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Preprint no. 2009-032



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ON CERTAIN CONVEX COMPACTIFICATIONS FOR RELAXATION IN EVOLUTION PROBLEMS

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Abstract. A general-topological construction of limits of inverse systems is applied to convex compactifications and furthermore to special convex compactifications of Lebesgue-space-valued functions parameterized by time. Application to relaxation of quasistatic evolution in phase-change-type problems is outlined.

AMS Classification: Primary: 54D35; Secondary: 26A45, 49J45.

Keywords: Limits of inverse systems, generalized Young measures, Helly principle.

1 Introduction

Phase-change problems typically involve non-convex multiwell potentials that give rise various microstructures. These microstructures are related to fast spatial oscillations of involved quantities, like deformation gradients, magnetization, polarization, etc. Thus special techniques for efficient treatment of such multiscale situations are desired. From a “mesoscopical-scale” viewpoint, conventional notions of (weak) solutions loose applicability at least as far as possible efficient numerical implementation concerns and, adopting certain modelling assumptions, some relaxation leading to notions of generalized solutions is desirable. Involving time is especially important for addressing evolution of such phase-change problems, cf. e.g. [2, 6, 18, 22, 23, 24, 29, 32, 39].

This is a relatively standardly used approach, especially if a correlation of microstructures between particular time instances is rather “macroscopical” only. Recently, some new construction occurs [10, 11, 12, 16, 17, 26] that allows for more intrinsic correlation between particular time levels, although sometimes it may be rather a question of modelling since relaxation in general is always a delicate approach [38]. Anyhow, such constructions are interesting mathematically and open new possibilities to identify limit information at least to some extent in case where standard methods do not meet sufficient data qualification.

The goal of this contribution is to develop a unifying theory of both such conventional and also new compactifications of evolution problems. In comparison with usual approaches focusing on topological aspects only, here also a convexity concept is systematically involved. Also, e.g. attainability, which is absolutely important for numerical implementation and not always addressed, will be under focus here. Moreover, the general construction is presented both in a broader context of general topology where it fits with the concept of so-called limits of inverse systems [1, 14, 15, 27] as well as in the context convex-compactification theory [35, 36, 37].

2 General-topology concept of limits of inverse systems

First, let us recall that an ordered set Ξ is called *directed* if for any $\xi_1, \xi_2 \in \Xi$ there is $\xi_3 \in \Xi$ such that both $\xi_1 \leq \xi_3$ and $\xi_2 \leq \xi_3$. A collection $(X_\xi)_{\xi \in \Xi}$ of topological spaces X_ξ

¹The author acknowledges fruitful discussions with profs. Miroslav Hušek, Martin Kružík, Alexander Mielke, and Ulisse Stefanelli. This research has been partly supported by the grants A 100750802 (GA AV ČR), 106/09/1573 (GA ČR), LC 06052 and MSM 21620839 (MŠMT ČR), and from the research plan AV0Z20760514 (ČR).

is called an *inverse system* if, for any $\zeta \geq \xi$, there is a continuous mapping $\pi_\xi^\zeta : X_\zeta \rightarrow X_\xi$ and these mappings, called *bonding mappings*, satisfy $\pi_{\xi_1}^{\xi_3} = \pi_{\xi_1}^{\xi_2} \circ \pi_{\xi_2}^{\xi_3}$ if $\xi_1 \leq \xi_2 \leq \xi_3$ and $\pi_{\xi_1}^{\xi_2} = \text{identity}$ if $\xi_1 = \xi_2$. Shortly, we write this inverse system as $\mathfrak{S} = (X_\xi, \pi_\xi^\zeta)_{\zeta, \xi \in \Xi, \zeta \geq \xi}$. Each $x \in \prod_{\xi \in \Xi} X_\xi$ is called a *thread* if $\pi_\xi^\zeta x_\zeta = x_\xi$ for any $\zeta \geq \xi$. The set

$$\varprojlim \mathfrak{S} := \left\{ x \in \prod_{\xi \in \Xi} X_\xi; x \text{ is a thread} \right\} \quad (1)$$

is called the *limit of the inverse system* $\mathfrak{S} = (X_\xi, \pi_\xi^\zeta)_{\zeta, \xi \in \Xi, \zeta \geq \xi}$. We further define the *projections* of the limit of the inverse system $\text{Pr}_\zeta : \varprojlim \mathfrak{S} \rightarrow X_\zeta$ as the restriction on $\varprojlim \mathfrak{S}$ of the projections $\prod_{\xi \in \Xi} X_\xi \rightarrow X_\zeta$. For all $\xi \leq \zeta$, it holds $\text{Pr}_\xi = \pi_\xi^\zeta \circ \text{Pr}_\zeta$.

If all X_ξ in $(X_\xi, \pi_\xi^\zeta)_{\zeta, \xi \in \Xi, \zeta \geq \xi}$ are Hausdorff topological spaces, then $\varprojlim \mathfrak{S}$ is closed in $\prod_{\xi \in \Xi} X_\xi$ equipped with the canonical product topology; cf. [15, Prop. 2.5.1]. This topology is compact by the Tikhonov theorem [40], so that we then have immediately:

Proposition 1 *The limit of the inverse system of compact spaces is itself compact in $\prod_{\xi \in \Xi} X_\xi$.*

The concept of limits of the inverse systems has been invented in early 30ties of the last century in [27] exploiting a bit modified definition in [1] and then developed in [14]; see e.g. [15, Sect.2.5] for a comprehensive exposition.

3 Convex compactifications and their inverse systems

A triple (K, Z, i) is called a *convex compactification* of a topological space (U, \mathcal{T}) if Z is a Hausdorff locally convex space, K is a convex, compact subset of Z , $i : U \rightarrow K$ is continuous, and $i(U)$ is dense in K . If i is also injective (resp. homeomorphical embedding), (K, Z, i) is called a Hausdorff (resp. \mathcal{T} -consistent) convex compactification. We define *ordering of convex compactifications* as follows: for two convex compactifications (K_1, Z_1, i_1) and (K_2, Z_2, i_2) of U , we say that (K_1, Z_1, i_1) is *finer* than (K_2, Z_2, i_2) (or (K_2, Z_2, i_2) is *coarser* than (K_1, Z_1, i_1)), and write $(K_1, Z_1, i_1) \succeq (K_2, Z_2, i_2)$, if there is an affine continuous mapping $\psi : K_1 \rightarrow K_2$ fixing U ; the adjective “affine” means $\psi(\frac{1}{2}z + \frac{1}{2}\tilde{z}) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi(\tilde{z})$ for any $z, \tilde{z} \in K_1$, while “fixing U ” means $\psi \circ i_1 = i_2$. We say that (K_1, Z_1, i_1) and (K_2, Z_2, i_2) are equivalent with each other, and write $(K_1, Z_1, i_1) \cong (K_2, Z_2, i_2)$, if simultaneously $(K_1, Z_1, i_1) \succeq (K_2, Z_2, i_2)$ and $(K_2, Z_2, i_2) \succeq (K_1, Z_1, i_1)$. We also say that (K_1, Z_1, i_1) is strictly finer than (K_2, Z_2, i_2) , and write $(K_1, Z_1, i_1) \succ (K_2, Z_2, i_2)$, if $(K_1, Z_1, i_1) \succeq (K_2, Z_2, i_2)$ and $(K_1, Z_1, i_1) \not\cong (K_2, Z_2, i_2)$. Note that $(K_1, Z_1, i_1) \cong (K_2, Z_2, i_2)$ does not mean that the locally convex spaces Z_1 and Z_2 are isomorphic with each other.

