Passage in the discrete energy inequality (4.52). It is just by weak lower semicontinuity and the assumption (4.4a). Thus the inequality "$\leq$" in the energy balance (3.18c) is obtained.

Step 7: Passage in the enthalpy equation (4.50). It is highly nontrivial because of the convergence of the dissipative heat $\bar{\xi}_\tau$. For execution of this convergence, it seems important (or rather necessary) to obtain inverse inequality for the mechanical energy balance

$$
\begin{aligned}
&\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(u(T),z(T),\nabla z(T))\mathrm{d}x + \mathrm{Var}_S(z;0,T) + 2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\geq \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0),z_0,\nabla z_0)\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t} - \mathscr{T}(w)\phi'\big(e(u)\big){:}e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{4.67}
$$

which is, under our assumptions and already proved results, further equivalent to energy equality (3.18c).

The most essential trick is to use the already proved "integral" semi-stability (3.18d), cf. [22, 27, 44, 45, 50] for this technique in a mere rate-independent context or, in the viscous context, [64, Proposition 5.4]. We consider now $\varepsilon > 0$, and a partition $0 = t_0^\varepsilon < t_1^\varepsilon < ... < t_{k_\varepsilon}^\varepsilon = T$ with $\max_{i=1,...,k_\varepsilon}(t_i^\varepsilon - t_{i-1}^\varepsilon) \leq \varepsilon$. Moreover, as (3.18d) holds a.e. $t \in I$ and also at $t = 0$ due to (3.16), we can consider the above partition so that the semi-stability holds at t_i^ε for each $i = 0, ..., k_\varepsilon - 1$. Using this semi-stability of z at time t_{i-1}^ε gives, when tested by $v := z(t_i^\varepsilon)$, the estimate

$$
\begin{aligned}
\int_\Omega \varphi\big(e(u(t_{i-1}^\varepsilon)), z(t_{i-1}^\varepsilon), \nabla z(t_{i-1}^\varepsilon)\big)\,\mathrm{d}x &\leq \int_\Omega \varphi\big(e(u(t_{i-1}^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) + \zeta_1\big(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon)\big)\,\mathrm{d}x\\
&= \int_\Omega \bigg(\varphi\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) + \zeta_1\big(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon)\big)\\
&\qquad - \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \varphi_e'\big(e(u(t)), z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}t\bigg)\mathrm{d}x\,;
\end{aligned}
\tag{4.68}
$$

again we used that φ_e' depends only on (e, z) due to (3.7). Summing (4.68) for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon-1}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$
\begin{aligned}
&\int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big) - \int_\Omega \varphi\big(e(u_0), z_0, \nabla z_0)\big)\,\mathrm{d}x + \mathrm{Var}_S(z;0,T)\\
&\qquad \geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t)), z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad \geq \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t)), z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t - \delta_\varepsilon\\
&\qquad = \sum_{i=1}^{k_\varepsilon-1} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\\
&\qquad\quad + \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'\big(e(u(t)), z(t_i^\varepsilon)\big) - \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)\big)\Big){:}e\Big(\frac{\partial u}{\partial t}\Big)\mathrm{d}x\mathrm{d}t\\
&\qquad\quad + \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t} - \Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}t - \delta_\varepsilon\\
&\qquad =: S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon - \delta_\varepsilon
\end{aligned}
\tag{4.69}
$$

where

$$
\delta_\varepsilon := \Big|\int_{t_{k_\varepsilon-1}^\varepsilon}^{T} \int_\Omega \varphi_e'\big(e(u(t)), z(T)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t\Big|.
\tag{4.70}
$$

As to S_2^ε, using the Lipschitz continuity of $\varphi_e'(\cdot,z):\mathbb{R}^{n\times n}_{\rm sym}\to\mathbb{R}^{n\times n}_{\rm sym}$ assumed in (3.14b) with ℓ denoting here the Lipschitz constant, we can estimate

$$\begin{aligned}|S_2^\varepsilon| &\le \sum_{i=1}^{k_\varepsilon}\int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \ell\big\|e\big(u(t)-u(t_i^\varepsilon)\big)\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\Big\|e\Big(\frac{\partial u}{\partial t}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}{\rm d}t\\ &\le \ell\max_{i=1,\ldots,k_\varepsilon}\ \max_{t\in[t_{i-1}^\varepsilon,t_i^\varepsilon]}\big\|e\big(u(t)-u(t_i^\varepsilon)\big)\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\big\|e(u)\big\|_{W^{1,1}(I;L^2(\Omega;\mathbb{R}^{n\times n}))}.\end{aligned}\tag{4.71}$$

Since certainly $e(u)\in W^{1,1}(I;L^2(\Omega;\mathbb{R}^{n\times n}))$, the "max max"-term tends to zero with $\varepsilon{\downarrow}0$, hence $\lim_{\varepsilon\downarrow0}S_2^\varepsilon=0$. As to S_3^ε, by Fubini's theorem, we can estimate

$$\begin{aligned}|S_3^\varepsilon| &= \bigg|\sum_{i=1}^{k_\varepsilon}\int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)),z(t_i^\varepsilon)\big):e\Big(u(t_i^\varepsilon)-u(t_{i-1}^\varepsilon)-(t_i^\varepsilon{-}t_{i-1}^\varepsilon)\Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\,{\rm d}x\bigg|\\ &\le \big\|\varphi_e'\big(e(u),z\big)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))}\sum_{i=1}^{k_\varepsilon}\Big\|e\Big(u(t_i^\varepsilon)-u(t_{i-1}^\varepsilon)-(t_i^\varepsilon{-}t_{i-1}^\varepsilon)\Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}.\end{aligned}\tag{4.72}$$