Let us remark that, if Z 's are forgotten, the concept of convex compactifications and their ordering reduces to the usual general-topology concept of compactifications and their ordering if one agrees with a broader concept that the continuous imbeddings need not be homeomorphical.

Let us agree to abbreviate occasionally e.g. $K_1 \succeq K_2$ instead of $(K_1, Z_1, i_1) \succeq (K_2, Z_2, i_2)$ when Z_1, Z_2, i_1 , and i_2 are obvious from a context.

The concept of convex compactifications has been invented in [35]; for a comprehensive treatment of convex-compactification theory we refer to [36].

Let us consider a collection of convex compactifications $\{(K_\xi, Z_\xi, i_\xi)\}_{\xi \in \Xi}$. If Ξ is directed and $\xi \mapsto K_\xi$ is monotone in the sense that bigger index ξ corresponds to a finer convex compactification K_ξ , the collection $\{(K_\xi, Z_\xi, i_\xi)\}_{\xi \in \Xi}$ equipped with the canonical

affine surjections $\pi_\zeta^\xi : K_\zeta \rightarrow K_\xi$ if $\zeta \geq \xi$ as the collection of bonding mappings forms an *inverse system*.

Proposition 2 *The limit (1) of any inverse system $\mathfrak{S} = ((K_\xi, Z_\xi, i_\xi), \pi_\xi^\zeta)_{\zeta, \xi \in \Xi, \zeta \geq \xi}$ of convex compactifications of U is again a convex compactification of U with the convex structure induced from the locally convex space $Z := \prod_{\xi \in \Xi} Z_\xi$ equipped with the canonical product topology and the affine continuous embedding $i : U \rightarrow \varprojlim \mathfrak{S}$ being defined by $i = (i_\xi)_{\xi \in \Xi}$. Moreover, it holds*

$$\sup_{\xi \in \Xi} K_\xi \cong \varprojlim \mathfrak{S}, \quad \text{with } \mathfrak{S} = ((K_\xi, Z_\xi, i_\xi), \pi_\xi^\zeta)_{\zeta, \xi \in \Xi, \zeta \geq \xi}. \quad (2)$$

Proof. First, let us prove that any thread $\eta \in \varprojlim \mathfrak{S}$ is attainable from U in the sense that, for some net $\{u_\alpha\}_{\alpha \in A}$ with A a directed set, it holds $\eta = \lim_{\alpha \in A} i(u_\alpha)$. Here it suffices to take A a base of the filter of the neighborhoods of 0 in Z and use that, for any finite selection $\xi_1, \dots, \xi_n \in \Xi$ and $N_{\xi_k} \in \mathcal{N}_{\xi_k}$, there is some $u \in U$ such that $i_{\xi_k}(u) - \text{Pr}_{\xi_k} \eta \in N_{\xi_k}$ for any $k = 1, \dots, n$ with Pr_{ξ_k} being the canonical projection $\prod_{\xi \in \Xi} Z_\xi \rightarrow Z_{\xi_k}$ and with \mathcal{N}_{ξ_k} denoting the set of the neighborhoods of 0 in Z_{ξ_k} . We also used that each $\text{Pr}_{\xi_k} \eta \in K_{\xi_k}$ and each (K_{ξ_k}, i_{ξ_k}) is a compactification of U .

To prove convexity of $\varprojlim \mathfrak{S}$ in Z , having any $u_1, u_2 \in U$, we have to show existence of a net $(u_\alpha)_{\alpha \in A}$ such that $\lim_{\alpha \in A} i(u_\alpha) = \frac{1}{2}i(u_1) + \frac{1}{2}i(u_2)$, cf. [36, Chap.2]. In other words, we need to prove that

$$\forall \zeta \in \Xi : \quad \lim_{\alpha \in A} i_\zeta(u_\alpha) = \frac{1}{2}i_\zeta(u_1) + \frac{1}{2}i_\zeta(u_2). \quad (3)$$

Let us now take $A = \{(\xi, O); \xi \in \Xi, O \in \mathcal{N}_\xi\}$ ordered by $(\xi_1, O_1) \leq (\xi_2, O_2)$ if $\xi_1 \leq \xi_2$ and $O_1 \supset \pi_{\xi_1}^{\xi_2}(O_2)$. The directedness of Ξ and the continuity of the bonding mappings $\pi_{\xi_k}^\xi$ make A directed because, for any (ξ_1, O_1) and (ξ_2, O_2) , there is $(\xi, O) \in A$ such that $(\xi, O) \geq (\xi_k, O_k)$, $k = 1, 2$, i.e. $\xi \geq \xi_k$ and both $O_1 \supset \pi_{\xi_1}^\xi(O)$ and $O_2 \supset \pi_{\xi_2}^\xi(O)$ if $O \in \mathcal{N}_\xi$ is taken small enough. As each (K_ξ, Z_ξ, i_ξ) is a convex compactification of U , for any

$$\forall O \in \mathcal{N}_\xi \exists u_{\xi, O} : \quad i_\xi(u_{\xi, O}) - \frac{1}{2}i_\xi(u_1) - \frac{1}{2}i_\xi(u_2) \in O. \quad (4)$$

Then, for $\zeta \in \Xi$ fixed and $N \in \mathcal{N}_\zeta$, there is $\alpha_0 = (\xi_0, O_0) \in A$ such that, for all $\alpha = (\xi, O) \geq \alpha_0$, it holds

$$i_\xi(u_\alpha) - \frac{1}{2}i_\xi(u_1) - \frac{1}{2}i_\xi(u_2) \in O; \quad (5)$$

indeed, it suffices to take $\xi_0 := \zeta$, $O_0 := N$, and $u_\alpha := u_{\xi, O}$ from (4). Then we apply π_ζ^ξ to (5): by the affinity and continuity of π_ζ^ξ , we obtain

$$\begin{aligned} i_\zeta(u_\alpha) - \frac{1}{2}i_\zeta(u_1) - \frac{1}{2}i_\zeta(u_2) &= \pi_\zeta^\xi i_\xi(u_\alpha) - \frac{1}{2}\pi_\zeta^\xi i_\xi(u_1) - \frac{1}{2}\pi_\zeta^\xi i_\xi(u_2) \\ &= \pi_\zeta^\xi \left(i_\xi(u_\alpha) - \frac{1}{2}i_\xi(u_1) - \frac{1}{2}i_\xi(u_2) \right) \in \pi_\zeta^\xi(O) \subset O_0 = N. \end{aligned}$$

As $\zeta \in \Xi$ and $N \in \mathcal{N}_\zeta$ are arbitrary, (3) is proved.

Obviously, $\varprojlim \mathfrak{S} \supseteq K_\xi$ for any $\xi \in \Xi$, the affine continuous surjection $\varprojlim \mathfrak{S} \rightarrow K_\xi$ which fixes U being just Pr_ξ . Moreover, if, for some convex compactification (K, j) of U ,

it holds $K \succeq K_\xi$ for each $\xi \in \Xi$ with $\psi_\xi : K \rightarrow K_\xi$ denoting the corresponding affine continuous surjection fixing U , we can see that $\psi := (\psi_\xi)_{\xi \in \Xi} : K \rightarrow \prod_{\xi \in \Xi} K_\xi$ is affine and continuous and $\psi(j(u)) = (\psi_\xi j(u))_{\xi \in \Xi} = (i_\xi(u))_{\xi \in \Xi} = i(u)$ so that ψ fixes U in the sense $\psi \circ j = i$. Since we have proved that the compact convex set $\overleftarrow{\lim} \mathfrak{S}$ is the closure of $i(U)$ in $\prod_{\xi \in \Xi} K_\xi$, the mapping $\psi : K \rightarrow \overleftarrow{\lim} \mathfrak{S}$ is surjective. Thus K is finer than $\overleftarrow{\lim} \mathfrak{S}$. Thus $\overleftarrow{\lim} \mathfrak{S}$ is also a least upper bound of the collection $(K_\xi)_{\xi \in \Xi}$, as claimed in (2). \square

4 Convex compactifications of sets in Lebesgue spaces

The Lebesgue spaces are definitely the most prominent function spaces occurring in applications. Following [36, 37], we briefly present a fairly universal construction of convex compactifications of their balls and convex locally compact envelopes of them.

Considering the Lebesgue space $L^p(\Omega; \mathbb{R}^m)$ with Ω a Lebesgue measurable set with a finite measure, we define a normed linear space

$$\begin{aligned} \text{Car}^p(\Omega; \mathbb{R}^m) := \{ & h : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}; \quad h(\cdot, s) \text{ measurable, } h(x, \cdot) \text{ continuous,} \\ & \exists a \in L^1(\Omega), b \in \mathbb{R} : |h(x, s)| \leq a(x) + b|s|^p \} \end{aligned} \quad (6)$$

of “test Carathéodory integrands”, and equip it with the norm

$$\|h\|_{\text{Car}^p(\Omega; \mathbb{R}^m)} := \inf_{|h(x,s)| \leq a(x) + b|s|^p} \|a\|_{L^1(\Omega)} + b; \quad (7)$$

more precisely, as usual, we consider equivalence classes up to zero-measure sets of such integrands. The essential trick is to consider a sufficiently large (but preferably still separable) linear subspace $H \subset \text{Car}^p(\Omega; \mathbb{R}^m)$, to define the embedding

$$i_H : L^p(\Omega; \mathbb{R}^m) \rightarrow H^* : u \mapsto \left(h \mapsto \int_{\Omega} h(x, u(x)) \, dx \right), \quad (8)$$

to put

$$Y_{H, \varrho}^p(\Omega; \mathbb{R}^m) := \text{the weak}^* \text{ closure of } i_H(\{u \in L^p(\Omega; \mathbb{R}^m); \|u\|_{L^p(\Omega; \mathbb{R}^m)} \leq \varrho\}), \quad (9)$$

and eventually to put

$$Y_H^p(\Omega; \mathbb{R}^m) := \bigcup_{\varrho \in \mathbb{N}} Y_{H, \varrho}^p(\Omega; \mathbb{R}^m). \quad (10)$$

The elements of $Y_H^p(\Omega; \mathbb{R}^m)$ are referred to as *Young functionals*.

Proposition 3 (See [37]) *For any linear subspace H of $\text{Car}^p(\Omega; \mathbb{R}^m)$, the couple $(Y_{H, \varrho}^p(\Omega; \mathbb{R}^m), i_H)$ forms a convex compactification of the ball of the radius ϱ in $L^p(\Omega; \mathbb{R}^m)$. Moreover, if H is rich enough, namely if one of the following situations holds:*

- (a) $H \supset \{g \otimes v; g \in C(\bar{\Omega}), v \text{ linear } \mathbb{R}^m \rightarrow \mathbb{R}\} \cup \{1 \otimes |s|^p\}$, $1 < p < \infty$,
- (b) $H \supset \{h_u; u \in U\}$ for a dense $U \subset L^p(\Omega; \mathbb{R}^m)$, where $h_u(x, s) = |s - u(x)|^p$,
- (c) $H \supset L^1(\Omega; C(\mathbb{R}^m)) \cup \{1 \otimes |s|^p\}$,

then the embedding i_H is even (norm, weak)-homeomorphical. If H is separable, this compactification is metrizable and thus $Y_{H, \varrho}^p(\Omega; \mathbb{R}^m)$ is sequentially compact.*