Note that $u(t_i^\varepsilon)-u(t_{i-1}^\varepsilon)\in W^{1,2}(\Omega;\mathbb{R}^n)$ although particular terms are in $W^{1,p}(\Omega;\mathbb{R}^n)$ and need not belong to $W^{1,2}(\Omega;\mathbb{R}^n)$ if $p<2$ and that the assumed growth (3.14a) of φ_e' together with (4.22a,b) indeed guarantees $\varphi_e'(e(u),z)\in L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))$. We have still a freedom to choose the partition $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ in such a way that both $\lim_{\varepsilon\downarrow0}S_3^\varepsilon=0$ and that the Riemann sum S_1^ε approaches the corresponding Lebesgue integral, namely

$$\lim_{\varepsilon\downarrow0}S_1^\varepsilon=\int_0^T\!\!\int_\Omega\varphi_e'\big(e(u(t)),z(t)\big):e\Big(\frac{\partial u}{\partial t}\Big)\,{\rm d}x{\rm d}t\,;\tag{4.73}$$

cf. [22, Lemma 4.12] or [27, Lemma 4.5], following the idea of Hahn [31]. Eventually, $\lim_{\varepsilon\downarrow0}\delta_\varepsilon=0$ because the integrand in (4.70) is absolutely continuous and $t_{k_\varepsilon-1}^\varepsilon{\uparrow}T$ for $\varepsilon{\downarrow}0$. This allows for a limit passage in (4.69) for $\varepsilon{\downarrow}0$, which gives the desired opposite inequality

$$\begin{aligned}&\int_\Omega\varphi\big(e(u(T)),z(T),\nabla z(T)\big)-\int_\Omega\varphi\big(e(u_0),z_0,\nabla z_0\big)\,{\rm d}x\\ &\qquad+{\rm Var}_S(z;0,T)\ge\int_0^T\!\!\int_\Omega\varphi_e'\big(e(u(t)),z(t)\big):e\Big(\frac{\partial u}{\partial t}\Big)\,{\rm d}x{\rm d}t\,.\end{aligned}\tag{4.74}$$

Further, we have also to prove $\frac{\partial^2u}{\partial t^2}\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)+L^1(I;L^2(\Omega;\mathbb{R}^n))$, which follows from $f\in L^1(I;L^2(\Omega;\mathbb{R}^n))$ and from the identity

$$\begin{aligned}\Big\|\frac{\partial^2u}{\partial t^2}-f\Big\|_{L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)} &= \sup_{\|v\|_{L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))}\le1}\Big\langle\frac{\partial^2u}{\partial t^2}-f,v\Big\rangle\\ &= \sup_{\|v\|_{L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))}\le1}\int_Q\frac{1}{\varrho}\Big(\mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big)+\varphi_e'(e(u),z)+\mathscr{T}(w)\phi'(e(u))\Big):e(v)\,{\rm d}x{\rm d}t\end{aligned}$$

from the estimate (4.22a,b) inherited for the limit (u,z) combined with (3.14a) and also by using $\int_Q\mathscr{T}(w)\phi'(e(u)):e(v){\rm d}x{\rm d}t\le C\|\mathscr{T}(w)\phi'(e(u))\|_{L^2(Q;\mathbb{R}^{n\times n})}\|e(v)\|_{L^2(Q;\mathbb{R}^{n\times n})}$. Hence $\frac{\partial u}{\partial t}\in L^2(I;W^{1,2}_{\Gamma_0}(\Omega;\mathbb{R}^n))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^n))$ is a legal test function for (3.18a) obtained already in Step 4. In particular, as $\frac{\partial^2u}{\partial t^2}$ and $\frac{\partial u}{\partial t}$ in mutually dual spaces, we can perform the by-part integration in time (3.21), and we obtain

$$\begin{aligned}\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 \mathrm{d}x + 2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big) + \varphi_e'(e(u(t)),z(t)) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \qquad&\\ = \int_\Omega \frac{\varrho}{2}|\dot u_0|^2\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t} - \mathscr{T}(w)\phi'\big(e(u)\big){:}e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t.&\end{aligned}\tag{4.75}$$

Summing (4.75) with (4.74) then gives (4.67).

Now, referring to the measure $\mathfrak{h}_z$ from Definition 3.1(ii), we have

$$\begin{aligned}\int_{\bar Q}\mathfrak{h}_z(\mathrm{d}x\mathrm{d}t) + 2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t =\ & \mathrm{Var}_S(z;0,T) + 2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\ \le \liminf_{\tau\downarrow 0}\int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big) &+ (2{-}\sqrt{\tau})\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\ \le \limsup_{\tau\downarrow 0}\int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big) &+ (2{-}\sqrt{\tau})\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\ \le \limsup_{\tau\downarrow 0}\bigg(\int_\Omega \frac{\varrho}{2}|\dot u_0|^2 &+ \varphi(u_{0,\tau},z_0,\nabla z_0) + \frac{\tau}{\gamma}|e(u_{0,\tau})|^\gamma\,\mathrm{d}x\\ - \int_\Omega\frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2 &+ \varphi(u_\tau(T),z_\tau(T),\nabla z_\tau(T)) + \frac{\tau}{\gamma}|e(u_\tau(T))|^\gamma \mathrm{d}x\\ + \int_Q \mathscr{T}(\bar w_\tau)\phi'\big(e(\bar u_\tau)\big){:}e\Big(\frac{\partial u_\tau}{\partial t}\Big) &- \bar f_\tau{\cdot}\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t\bigg)\\ \le \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 - \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 &+ \varphi(e(u_0),z_0,\nabla z_0) - \varphi(u(T),z(T),\nabla z(T))\,\mathrm{d}x\\ + \int_Q \mathscr{T}(w)\phi'\big(e(u)\big){:}e\Big(\frac{\partial u}{\partial t}\Big) &- f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t\\ \le\ \mathrm{Var}_S(z;0,T) + 2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)&\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.76}$$

The inequalities in (4.76) are successively by the lower weak* semicontinuity, by general comparison "liminf≤limsup", by the discrete mechanical-energy inequality (4.6) for $k = K_\tau$, by the upper weak* semicontinuity and the obvious non-negativity $\frac{\tau}{\gamma}|e(u_\tau(T))|^\gamma \ge 0$ and the convergence

$$\mathscr{T}(\bar w_\tau)\phi'\big(e(\bar u_\tau)\big) : e\Big(\frac{\partial u_\tau}{\partial t}\Big) \to \mathscr{T}(w)\phi'\big(e(u)\big) : e\Big(\frac{\partial u}{\partial t}\Big) \quad \text{weakly in } L^1(Q)\tag{4.77}$$

because of (4.55) and of the strong convergence of $e(\bar u_\tau)$ proved in Step 3, and finally by (4.67). Thus we have equality in the above chain of inequalities (4.76).

Realizing the weak* lower-semicontinuity of both parts of the dissipation energy separately, this implies both the convergence

$$\int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)\mathrm{d}x\mathrm{d}t \to \mathrm{Var}_S(z;0,T) = \int_{\bar Q}\mathfrak{h}_z(\mathrm{d}x\mathrm{d}t)\tag{4.78}$$

and the convergence

$$\int_Q \zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\mathrm{d}x\mathrm{d}t \to \int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\mathrm{d}x\mathrm{d}t.\tag{4.79}$$

Further, we show that (4.78) implies the convergence

$$\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big) \overset{*}{\rightharpoonup} \mathfrak{h}_z \quad \text{weakly* in } \mathscr{M}(\bar Q) \cong C(\bar Q)^*.\tag{4.80}$$

We use the a-priori estimate (4.22b) and, for a moment, assume that (in terms of a subsequence) w*-$\lim_{\tau\downarrow0}\zeta_1(\frac{\partial z_\tau}{\partial t}) = \mu \neq \mathfrak{h}_z$ and define the Borel set $B := \mathrm{supp}(\mathfrak{h}_z-\mu)^+ \subset \bar{Q}$ where $(\cdot)^+$ denotes the positive variation. The convergence (4.78) would imply $[\mathfrak{h}_z-\mu](B) > 0$ because otherwise, if $[\mathfrak{h}_z-\mu](B) = 0$ and $\mu \neq \mathfrak{h}_z$, $[\mu-\mathfrak{h}_z](\bar{Q}) > 0$ which would contradict (4.78). Thus $\lim_{\tau\downarrow0}\int_B \zeta_1(\frac{\partial z_\tau}{\partial t})\mathrm{d}x\mathrm{d}t = \int_B \mu\mathrm{d}x\mathrm{d}t < \int_B \mathfrak{h}_z\mathrm{d}x\mathrm{d}t$, which would contradict the weak* lower-semicontinuity of $z \mapsto \int_B \mathfrak{h}_z\mathrm{d}x\mathrm{d}t$. Thus (4.80) is proved.

Also, (4.79) implies $\zeta_2(e(\frac{\partial u_\tau}{\partial t})) \to \zeta_2(e(\frac{\partial u}{\partial t}))$ in $L^1(Q)$ because, having assumed ζ_2 coercive by (3.6c), we can re-norm $L^2(Q;\mathbb{R}^{n\times n})$ suitably so that its norm is just $(\int_Q \zeta_2(\cdot)\mathrm{d}x\mathrm{d}t)^{1/2}$ and obtain strong convergence of $e(\frac{\partial u_\tau}{\partial t})$ in $L^2(Q;\mathbb{R}^{n\times n})$ by usual arguments; note that we proved that, in fact, the convergence in (4.77) is strong.

Limit passage in the enthalpy equation (4.50) is by the strong convergence (4.55) of $\bar{w}_\tau \to w$ and similarly also of $\underline{w}_\tau \to w$ and by the weak* convergence of the dissipative heat already discussed. As for the adiabatic term $\mathscr{T}(\bar{w}_\tau)\phi'(e(\bar{u}_\tau)) : \frac{\partial}{\partial t}e(u_\tau)$, we use again (4.77). $\square$

Remark 4.4 (*More general heat production.*) The dissipation rate ξ in (2.4) may easily involve a more general nonlocal contribution of the type $\xi_{\mathrm{nonloc}} = \xi_{\mathrm{nonloc}}(t,z,\theta)$ with $\xi_{\mathrm{nonloc}} : I \times L^{q^*-\epsilon}(\Omega;\mathbb{R}^m) \times L^{(n+2)/n-\epsilon}(\Omega) \to L^1(\Omega)$ bounded and such that $\xi_{\mathrm{nonloc}}(t,\cdot,\cdot,\cdot)$ is continuous and $\xi_{\mathrm{nonloc}}(\cdot,u,z,\theta)$ is measurable. Let us outline the modifications. Due to the assumed boundedness, we can easily use a semi-implicit time discretization, i.e. augmentation of the right-hand side of (4.1c) by $\xi_{\mathrm{nonloc}}(t,z_\tau^{k-1},\mathscr{T}(w_\tau^{k-1}))$, and then converge the corresponding term $\xi_{\mathrm{nonloc}}(\underline{z}_\tau,\mathscr{T}(\underline{w}_\tau))$ by using the strong convergence $\underline{z}_\tau \to z$ in $L^{q^*-\epsilon}(Q;\mathbb{R}^m)$ and $\underline{w}_\tau \to w$ in $L^{(n+2)/n-\epsilon}(Q)$ like (4.58) and (4.55) at the very end of Step 7 of the proof of Proposition 4.3.

Remark 4.5 (*Modification for omitting gradient theory for z.*) In special cases when $\varphi = \varphi(e,z)$ is quadratic and ϕ linear in (1.1), one can avoid gradient theory for z-variable. Then, in particular, one must avoid Step 3 as (4.56), which relies on a strong convergence $\bar{z}_\tau \to z$, does not work now. Yet, on the other hand, the weak convergence suffices for other limit passages, in particular (4.77), for which ϕ' constant is now needed. We refer to Example 5.1 with [9].

Remark 4.6 (*Weakening kinetic effects.*) Omitting kinetic effects (i.e. $\varrho = 0$) brings just routine modifications and simplifications. Likewise, splitting the inertial variable u into two components, one still subjected to inertia and the other not, is just an obvious compromise. For applications see Examples 5.2 and 5.3 below.

Remark 4.7 (*Modification for $\psi(\cdot,z,Z,\theta)$ nonconvex: higher-gradient theory for u.*) The (e)-semiconvexity still cannot avoid convexity of $\psi(\cdot,z,Z,\theta)$ due to the assumptions (3.8) and (3.11a). Anyhow, some applications are ultimately based on nonconvexity of $\psi(\cdot,z,Z,\theta)$, cf. Example 5.5 and, in fact, also Example 5.3. Corresponding modification relies on introducing a gradient theory (also) for strains, augmenting of ψ by "bending" (or "capillarity") terms like $\frac{1}{2}|\nabla e|^2$ or $\frac{1}{2}|\nabla^2 u|^2$ and assuming $p < 2^* =: 2n/(n-2))$ (or just $p < \infty$ if $d \le 2$). The viscosity potential ζ_2 should then involve also term like $\frac{1}{2}|\nabla e(\frac{\partial u}{\partial t})|^2$ or $\frac{1}{2}|\nabla^2\frac{\partial u}{\partial t}|^2$, while the term $\frac{1}{2}\mathbb{D}e(\frac{\partial u}{\partial t}) : e(\frac{\partial u}{\partial t})$ either can or need not be involved. Then both $\varphi(\cdot,z,Z)$ and $\phi(\cdot)$ in (1.1) may be nonconvex. The respective modification would then be in replacing $W^{1,p}(\Omega)$ by $W^{2,2}(\Omega)$, in the corresponding modification of the boundary conditions (2.17), and in (4.56) which would use uniform monotonicity of the new higher-order terms in (4.56) while $\psi_e'(e(\bar{u}_\tau),\bar{z}_\tau,\mathscr{T}(\bar{w}_\tau))$ would be in the position of a lower-order term and converge by Aubin-Lions' compact-embedding theorem. The higher-order term involved in viscosity causes that the interpolation (4.37) can be performed more gently to weaken (3.13) (cf. also [63, Remark 4.10]) and even (4.56) itself is not needed because the convergence in (4.77) would be via compactness although, like in (4.76) and (4.79), we would get the strong convergence in $\nabla^2\frac{\partial u_\tau}{\partial t}$ anyhow.