Remark 1 (Ordering of $Y_{H,\varrho}^p(\Omega; \mathbb{R}^m)$) Due to a large variety of such envelopes, it is worth applying the ordering defined in Section 3. Having two subspaces $H_2 \subset H_1$ of $\text{Car}^p(\Omega; \mathbb{R}^m)$, $Y_{H_1,\varrho}^p(\Omega; \mathbb{R}^m)$ is a finer convex compactification than $Y_{H_2,\varrho}^p(\Omega; \mathbb{R}^m)$, the affine continuous surjection being (the restriction of) the adjoint mapping of the inclusion $H_2 \rightarrow H_1$. In this context, a natural topology of $\text{Car}^p(\Omega; \mathbb{R}^m)$ is the locally convex topology on $\text{Car}^p(\Omega; \mathbb{R}^m)$ induced via the seminorm $|\cdot|_\varrho$ defined by $|h|_\varrho := \sup_{\|u\| \leq \varrho} \left| \int_\Omega h(x, u(x)) \, dx \right|$, rather than the norm topology induced by (7). The class $\{Y_{H,\varrho}^p(\Omega; \mathbb{R}^m); H \subset \text{Car}^p(\Omega; \mathbb{R}^m)\}$ is a *lattice*, the supremum and the infimum being given respectively by

$$\sup \left\{ Y_{H_1,\varrho}^p(\Omega; \mathbb{R}^m), Y_{H_2,\varrho}^p(\Omega; \mathbb{R}^m) \right\} = Y_{H_1+H_2,\varrho}^p(\Omega; \mathbb{R}^m), \quad (11a)$$

$$\inf \left\{ Y_{H_1,\varrho}^p(\Omega; \mathbb{R}^m), Y_{H_2,\varrho}^p(\Omega; \mathbb{R}^m) \right\} = Y_{\bar{H}_1 \cap \bar{H}_2,\varrho}^p(\Omega; \mathbb{R}^m), \quad (11b)$$

where $\bar{H}_j :=$ the closure of H_j in $\text{Car}^p(\Omega; \mathbb{R}^m)$ with respect to the mentioned seminorm.

Remark 2 (Convex locally compact envelopes of L^p -spaces) It is interesting that the whole $Y_H^p(\Omega; \mathbb{R}^m)$ from (10) is always convex in H^* . Assuming that H is rich enough as, e.g., specified in Proposition 3, then $Y_H^p(\Omega; \mathbb{R}^m)$ is locally compact in H^* and $L^p(\Omega; \mathbb{R}^m)$ itself is embedded into it (norm, weak*)-homeomorphically via i_H defined by (8). Again, if H is separable, then $Y_H^p(\Omega; \mathbb{R}^m)$ locally sequentially compact. Such envelopes $Y_H^p(\Omega; \mathbb{R}^m)$ of $L^p(\Omega; \mathbb{R}^m)$ imitate thus most of the important properties of Euclidean spaces (with the exception that these envelopes are not linear spaces but only convex subsets of those).

Example 1 (DiPerna-Majda's measures) Let us consider some metrizable compactification $\gamma(\mathbb{R}^m)$ of \mathbb{R}^m , e.g. the Alexandroff one-point compactification $\alpha(\mathbb{R}^m)$ of the locally compact space \mathbb{R}^m or a compactification of $\mathbb{R}^{n \times m}$ by a sphere S^{m-1} . In principle, one could think also about a Čech-Stone compactification $\beta(\mathbb{R}^{n \times m})$ but it would not yield metrizable hulls. Based e.g. on $\gamma(\mathbb{R}^{n \times m})$, DiPerna-Majda measures [13] and the related convex compactifications (cf. [36, Sect.3.2.c and Examples 3.3.10 and 3.4.6] for details), H then taken as

$$H_\gamma = C(\bar{\Omega}) \otimes V_\gamma \quad \text{with} \quad (12a)$$

$$V_\gamma = \left\{ v \in C(\mathbb{R}^m); \frac{v(\cdot)}{1 + |\cdot|^p} \text{ has a continuous extension on } \gamma(\mathbb{R}^m) \right\}. \quad (12b)$$

For further purposes, for this H_γ , let us denote $\text{DM}_\gamma(\Omega; \mathbb{R}^m) := Y_{H_\gamma}^p(\Omega; \mathbb{R}^m)$.

5 Convex compactifications of sets in $B(I; L^p)$ -spaces

Distributed-parameter evolution problems standardly involve time $t \in I \subset \mathbb{R}$ and, most typically, time-dependent states valued in the Lebesgue space, say $L^p(\Omega; \mathbb{R}^m)$. Sometimes, classical weak solutions of governing equations do not exist and one can thus think about a sensible extension, so-called relaxation. Thus, for evolution problems, it is worth developing a sufficiently wide class of convex compactifications of the space of bounded mappings $I \rightarrow L^p(\Omega; \mathbb{R}^m)$, denoted by $B(I; L^p(\Omega; \mathbb{R}^m))$. Having in mind some available a-priori estimates, it suffices to compactify only the ball

$$B_\rho := \{u \in B(I; L^p(\Omega; \mathbb{R}^m)); \forall t \in I : \|u(t)\|_{L^p(\Omega; \mathbb{R}^m)} \leq \rho\}. \quad (13)$$

Coarse compactifications handle spatial oscillations/concentrations on particular time levels separately but finer compactifications can handle possible correlations of such oscillations/concentrations at various time instances. Always, a finite (although not a-priori

given) number of those time instances suffices to be in correlation. To this goal, we systematically exploit the theory of inverse systems of convex compactifications from Section 3.

Let us denote by $\Pi = (t_1, t_2, \dots, t_{\#\Pi})$ a finite partition of $I \subset \mathbb{R}$ with $t_1 < t_2 < \dots < t_{\#\Pi}$ where $\#\Pi$ denotes the number of elements of Π and where $t_i \in I$ for all $i = 1, \dots, \#\Pi$. Let us denote by $\mathfrak{F}(I)$ the collection of all such partitions ordered by inclusion. It makes $\mathfrak{F}(I)$ directed. Let us further take, for any $\Pi \in \mathfrak{F}(I)$, some normed linear subspace $H_\Pi \subset \text{Car}^p(\Omega; \mathbb{R}^{\#\Pi \times m})$. Let us further define

$$\begin{aligned} e_{H_\Pi} : B(I; L^p(\Omega; \mathbb{R}^m)) &\rightarrow H_\Pi^* \quad \text{by} \\ \langle e_{H_\Pi}(u), h \rangle &:= \int_\Omega h(x, u(t_1, x), u(t_2, x), \dots, u(t_{\#\Pi}, x)) \, dx. \end{aligned} \quad (14)$$