Important fact is that the relation between (4.1a,b) and (4.14) through (4.13) is based on semiconvexity of φ only, while ϕ is eliminated from (4.16). Especially in the case when $\frac{1}{2}\mathbb{D}\frac{\partial e}{\partial t}:\frac{\partial e}{\partial t}$ is omitted, then the convexity of the functionals in the auxiliary minimization problem (4.13) and in (4.16) is to be proved in the integral form rather than pointwise, using that the pointwise (e)-semiconvexity (3.9) yields some ℓ and a possibility to take $\tau>0$ so small that, instead of (4.5), it satisfies

$$\forall u\in W^{2,2}(\Omega;\mathbb{R}^n),\ u|_{\Gamma_0}=0: \qquad \int_\Omega \frac{c_2}{\tau}|\nabla e(u)|^2-\ell|e(u)|^2\mathrm{d}x\ge 0, \tag{4.81}$$

or alternatively $\int_\Omega \frac{1}{\tau}|\nabla^2 u|^2-\ell|e(u)|^2\mathrm{d}x>0$. Furthermore, the condition (3.14) requiring so far essentially $p\le 2$ can now be weakened to

$$\big|\varphi_e'(e,z)\big|\le C\big(1+|e|^5+|z|^{5q^*/6}\big), \tag{4.82a}$$

$$\big|\varphi_e'(e,z)-\varphi_e'(\tilde e,z)\big|\le \ell\big(1+|e|^4+|\tilde e|^4+|z|^{2q^*/3}\big)|e-\tilde e| \tag{4.82b}$$

to be used for (4.71)–(4.73) modified by replacing $e(u)\in L^2(\Omega;\mathbb{R}^{n\times n})$ by $\nabla u\in L^6(\Omega;\mathbb{R}^{3\times 3})$; here for simplicity we consider only the physically relevant case $n=3$.

Such a modification would obviously allow for *large strains* by replacing the small-strain tensor $e(u)$ by ∇u, cf. Example 5.5 below.

Remark 4.8 (*Omitting gradient theory for z once again.*) An alternative to Remark 4.5 in the situation of Remark 4.7 may rely on affinity of $\varphi_e'(e,\cdot)$, i.e. the ansatz $\varphi(e,z)=\varphi_0(e)+\varphi_1(e)z+\varphi_2(z)$, together with the assumption φ_2 quadratic to use the binomial trick like (5.11) or (5.14) below. Then (4.48) and (4.56) work under only weak convergence $z_\tau\to z$, too. We refer to Examples 5.3 and 5.5.

Remark 4.9 (*General difficulties.*) More general coupling with $\phi=\phi(e,z)$ in (1.1) would yield the adiabatic term $\theta\phi(e,z)$ which, however, seems very difficult because it would lead to the term $\mathscr{T}(w)\phi_z'(e,z)\frac{\partial z}{\partial t}$. Yet, the L^1-character of $\frac{\partial z}{\partial t}$ would ultimately need L^∞-estimates (or even compactness) for w, which does not seem not realistic, however. For the same reason, also temperature dependence in ζ_1 seems very difficult. Altogether, the flow rule (2.15b) for z had to be considered as temperature independent. Also temperature-dependent viscosity (i.e. $\mathbb{D}=\mathbb{D}(\theta)$) seems to bring serious troubles because (4.57) would not work. More general coupling of the type $\phi(e,\theta)$ instead of $\theta\phi(e)-\phi_0(\theta)$ in (1.1) would lead to $c_\mathrm{v}=c_\mathrm{v}(e,\theta)=\theta\phi_{\theta\theta}''(e,\theta)$ and too high-order adiabatic terms $\theta\big(\phi_e'(e(u),\theta)-\phi_{e\theta}''(e(u),\theta)\big):e(\frac{\partial u}{\partial t})$, and also the enthalpy transformation does not seem to work in the term $\mathrm{div}(\mathbb{K}(e,z,\theta)\nabla\theta)$. On the other hand, if rate-independent rule for z were combined with a quadratic "viscous-like" term by $\zeta_1(\dot z)=\delta_S^*(\dot z)+|\dot z|^2$, then $\frac{\partial z}{\partial t}$ would get an L^2-character and both temperature dependence of ζ_1 and adiabatic coupling $\theta\phi(e,z)$ would become possible, cf. [8] for the former option in the context of plasticity.

5 Examples

We illustrate the presented general theory for system (1.2) and (2.13) by several nontrivial examples of rate-independent processes in the bulk. Other examples could involve rate-independent processes on the boundary, like adhesive contacts or debonding, but this would require, however, a modification of the general framework (1.2) and thus we will not present it here.

Example 5.1 (*Thermoplasticity with hardening.*) The internal variable $z=(\pi,\eta)\in\mathbb{R}^{n\times n}_{\mathrm{sym},0}\times\mathbb{R}$ has now the meaning of the plastic deformation π and the hardening parameter η, where $\mathbb{R}^{n\times n}_{\mathrm{sym},0}:=\big\{A\in\mathbb{R}^{n\times n}_{\mathrm{sym}};\ \mathrm{tr}(A)=0\big\}$. In the linearized version, we can apply Remark 4.5 and consider

$$\psi(e,\pi,\eta,\theta)=\frac{1}{2}\mathbb{C}\big(e-\pi-\mathbb{E}\theta\big):\big(e-\pi-\mathbb{E}\theta\big)+\frac{b}{2}\eta^2-\frac{\theta^2}{2}\mathbb{C}\mathbb{E}:\mathbb{E}-\phi_0(\theta),\tag{5.1}$$

where $\mathbb{C}$ is the positive-definite elasticity tensor exhibiting the usual symmetries $\mathbb{C}_{ijkl}=\mathbb{C}_{jikl}=\mathbb{C}_{klij}$, $b>0$ a hardening parameter, $\mathbb{E}$ a matrix of thermal-expansion coefficients. It is important that it complies with (1.1) provided the material is *isotropic*, i.e.

$$\mathbb{C}_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\big(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\big),\qquad \mathbb{E}_{ij}=\alpha\delta_{ij}\tag{5.2}$$

with δ denoting here the Kronecker symbol, $\lambda>-2\mu/n$ and $\mu>0$ the Lamé constants and α the thermal-expansion coefficient, because then one has the *orthogonality*

$$\mathbb{C}\pi:\mathbb{E}=\alpha\big(\lambda\mathrm{tr}(\pi)\mathbb{I}+2\mu\pi\big):\mathbb{I}=\alpha\big(n\lambda+2\mu\big)\mathrm{tr}(\pi)=0,\tag{5.3}$$

where $\mathbb{I}=[\delta_{ij}]$ denotes the unit matrix. Then, obviously, $\phi(e)=\mathbb{C}(\pi-e):\mathbb{E}=\mathbb{C}e:\mathbb{E}$ in (1.1). Note also that the linearity of ϕ is important, as emphasized in Remark 4.5.

Let $S_1\subset\mathbb{R}^{n\times n}_{\mathrm{sym},0}$ be a convex closed neighbourhood of the origin, δ_{S_1} is its indicator function, and $\delta^*_{S_1}$ the conjugate functional to δ_{S_1} with respect to the duality pairing $\sigma:e=\sum_{i,j=1}^n\sigma_{ij}e_{ij}$. Then we consider the cone $K:=\{z=(\pi,\eta);\ \eta\geq\delta^*_{S_1}(\pi)\}$. The degree-1 homogeneous dissipation potential is

$$\zeta_1(\dot\pi,\dot\eta):=\delta^*_{S_1}(\dot\pi)+\delta_K(\dot\pi,\dot\eta).\tag{5.4}$$

Choosing the initial conditions $\eta_0=1$ makes S_1 the initial elasticity domain that may be "inflated" within evolution of the hardening. Then the initial condition π_0 such that $z_0:=(\pi_0,\eta_0)\in K$ a.e. on Ω ensures that $z\in K$ holds also during the evolution a.e. on Q. Then we can consider φ restricted on K, which makes it coercive as (3.6a). Note that ζ_1 is not continuous and even does not satisfy (3.6b) but (4.22b) still holds with the help of coercivity of φ. Altogether, (5.1)–(5.4) fits with the ansatz (3.10b) and, since $\psi(\cdot,\cdot,\cdot,w)$ is convex and quadratic, with Remark 4.5, as already said. For the linearized plasticity in the isothermal case see e.g. [1, 18, 19, 33, 44, 57].

Example 5.2 (*Shape-memory alloys.*): A popular simple model of so-called shape-memory alloys takes a "mixture" of quadratic energies with equal the elastic-moduli tensors in the form

$$\psi(e,z,\nabla z,\theta)=\frac{1}{2}\mathbb{C}\big(e-e_{\mathrm{tr}}(z)\big):\big(e-e_{\mathrm{tr}}(z)\big)+\delta_K(z)+\frac{\kappa}{2}|\nabla z|^2+\psi_0(z,\theta)$$
$$\text{with}\quad e_{\mathrm{tr}}(z)=\sum_{\ell=1}^m z_\ell e_\ell\quad\text{where}\quad e_\ell:=\frac{U_\ell^\top+U_\ell}{2},\tag{5.5}$$

where $e_{\mathrm{tr}}(z)$ is the so-called *transformation strain* with the prescribed distortion matrices U_ℓ of particular pure phases (or phase variants), and $K:=\{z\in\mathbb{R}^m;\ z_\ell\geq 0\ \&\ \sum_{\ell=1}^m z_\ell=1\}$. For models of this type we refer to [2–5, 17, 34, 40, 42, 43, 60, 66]. The dissipation usually involves volume fractions z's, sometimes in a rate-independent manner (though in an isothermal case), see [4, 17, 30, 34, 37, 75]. This is rather an example of how our structure qualification is unpleasantly strong because the ansatz (1.1) would require $\psi_0(z,\theta)=\phi_1(z)+\phi_0(\theta)$ but then the mechanical and the thermal parts would be completely decoupled one from each other. Therefore, we apply a *regularization* by introducing an auxiliary "phase field" λ subjected to (at least small) viscous dissipation $\varepsilon\Delta\frac{\partial\lambda}{\partial t}$ and then we consider the free energy

$$\psi(e,z,\nabla z,\lambda,\theta)=\frac{1}{2}\mathbb{C}\big(e-e_{\mathrm{tr}}(z)\big):\big(e-e_{\mathrm{tr}}(z)\big)+\phi_1(z)+\delta_K(z)$$
$$+\frac{\kappa}{2}\big|\nabla z\big|^2+\psi_0(\lambda,\theta)+\frac{1}{2\varepsilon}\big|z-\lambda\big|^2+\frac{\varepsilon}{2}\big|\nabla\lambda\big|^2.\tag{5.6}$$

The ansatz (1.1) then requires $\psi_0(\lambda,\theta)=\theta\phi(\lambda)-\phi_0(\theta)$. For such a linearized term $\theta\phi(\lambda)$ we refer e.g. to [30, 37, 58, 67, 69, 70]. Thus we eventually come to a regularized model that fits with (3.10a), namely

$$\varrho\frac{\partial^2u}{\partial t^2}-\mathrm{div}\,\mathbb{D}e\big(\frac{\partial u}{\partial t}\big)-\mathrm{div}\,\mathbb{C}\big(e{-}e_{\rm tr}(z)\big)=f,\tag{5.7a}$$

$$-\,\varepsilon\Delta\frac{\partial\lambda}{\partial t}-\varepsilon\Delta\lambda+\frac{1}{\varepsilon}(\lambda{-}z)+\theta\phi'(\lambda)=0,\tag{5.7b}$$

$$\partial\zeta_1\big(\frac{\partial z}{\partial t}\big)-\mathbb{C}e_{\rm tr}'(z){:}\big(e{-}e_{\rm tr}(z)\big)+\phi_1'(z)-\kappa\nabla z+\frac{1}{\varepsilon}(z{-}\lambda)+N_K(z)\ni 0,\tag{5.7c}$$

$$c_{\rm v}(\theta)\frac{\partial\theta}{\partial t}-\mathrm{div}\big(\mathbb{K}(\theta)\nabla\theta\big)=\zeta_1\big(\frac{\partial z}{\partial t}\big)+\mathbb{D}e\big(\frac{\partial u}{\partial t}\big){:}e\big(\frac{\partial u}{\partial t}\big)+\varepsilon\Big|\nabla\frac{\partial\lambda}{\partial t}\Big|^2+\theta\phi'(\lambda){\cdot}\frac{\partial\lambda}{\partial t},\tag{5.7d}$$

where $N_K(z)$ stands for the normal cone to the convex set K at z.