We consider simply the Cartesian product $\prod_{\Pi \in \mathfrak{F}(I)} H_\Pi^*$ equipped with the Tikhonov product topology here counting the weak* topologies of each H_Π^* . Then we embed B_ρ into $\prod_{\Pi \in \mathfrak{F}(I)} H_\Pi^*$ by $i := (e_{H_\Pi})_{\Pi \in \mathfrak{F}(I)}$ and define

$$\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)) := \text{the closure of } (e_{H_\Pi}(B_\rho))_{\Pi \in \mathfrak{F}(I)}, \quad (15)$$

where the closure refers to the Tikhonov product topology counting the weak* topologies on each particular H_Π^* and where \mathcal{H} abbreviates the collection $(H_\Pi)_{\Pi \in \mathfrak{F}(I)}$. Also, we can consider $\tilde{Y}_{H_\Pi, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)) := \text{the closure of } e_{H_\Pi}(B_\rho)$, which is a compact subset of H_Π^* . It is exactly the set $Y_{H_\Pi, \rho}^p(\Omega; \mathbb{R}^{\#\Pi \times m})$ defined by (9). Defining the mapping $j_\Pi : B(I; L^p(\Omega; \mathbb{R}^m)) \rightarrow L^p(\Omega; \mathbb{R}^{\#\Pi \times m}) \cong L^p(\Omega; \mathbb{R}^m)^{\#\Pi}$ by $j_\Pi(u) = (u(t_1, \cdot), \dots, u(t_{\#\Pi}, \cdot))$, we can see that $e_{H_\Pi} = i_{H_\Pi} \circ j_\Pi$ with i_{H_Π} defined in (8) with H_Π in place of H . As the triple $(Y_{H_\Pi, \rho}^p(\Omega; \mathbb{R}^{\#\Pi \times m}), i_{H_\Pi}, H_\Pi^*)$ forms a convex compactification of the set $\{u \in L^p(\Omega; \mathbb{R}^{\#\Pi \times m}); \max_{i=1, \dots, \#\Pi} \int_\Omega \sum_{j=1}^m |u_{ij}(x)|^p \, dx \leq \varrho^p\}$ just as explained in Section 4, $(\tilde{Y}_{H_\Pi, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)), e_{H_\Pi}, H_\Pi^*)$ makes a convex compactification of B_ρ . If H_Π are rich enough (as, e.g., in Proposition 3), each i_{H_Π} is injective but e_{H_Π} is not because j_Π is not injective (except a trivial case that I itself is finite). Note that, in spite of it, $i = (e_{H_\Pi})_{\Pi \in \mathfrak{F}(I)}$ is injective. Also note that $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)) \subset \prod_{\Pi \in \mathfrak{F}(I)} Y_{H_\Pi, \rho}^p(\Omega; \mathbb{R}^{\#\Pi \times m})$ and, as each $Y_{H_\Pi, \rho}^p(\Omega; \mathbb{R}^{\#\Pi \times m})$ is compact, by Tikhonov theorem [40], $\prod_{\Pi \in \mathfrak{F}(I)} Y_{H_\Pi, \rho}^p(\Omega; \mathbb{R}^{\#\Pi \times m})$ and thus also $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ itself is compact, too. Hence, $(\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)), i)$ forms a compactification of B_ρ . This compactification is not metrizable except trivial cases.

Let us now assume existence of the collections $(P_{\Pi_2}^{\Pi_1})_{\Pi_1, \Pi_2 \in \mathfrak{F}(I)}$ of linear operators such that

$$\Pi_1 \supset \Pi_2 \quad \Rightarrow \quad P_{\Pi_2}^{\Pi_1} \in \mathcal{L}(H_{\Pi_2}, H_{\Pi_1}) \text{ is injective,} \quad (16a)$$

$$\Pi_1 \supset \Pi_2 \supset \Pi_3 \quad \Rightarrow \quad P_{\Pi_3}^{\Pi_2} \circ P_{\Pi_2}^{\Pi_1} = P_{\Pi_3}^{\Pi_1}, \quad (16b)$$

$$\Pi_1 = \Pi_2 \quad \Rightarrow \quad P_{\Pi_2}^{\Pi_1} = \text{identity,} \quad (16c)$$

where $\mathcal{L}(\cdot, \cdot)$ denotes the space of linear operators. By (16a), the adjoint mapping

$$\psi_{\Pi_2}^{\Pi_1} := [P_{\Pi_2}^{\Pi_1}]^* : H_{\Pi_1}^* \rightarrow H_{\Pi_2}^* \quad (17)$$

is surjective, and then also $\psi_{\Pi_2}^{\Pi_1} : Y_{H_{\Pi_1}, \rho}^p(\Omega; \mathbb{R}^{\#\Pi_1 \times m}) \rightarrow Y_{H_{\Pi_2}, \rho}^p(\Omega; \mathbb{R}^{\#\Pi_2 \times m})$ is a surjection, just showing that

$$(Y_{H_{\Pi_1}, \rho}^p(\Omega; \mathbb{R}^{\#\Pi_1 \times m}), e_{H_{\Pi_1}}, H_{\Pi_1}^*) \supseteq (Y_{H_{\Pi_2}, \rho}^p(\Omega; \mathbb{R}^{\#\Pi_2 \times m}), e_{H_{\Pi_2}}, H_{\Pi_2}^*), \quad (18)$$

i.e. the convex compactification $Y_{H_{\Pi_1}, \rho}^p(\Omega; \mathbb{R}^{\#(\Pi_1) \times m})$ of B_ρ is finer than the convex compactification $Y_{H_{\Pi_2}, \rho}^p(\Omega; \mathbb{R}^{\#(\Pi_2) \times m})$. By (16a), $\psi_{\Pi_2}^{\Pi_1}$ is continuous, and by (16b) it satisfies $\psi_{\Pi_3}^{\Pi_2} \circ \psi_{\Pi_2}^{\Pi_1} = \psi_{\Pi_3}^{\Pi_1}$, and eventually (16c) ensures $\psi_{\Pi}^{\Pi} = \text{identity}$. The collection $\eta = (\eta_{\Pi})_{\Pi \in \mathfrak{F}(I)}$ of $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ then satisfies the property that

$$\eta_{\Pi_2} = \psi_{\Pi_2}^{\Pi_1} \eta_{\Pi_1} \quad \text{whenever } \Pi_1 \supset \Pi_2. \quad (19)$$

Altogether, the operators $\psi_{\Pi_2}^{\Pi_1}$ play the role of the bonding mappings and thus

$$\mathfrak{S} = \left((Y_{H_{\Pi_1}, \rho}^p(\Omega; \mathbb{R}^{\#(\Pi_1) \times m}), e_{H_{\Pi_1}}, H_{\Pi_1}^*), \psi_{\Pi_2}^{\Pi_1} \right)_{\Pi_1, \Pi_2 \in \mathfrak{F}(I), \Pi_1 \supset \Pi_2} \quad (20)$$

is an inverse system of convex compactifications of B_ρ , and $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ is its limit, i.e.

$$\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)) = \varprojlim \mathfrak{S} \quad \text{with } \mathfrak{S} \text{ from (20).}$$

By Proposition 2, $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ itself is a convex compactification of B_ρ ; the embedding is $i = (e_{H_{\Pi}})_{\Pi \in \mathfrak{F}(I)}$ and the linear space inducing its convex structure is now $\prod_{\Pi \in \mathfrak{F}(I)} H_{\Pi}^*$. By (2), also

$$\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m)) \cong \sup_{\Pi \in \mathfrak{F}(I)} Y_{H_{\Pi}, \rho}^p(\Omega; \mathbb{R}^{\#(\Pi) \times m}). \quad (21)$$

The threads, i.e. the elements of $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$, cf. (1), are also called *compatible systems of Young functionals*. For a special choice of the system \mathcal{H} as in Example 3 below and $p = 1$, such systems have been invented in [10, Sect.7] (under the name ‘‘compatible systems of generalized Young measures’’) and further used in [11, 12, 16, 17]. A general ansatz based on Example 3 below has been scrutinized in [26].