Example 5.3 (*Magnetostriction: a phase-field type model.*) Beside small strains, in magnetostrictive materials the state involves also the magnetization vector $\vec m\in\mathbb{R}^n$ which has partly a viscous and partly a rate-independent character. A peculiarity is that $\vec m$ does not exhibit any inertia but is involved in a non-potential non-dissipative gyroscopic term causing a precession movement within evolving $\vec m$. In view of Remark 4.9, we adopt the concept of a phase-field parameter $z\in\mathbb{R}^m$ that is related only rather vaguely with $\mathscr{L}(\vec m)$ with $\mathscr{L}:\mathbb{R}^n\to\mathbb{R}^m$ a continuous mapping that allows to distinguish particular "phases", i.e. here directions of easy magnetizations. We consider a so-called anisotropic energy $\varphi_{\rm an}:\mathbb{R}^m\to\mathbb{R}$ and a function $e_{\rm p}:\mathbb{R}^n\to\mathbb{R}^{n\times n}_{\rm sym}$ describing dependence of the preferred strain $e_{\rm p}$ on magnetization. In contrast to most of mathematical literature, we do not count exactly with the so-called Heisenberg constraint $|\vec m|=m_{\rm s}$ with $m_{\rm s}\ge 0$ being the temperature-dependent saturation magnetization, which is relevant rather for temperatures close to absolute zero while for temperatures around the Curie point (where $m_{\rm s}$ falls to zero and ferro-to-para-magnetic transition occurs), this constraint is substantially deviated in external fields, cf. [11, Fig.5.4], and is rather to be involved in $\varphi_{\rm an}$, cf. [61]. The free energy in magnetostriction (with a demagnetizing field neglected) is then considered as

$$\begin{aligned}\psi(e,\vec m,\nabla\vec m,z,\theta):=&\ \frac12\mathbb{C}(e{-}e_{\rm p}(\vec m)){:}(e{-}e_{\rm p}(\vec m))\\&+\varphi_{\rm an}(\vec m)+\theta\phi(e,\vec m)+\frac12L\big|z{-}\mathscr{L}(\vec m)\big|^2+\frac{\kappa}{2}\big|\nabla\vec m\big|^2-\phi_0(\theta)\end{aligned}\tag{5.8}$$

where the κ-term is the so-called exchange energy, cf. e.g. [36, 72] or also [44], and where L is assumed large so that practically $z\sim\mathscr{L}(\vec m)$. The evolution is here governed by the system

$$\varrho\frac{\partial^2u}{\partial t^2}-\mathrm{div}\Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big)+\mathbb{C}(e{-}e_{\rm p}(\vec m))+\theta\phi_e'(e,\vec m)\Big)=f,\tag{5.9a}$$

$$\alpha_0\frac{\partial\vec m}{\partial t}-\alpha_1\Delta\frac{\partial\vec m}{\partial t}+\frac{\vec m}{\gamma(|\vec m|)}\times\frac{\partial\vec m}{\partial t}+\psi_{\vec m}'\big(e(u),\vec m,z,\theta\big)-\kappa\Delta\vec m=h_{\rm ext},\tag{5.9b}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+L\big(z-\mathscr{L}(\vec m)\big)\ni 0,\tag{5.9c}$$

$$\begin{aligned}c_{\rm v}(\theta)\frac{\partial\theta}{\partial t}-\mathrm{div}\big(\mathbb{K}(\theta)\nabla\theta\big)=&\ \zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\mathbb{D}e\big(\frac{\partial u}{\partial t}\big):e\big(\frac{\partial u}{\partial t}\big)+\alpha_0\Big|\frac{\partial\vec m}{\partial t}\Big|^2+\alpha_1\Big|\nabla\frac{\partial\vec m}{\partial t}\Big|^2\\&+\theta\phi_e'(e(u),\vec m):\frac{\partial e(u)}{\partial t}+\theta\phi_{\vec m}'(e(u),\vec m)\cdot\frac{\partial\vec m}{\partial t},\end{aligned}\tag{5.9d}$$

where $\alpha_0,\alpha_1>0$ are constants determining attenuation of the magnetization oscillations (for α_1-term see [12]), γ is a so-called gyromagnetic moment (depending on $|\vec m|$), $h_{\rm ext}$ is the external magnetic field, cf. [61] for details in the rigid case $u=0$, $z=0$.

The potential ζ_1 may describe activation energy for re-magnetization, which is related in particular to the so-called "pinning" effect within domain-wall evolution and which contributes to hysteretic response of the ferromagnet due to, e.g., various impurities that can phenomenologically be described just by ζ_1; a similar rate-independent contribution has been proposed in [6, 10, 73]. This energy is finite, i.e. S is bounded, and hence ζ_1 continuous.