Example 2 (Non-correlated threads) A rather standard but coarse convex compactification is obtained simply by copying the construction from Section 4 constantly at each time instant, obtaining thus the convex compact subset $Y_{H, \rho}^p(\Omega; \mathbb{R}^m)^I$ of the product $(H^*)^I$ into which B_ρ is embedded simply by $u \mapsto (i_H(u(t)))_{t \in I}$. In fact, such type of compactification has been used for relaxation of various evolutionary phase-change problems e.g. in [2, 4, 6, 7, 18, 19, 22, 23, 24, 29, 32, 39] and many others. Up to an equivalence introduced in Section 3, we can obtain this convex compactification in the above framework, too. To this goal, let us put

$$H_{\Pi} := \left\{ (x, s_1, \dots, s_{\#(\Pi)}) \mapsto \sum_{k=1}^{\#(\Pi)} h_k(x, s_k), h_k \in H \right\}. \quad (22)$$

Obviously, H_{Π} is a subspace of $\text{Car}^p(\Omega; \mathbb{R}^{\#(\Pi) \times m})$ if H is a subspace of $\text{Car}^p(\Omega; \mathbb{R}^m)$. Thus, due to the special choice (22), here $H_{\Pi}^* \cong (H^*)^{\Pi} \cong (H^*)^{\#(\Pi)}$ and we can define the linear injective mapping $\psi : (H^*)^I \rightarrow \prod_{\Pi \in \mathfrak{F}(I)} H_{\Pi}^*$ by $\eta \mapsto (\eta|_{\Pi})_{\Pi \in \mathfrak{F}(I)}$ which is also a homeomorphical embedding. Defining the system $\mathcal{H} := (H_{\Pi})_{\Pi \in \mathfrak{F}(I)}$ by taking H_{Π} from (22), we obtain the convex compactification $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ by (15). The bonding mappings are defined as $\psi_{\Pi_2}^{\Pi_1}$ with $P_{\Pi_2}^{\Pi_1} : H_{\Pi_2} \rightarrow H_{\Pi_1}$ given by

$$[P_{\Pi_2}^{\Pi_1} h](x, s_1, \dots, s_{\#(\Pi_1)}) := h(x, s_{j_1}, \dots, s_{j_{\#(\Pi_2)}}) \quad (23)$$

where here $j : \Pi_2 \rightarrow \Pi_1$ is just the inclusion $\Pi_2 \subset \Pi_1$. The mapping ψ restricted on $Y_{H, \rho}^p(\Omega; \mathbb{R}^m)^I$ then realizes the homeomorphism between $Y_{H, \rho}^p(\Omega; \mathbb{R}^m)^I$ and $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$, which makes these convex compactifications equivalent to each other.

Example 3 (Correlated threads based on DiPerna-Majda's measures) Based on the DiPerna-Majda measures, the construction of threads nontrivially correlated have essentially been invented in [26]. For $\Pi \in \mathfrak{F}(I)$, one can use $\text{DM}_\gamma(\Omega; \mathbb{R}^{\#\Pi \times m})$ with γ referring either to the one-point compactification or the compactification by a sphere as in from Example 1. The collection $\mathcal{H} = (H_\Pi)_{\Pi \in (I)}$ is then taken as

$$H_\Pi = C(\bar{\Omega}) \otimes V_{\gamma, \#\Pi} \quad \text{with} \quad (24a)$$

$$V_{\gamma, n} = \left\{ v \in C(\mathbb{R}^{n \times m}); \frac{v(\cdot)}{1+|\cdot|^p} \text{ has a continuous extension on } \gamma(\mathbb{R}^{n \times m}) \right\}. \quad (24b)$$

The bonding mappings are again determined as in Example 2 by means of (23). Here, for any $\Pi_2 \subset \Pi_1 \in \mathfrak{F}(I)$, it is important that

$$\forall g \in C(\bar{\Omega}) \quad \forall v \in V_{\gamma, \#\Pi_2} : \quad P_{\Pi_2}^{\Pi_1}(g \otimes v) \in C(\bar{\Omega}) \otimes V_{\gamma, \#\Pi_1}, \quad (25)$$

which indeed holds true for (23).

6 Threads in $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ with a bounded variation in time

Let us consider the distance $d(x, \cdot, \cdot) : (\mathbb{R}^m)^2 \rightarrow \mathbb{R}^+$ parameterized by $x \in \Omega$, i.e. the triangle inequality $d(x, s_1, s_3) \leq d(x, s_1, s_2) + d(x, s_2, s_3)$ is satisfied for a.a. $x \in \Omega$ and all $s_1, s_2, s_3 \in \mathbb{R}^m$. If

$$\forall \Pi \in \mathfrak{F}(I), \quad \#\Pi \geq 2 : \quad \left(d_\Pi : (x, s_1, \dots, s_{\#\Pi}) \mapsto \sum_{i=2}^{\#\Pi} d(x, s_{i-1}, s_i) \right) \in H_\Pi, \quad (26)$$

like in [10, Def.8.1 and 8.6] we can define the *dissipation of a thread* η over I with respect to the distance d by

$$\text{Diss}_d(\eta; I) := \sup_{\Pi \in \mathfrak{F}(I)} \langle \text{Pr}_\Pi \eta, d_\Pi \rangle \quad (27)$$

where d_Π is from (26). One can realize that $\langle \eta_\Pi, d_\Pi \rangle = \sum_{i=2}^{\#\Pi} \langle \eta_{\{t_{i-1}, t_i\}}, d \rangle$ if d_Π is from (26). Thus, equivalently, like in [10, Rem.8.3], one can write (27) as

$$\text{Diss}_d(\eta; I) = \sup_{\Pi \in \mathfrak{F}(I)} \sum_{i=2}^{\#\Pi} \langle \text{Pr}_{\{t_{i-1}, t_i\}} \eta, d \rangle; \quad (28)$$

note that always $d \in H_{\{t_{i-1}, t_i\}}$ due to (26). In fact, ‘‘sup’’ in (27) and (28) can be replaced by ‘‘lim’’ with the same effect.