The system (5.9) fits with the presented theory with $p = 2 = q$ only through suitable combination of Remarks 4.6, 4.7, and 4.8. The $(e, \vec{m})$-semiconvexity of $\varphi(e, \vec{m}, z) := \frac{1}{2}\mathbb{C}(e-e_{\rm p}(\vec{m})){:}(e-e_{\rm p}(\vec{m})) + \varphi_{\rm an}(\vec{m}) + \frac{1}{2}L|z-\mathscr{L}(\vec{m})|^2$ is easily guaranteed if $e_{\rm p}$ and $\mathscr{L}$ are Lipschitz continuous and $\varphi_{\rm an}$ is semiconvex in the usual sense. The peculiarity is also in time-discretization of the gyroscopic term $\frac{\vec{m}}{\gamma(|\vec{m}|)} \times \frac{\partial\vec{m}}{\partial t}$ which must be done by a semi-implicit way as $\frac{\vec{m}_\tau^{k-1}}{\gamma(|\vec{m}_\tau^{k-1}|)} \times \frac{\vec{m}_\tau^k - \vec{m}_\tau^{k-1}}{\tau}$ so that it will not destroy the convexity of the incremental problem corresponding to (4.13) in this special case. The limit passage in semi-stability without any gradient term like ∇z (indeed omitted in (5.9c)), i.e. here

$$\forall v{\in}L^2(Q;\mathbb{R}^m): \quad \int_Q \frac{1}{2}L\big|z - \mathscr{L}(\vec{m})\big|^2 \,\mathrm{d}x\mathrm{d}t \le \int_Q \frac{1}{2}L\big|v - \mathscr{L}(\vec{m})\big|^2 + \zeta_1(v - z)\,\mathrm{d}x\mathrm{d}t, \tag{5.10}$$

can rely on the quadratic form of $\psi(e, \vec{m}, \nabla\vec{m}, \cdot, \theta)$ in (5.8) and be done by the binomial trick (4.66) modified to result in

$$\begin{aligned}\int_Q \big|z{-}\mathscr{L}(\vec{m})\big|^2 \,\mathrm{d}x\mathrm{d}t - \int_Q \big|v{-}\mathscr{L}(\vec{m})\big|^2 \,\mathrm{d}x\mathrm{d}t &= \int_Q |z|^2 - |v|^2 - 2(z{-}v)\cdot\mathscr{L}(\vec{m})\,\mathrm{d}x\mathrm{d}t \\ &= \int_Q (z{-}v)\cdot\big(z{+}v{-}2\mathscr{L}(\vec{m})\big)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.11}$$

The strong convergence in $e_{\rm p}(\vec{m})$, guaranteed through Aubin-Lions' theorem, is then used both for the strong convergence (4.56) as well as for the gyroscopic term $\frac{\vec{m}}{\gamma(|\vec{m}|)} \times \frac{\partial\vec{m}}{\partial t}$. Of course, instead of balancing mechanical energy in (4.67), one must balance the magneto-mechanical energy for which it is important that $(\frac{\vec{m}}{\gamma(|\vec{m}|)} \times \frac{\partial\vec{m}}{\partial t}) \cdot \frac{\partial\vec{m}}{\partial t} = 0$ as well as $(\frac{\vec{m}_\tau^{k-1}}{\gamma(|\vec{m}_\tau^{k-1}|)} \times \frac{\vec{m}_\tau^k - \vec{m}_\tau^{k-1}}{\tau}) \cdot \frac{\vec{m}_\tau^k - \vec{m}_\tau^{k-1}}{\tau} = 0$.

Augmenting of the stored energy by the demagnetizing-field energy, which is a nonlocal but quadratic term of the form $\int_{\mathbb{R}^n} |\nabla\Delta^{-1}{\rm div}(\chi_\Omega\vec{m})|^2\,\mathrm{d}x$ with χ_Ω the characteristic function of Ω, does not bring any essential problems into the above presented theory.

Example 5.4 (*Damage.*) Our assumptions allow for a rather special situation in damaging materials, namely a "mixture" of two materials, one with the elastic moduli $\mathbb{C}_1$ undergoing (for simplicity isotropical) damage described by a scalar parameter z valued in $[0, 1]$ (i.e. $m = 1$), the other one with the elastic moduli $\mathbb{C}_2$ undergoing thermal expansion. Thus we consider

$$\begin{aligned}\psi(e, z, \nabla z, \theta) := \frac{1{-}z}{2}\mathbb{C}_1 e : e + \frac{1}{2}\mathbb{C}_2(e{-}\theta\mathbb{E}) : (e{-}\theta\mathbb{E}) \\ - a_0 z + \delta_{[0,1]}(z) + \frac{\kappa}{2}|\nabla z|^2 - \frac{\mathbb{C}_2\mathbb{E} : \mathbb{E}}{2}\theta^2 - \phi_0(\theta),\end{aligned} \tag{5.12}$$

where $\mathbb{E}$ is the matrix of thermal-expansion coefficients and $a_0 > 0$ is the part of the energy deposited through the damage into the change of structure of the material (not dissipated into the heat). We consider damage with a possible "healing", i.e. $S := [-a_1, a_2]$ where $a_1 > 0$ and $a_2 > a_0$ so that $a_1 + a_0$ is an activation threshold for damage evolution and $a_2 - a_0$ an activation threshold for healing of damage. Certain healing may indeed occur in various biomaterials or polymer adhesives, cf. [7,41,76]. Mathematically, healing was used e.g. in [46,68]. Usually, $a_2 \gg a_1$ and, if $a_2 = +\infty$, damage becomes a unidirectional process without any healing possible and then $S := [-a_1, +\infty)$ and $\zeta_1(\dot z) = \delta_S^*(\dot z) = -a_1\dot z + \delta_{(-\infty,0]}(\dot z)$. Natural initial condition is $z_0 = 1$, i.e. undamaged material. The so-called factor of influence $\kappa > 0$ is related with certain "hardening" effects: activation threshold a is effectively increased/decreased at a given point if its surrounding is less/more damaged, respectively, cf. also [16,28,29,46,50,53]. We assume here $\mathbb{C}_2$ positive definite so that the material cannot completely disintegrate even for $z = 0$; we just remark that a complete damage is very difficult even in the isothermal case, see [16,53].

Since z ranges a bounded interval $[0,1]$ only, the nonconvex term $\frac{1-z}{2}\mathbb{C}_1 e{:}e$ is (e)-semiconvex, as required in (3.9). Note also that φ is not convex but $\varphi(\cdot, z, \cdot)$ is convex quadratic and complies with (3.8) and (3.10a) for $p = 2 = q$ provided the healing threshold a_2 is finite. Without healing, the unidirectional damage with $a_2 = +\infty$ is, unfortunately, not covered by any of the previous results because both ζ_1 and φ are simultaneously discontinuous. For some special techniques in the isothermal case, we refer to [16,50,53].