Example 4 (Variation of threads) Taking the special distance $d(x, s_1, s_2) = |s_1 - s_2|$, it is natural to define the *variation* of η as $\text{Var}(\eta; I) := \text{Diss}_d(\eta; I)$. Then, for any $u \in \text{B}(I; L^p(\Omega; \mathbb{R}^m))$, one just have

$$\text{Var}(i(u); I) = \left\| \frac{du}{dt} \right\|_{M(I; L^1(\Omega; \mathbb{R}^m))}, \quad (29)$$

so that $\text{Var}(i(u); I)$ is the standard variation of u in the sense of $\text{BV}(I; L^1(\Omega; \mathbb{R}^m))$; in (29), $M(I; X)$ stands for the space of X -valued measures on I , $\frac{du}{dt}$ denotes the distributional derivative of u , and i is the embedding defined in Section 5.

Example 5 () For a special case $d(x, s_1, s_2) = g(x)\delta(s_1, s_2)$ with $g \in C(\bar{\Omega})$ and $\delta \in V_{\gamma,2}$ from (24b), the property (26) holds for H_{Π} from (24) because $(s_1, \dots, s_{\#(\Pi)}) \mapsto \sum_{i=2}^{\#(\Pi)} \delta(s_{i-1}, s_i) \in V_{\gamma, \#(\Pi)}$ whenever $\#(\Pi) \geq 2$. In particular, it holds for $d(x, s_1, s_2) = |s_1 - s_2|$ from Example 4.

The lower-semicontinuity and convexity of $\text{Diss}_d(\cdot; I)$ is often useful:

Proposition 4 For any sequence of threads $\{\eta_k\}_{k \in \mathbb{N}}$ converging to some η in $\mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ and any distance d , it holds

$$\liminf_{k \rightarrow \infty} \text{Diss}_d(\eta_k; I) \geq \text{Diss}_d(\eta; I). \quad (30)$$

Moreover, the functional $\text{Diss}_d(\cdot; I)$ is convex.

Proof. For any $\Pi \in \mathfrak{F}(I)$ we have $\langle \text{Pr}_{\Pi} \eta_k, d_{\Pi} \rangle \leq \text{Diss}_d(\eta_k; I)$ due to (27). By continuity of Pr_{Π} , we have $\text{Pr}_{\Pi} \eta_k \rightarrow \text{Pr}_{\Pi} \eta$ weakly* in H_{Π}^* and we therefore have $\lim_{k \rightarrow \infty} \langle \text{Pr}_{\Pi} \eta_k, d_{\Pi} \rangle = \langle \text{Pr}_{\Pi} \eta, d_{\Pi} \rangle$. Thus $\langle \text{Pr}_{\Pi} \eta, d_{\Pi} \rangle \leq \liminf_{k \rightarrow \infty} \text{Diss}_d(\eta_k; I)$. Taking the supremum over $\Pi \in \mathfrak{F}(I)$, we obtain (30).

As for the convexity, it suffices to realize that, by (27), $\text{Diss}_d(\cdot; I)$ is the supremum of a collection of affine functionals. \square

We further consider a special case “homogeneous” in t - and (approximately) uniformly continuous in s -variables:

$$H_{\Pi} \text{ depends only on } \#(\Pi) \text{ but not on } \Pi \text{ itself,} \quad (31a)$$

$$\forall \Pi \in \mathfrak{F}([0, T]) \exists \tilde{H}_{\Pi} \subset H_{\Pi} \text{ densely } \forall h \in \tilde{H}_{\Pi} \exists K \in \mathbb{R} :$$

$$\left| h(x, s_1, \dots, s_{\#(\Pi)}) - h(x, \tilde{s}_1, \dots, \tilde{s}_{\#(\Pi)}) \right| \leq K \sum_{i=1}^{\#(\Pi)} d(x, s_i, \tilde{s}_i), \quad (31b)$$

where the density refers to the topology of $\text{Car}^p(\Omega; \mathbb{R}^{\#(\Pi) \times m})$. Let us note that Examples 2–3 satisfy (31a). As for (31b), a special investigations and assumptions are needed in particular cases:

Example 6 Taking d from Example 4, (31b) represents approximability by Lipschitz-continuous functions. For $p = 1$ and some special compactifications $\gamma(\mathbb{R}^m)$ or $\gamma(\mathbb{R}^{n \times m})$ in (12b) and (24b), H_{γ} from (12) and similarly $H_{\Pi} = C(\bar{\Omega}) \times V_{\gamma, n(\Pi)}$ from (24) has this Lipschitz-approximability property and thus Examples 2 and 3 satisfy (31b). Certainly, it holds for the one-point compactification $\gamma(\mathbb{R}^{n \times m}) = \alpha(\mathbb{R}^{n \times m})$ and also for $\gamma(\mathbb{R}^{n \times m})$ being a compactification of $\mathbb{R}^{n \times m}$ by the sphere $S^{n \times m - 1}$. More specifically, one can use the argument that, for any $h = g \otimes v \in H_{\Pi} = C(\bar{\Omega}) \times V_{\gamma, n(\Pi)}$ from (24b), the function $v \in V_{\gamma, n}$ admits a decomposition $v = v_1 + v_{\infty}$ with the positively homogeneous (so-called recession) function $v_{\infty}(s) := \lim_{\xi \rightarrow +\infty} v(\xi s)/\xi$ which is essentially defined by the values on the unit sphere $S^{n \times m - 1}$ of $\mathbb{R}^{n \times m}$, and then one makes a Lipschitz approximation both of $v_{\infty}|_{S^{n \times m}}$ and of $v_0 := v_1(\cdot)/(1 + |\cdot|) \in C_0(\mathbb{R}^{n \times m})$, let us denote the resulted functions by v_{∞}^L and v_0^L , respectively, and eventually compose $h^L = g \otimes v_0^L(\cdot)(1 + |\cdot|) + v_{\infty}^L(\cdot)$. Thus one can approximate $h = g \otimes v$ in the norm $\|h\| := \sup_{(x, s) \in \Omega \times \mathbb{R}^{n \times m}} |h(x, s)/(1 + |s|)|$ which is even stronger than the natural norm of $\text{Car}^1(\Omega; \mathbb{R}^{n \times m})$ from (7).

Let us emphasize that, on the other hand, for $p > 1$, the approximability (31b) cannot hold because any Lipschitz-continuous function has at most linear growth and cannot satisfy (31b) if H_{Π} contains functions with p -growth.