Example 5.5 (*Shape-memory alloys at large strains.*) Shape-memory alloys typically have multi-well nonconvex stored energy which, however, requires further regularizing gradient theory like in Remark 4.7. Then we can work in terms of large strains, adopting also the concept of the *phase-field* model with the vectorial *order parameter* z being related with particular phases identified through a mapping $\mathscr{L} : \mathbb{R}^{n\times n} \to \{z \in (\mathbb{R}^+)^m;\ \sum_{i=1}^m z_i = 1\}$. The free energy can then be considered as

$$\psi(\nabla u, z, \theta) = \varphi_0(\nabla u) + \theta\phi(\nabla u) - \phi_0(\theta) + L\big|z - \mathscr{L}(\nabla u)\big|^2 + \varepsilon\big|\nabla^2 u\big|^2. \tag{5.13}$$

Note that it complies with Remark 4.8. For particular examples for φ_0 and ϕ and a construction method based on cubic C^2-splines fitted with experimentally-measured wells and elastic moduli in specific shape-memory materials we refer to [35]. It is assumed that L is large so that z is presumably mostly close to $\mathscr{L}(\nabla u)$, while $\varepsilon > 0$ is small, determining rather some internal scale than influencing a macroscopical response itself. Note also that $|\nabla z|^2$ is not involved in (5.13), similarly like in Example 5.1. We also need (e)-semiconvexity (or, here, rather ∇u-semiconvexity) of the term $L(|z|^2 - 2z{\cdot}\mathscr{L}(\nabla u))$, for which it suffices to assume $\mathscr{L}'$ bounded. Like (5.11), limit passage in semi-stability can rely on the quadratic form of $\psi(\nabla u, \cdot, \theta)$ in (5.13) and the identity

$$\begin{aligned}\int_Q \big|z - \mathscr{L}(\nabla u)\big|^2 \,\mathrm{d}x\mathrm{d}t - \int_Q \big|v - \mathscr{L}(\nabla u)\big|^2 \,\mathrm{d}x\mathrm{d}t &= \int_Q |z|^2 - |v|^2 - 2(z-v)\cdot\mathscr{L}(\nabla u)\,\mathrm{d}x\mathrm{d}t \\ &= \int_Q (z-v)\cdot\big(z+v-2\mathscr{L}(\nabla u)\big)\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.14}$$

and then we can use the trick like in (4.66). Further, we employ Remark 4.7 with (4.82), in which q^* is replaced by 2, however.

Remark 5.6 (*Heat production in electric conductors.*) Electrically conductive materials under external voltage may produce *Joule heat* that serves as an example of $\xi_{\rm nonloc}$ from Remark 4.4. More specifically, $\xi_{\rm nonloc}(t, z, \theta) := \vec{j}\cdot\nabla\phi$ induced by the electric current $\vec{j} = \mathbb{S}(z,\theta)\nabla\phi$, where $\mathbb{S} = \mathbb{S}(z,\theta)$ is the electric-conductivity tensor with $\phi \in W^{1,2}(\Omega)$ solving the boundary-value problem for the equation $\mathrm{div}(j) = 0$ with the boundary conditions $j\cdot\vec{n} = 0$ on the electrically isolated part of Γ and $\phi = \phi_{\mathrm{ext},i}(t)$ on the parts Γ_i of Γ that are electrodes with a prescribed external potential $\phi_{\mathrm{ext},i}$. Even more realistically, conditions like $\phi - \phi_{\mathrm{ext},i}(t) = \frac{1}{R}\int_{\Gamma_i} j\cdot\vec{n}\,\mathrm{d}S$ where R denotes the internal resistance of the external voltage source. Assuming $\mathbb{S} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{n\times n}$ continuous, bounded, and uniformly positive definite, we obtain

$(z,\theta)\to\phi : L^q(\Omega;\mathbb{R}^m)\times L^1(\Omega)\to W^{1,2}(\Omega)$ continuous, which follows from the estimate

$$\alpha\big\|\nabla(\phi_k-\phi)\big\|^2_{L^2(\Omega;\mathbb{R}^n)}\le\int_\Omega\mathbb{S}(z_k,\theta_k)\nabla(\phi_k-\phi)\cdot\nabla(\phi_k-\phi)\,\mathrm{d}x=\int_\Omega\mathbb{S}(z_k,\theta_k)\phi\cdot\nabla(\phi-\phi_k)\,\mathrm{d}x\to 0$$

with $\alpha:=\inf_{|g|=1,z,\theta}\mathbb{S}(z,\theta)g\cdot g$, advancing thus the weak convergence of $\phi_k\to\phi$ to the desired strong convergence with ϕ_k corresponding to (z_k,θ_k) with a sequence $\{(z_k,\theta_k)\}_{k\in\mathbb{N}}$ converging to (z,θ). In particular, in shape-memory alloys or magnetostrictive materials, $\mathbb{S}$ may depend on volume fraction and temperature, cf. [59] or also [65], thus, in particular, may be different in austenite and in martensite. Dependence of $\mathbb{S}$ on the damage parameter is natural as damaged material conducts electric current harder than if it is nondamaged.

Remark 5.7 (*Growth of* $c_\mathrm{v}(\cdot)$ *and* $\mathbb{K}(e,z,\cdot)$.) Comparing our results with a conventional thermo-visco-elasticity in thermally expanding materials (using $p_1=0$ and $p_2=2$ and z avoided in Example 5.1) in the case $n=3$, we can see that (3.13) yields $\omega>6/5$, i.e. a polynomial growth of c_v of order $>1/6$ only. On the other hand, in case of $\mathbb{K}$ constant, the condition $\omega>3/2$ is obtained [13, 63]. To explain this “optical” discrepancy, let us mention some other results showing that, if the heat capacity c_v is constant (i.e. $\omega=1$ in (3.12b)), a polynomial growth of $\theta\mapsto\mathbb{K}(\theta)$ bigger than 1/3 helps, see [23] or also [24, Sect.5.4.2.1]. Our condition (3.12d) requires, in view of the definition (3.3), a certain growth of $\mathbb{K}(e,z,\cdot)$ as $\mathscr{T}$ must inevitably decay if $\omega>1$, cf. the formula (4.12); here the growth of $\mathbb{K}(e,z,\cdot)$ should be bigger than the decay of the factor $1/(c_\mathrm{v}\circ\mathscr{T})$, cf. the definition (3.3) of $\mathbb{K}$, i.e. $1-1/\omega$. In the 3D case, both $c_\mathrm{v}(\cdot)$ and $\mathbb{K}(e,z,\cdot)$ should thus growth polynomially at least as $>1/6$. In view of this, the enthalpy-transformation resultes in a certain compromise between growth of $c_\mathrm{v}(\cdot)$ and of $\mathbb{K}(e,z,\cdot)$ which both are thus allowed to be relatively mild.

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