Proposition 5 (Helly principle I) *Let I be an interval, say $[0, T]$, let all H_Π be separable and satisfy (26) and (31), the distance d be coercive in the sense that $d(x, s_1, s_2) = 0$ implies $s_1 = s_2$, and let $(\eta_k)_{k \in \mathbb{N}} \subset \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ be a sequence with a bounded dissipation (independently of k). Then there is a subsequence and a thread $\eta \in \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ such that $\text{Pr}_\Pi \eta_k \rightarrow \text{Pr}_\Pi \eta$ weakly* for any $\Pi \in \mathfrak{F}([0, T])$. In particular, this η is attainable by a sequence from B_ϱ .*

Proof. We divide it into four steps.

Step 1. Define $\delta_k : [0, T] \rightarrow \mathbb{R}^+$ by $\delta_k(t) = \text{Diss}_d(\eta_k; [0, t])$. These functions are obviously nondecreasing and, by assumption, bounded by $\sup_{k \in \mathbb{N}} \text{Diss}_d(\eta_k; [0, T]) < \infty$. By the classical Helly theorem [21], there is a subsequence (indexed for notational simplicity again by k) and a nondecreasing bounded function $\delta_\infty : [0, T] \rightarrow \mathbb{R}^+$ such that $\delta_k(t) \rightarrow \delta_\infty(t)$ for all $t \in [0, T]$. Since all δ_k and δ_∞ are nondecreasing, they are continuous with the exception at most countable number of t 's; let us denote this set of their discontinuities by D .

Step 2. Further, we can choose a dense countable subset $C \subset [0, T]$ containing $D \cup \{0\}$ and (if needed) another subsequence such that, for any $\Pi \in \mathfrak{F}(C)$, there is some $\eta_\Pi \in Y_{H_\Pi}^p(\Omega; \mathbb{R}^m)$ such that $\text{Pr}_\Pi \eta_k \rightarrow \eta_\Pi$ weakly* in H_Π^* ; here we used the assumption about separability of each particular H_Π , the countability of $\mathfrak{F}(C)$, and Cantor's diagonalization procedure.

Step 3. Let us now take a general $\Pi = (t_1, \dots, t_{\#\Pi}) \in \mathfrak{F}([0, T])$. Again, we can select a further subsequence (depending possibly on Π) such that $\text{Pr}_\Pi \eta_k$ converges weakly* in H_Π^* to some η_Π . However, we will show that this η_Π is, in fact, determined already by the collection $(\eta_\Pi)_{\Pi \in \mathfrak{F}(C)}$ from Step 2 and therefore no further selection of subsequence is needed, which is important because $\mathfrak{F}([0, T])$ is not countable.

For any $n \in \mathbb{N}$, let us consider $\Pi_n = (t_1^n, \dots, t_{\#\Pi}^n) \in \mathfrak{F}(C)$, $\#\Pi_n = \#\Pi$, so that we have η_{Π_n} from Step 2 at our disposal. Moreover, we assume

$$t_1^n \leq t_1 < t_2^n \leq t_2 < \dots < t_{\#\Pi}^n \leq t_{\#\Pi} \quad \text{and} \quad (32a)$$

$$\forall i = 1, \dots, \#\Pi : \quad t_i^n \begin{cases} = t_i & \text{if } t_i \in C \cup \{0\}, \\ < t_i & \text{otherwise.} \end{cases} \quad (32b)$$

Since C is dense in $[0, T]$, we can assume $\lim_{n \rightarrow \infty} t_i^n = t_i$ for any $i = 1, \dots, \#\Pi$. Note that always $\lim_{n \rightarrow \infty} \delta_k(t_i^n) = \delta_k(t_i)$ because either t_i is a continuity point of δ_k due to (32b) if $t_i \in (0, T] \setminus C \subset [0, T] \setminus D$ or the sequence $\{t_i^n\}_{n \in \mathbb{N}}$ is taken simply constant.

Let us denote $\eta_\Pi^k = \text{Pr}_\Pi \eta_k$. For any $h \in \tilde{H}_\Pi$, as η_k has the thread property (19), it holds

$$\left| \langle \eta_{\Pi_n}^k - \eta_\Pi^k, h \rangle \right| \leq K \sum_{\substack{t_i \notin C \\ i=1, \dots, \#\Pi}} \langle \text{Pr}_{\{t_i^n, t_i\}} \eta_k, d \rangle \leq K \sum_{i=1, \dots, \#\Pi} \delta_k(t_i) - \delta_k(t_i^n), \quad (33)$$

where the first inequality in (33) is just by a continuous extension of the analogous conventional estimate obtained for a net $\{u_\alpha\}_{\alpha \in A} \subset B_\varrho \subset \text{B}([0, T]; L^p(\Omega; \mathbb{R}^m))$ attaining the thread η_k , namely

$$\begin{aligned} & \left| \int_\Omega h(x, u_\alpha(t_1^n), \dots, u_\alpha(t_{\#\Pi}^n)) - h(x, u_\alpha(t_1), \dots, u_\alpha(t_{\#\Pi})) \, dx \right| \\ & \leq K \sum_{\substack{t_i \notin C \\ i=1, \dots, \#\Pi}} \left| \int_\Omega d(x, u_\alpha(t_i^n), u_\alpha(t_i)) \, dx \right|. \end{aligned} \quad (34)$$

Having (33) at our disposal, we can pass $k \rightarrow \infty$, obtaining

$$|\langle \eta_{\Pi_n} - \eta_{\Pi}, h \rangle| \leq K \sum_{i=1, \dots, \#(\Pi)} \delta_{\infty}(t_i) - \delta_{\infty}(t_i^n) \rightarrow 0. \quad (35)$$

The convergence to 0 claimed in (35) was already discussed. As h ranges over \tilde{H}_{Π} which is supposed dense in H_{Π} by (31b), we proved that $\eta_{\Pi_n} \rightarrow \eta_{\Pi}$ weakly* in H_{Π}^* , and thus η_{Π} is indeed determined by the choice in Steps 1 and 2.

Step 4. It remains to show that $(\eta_{\Pi})_{\Pi \in \mathfrak{F}([0, T])}$ is a thread, i.e. (19) holds. Yet, it just suffices to pass to the limit in $\psi_{\Pi_2}^{\Pi_1} \eta_{\Pi_1}^k = \eta_{\Pi_2}^k$ which holds because each $\eta_k = (\eta_{\Pi}^k)_{\Pi \in \mathfrak{F}([0, T])}$ itself is a thread. \square

We already mentioned that the case $p > 1$ is not compatible with (31b). Thus a weakened variant of both (31b) and Proposition 5 have naturally to be used, based on the notion of p -nonconcentrating threads. We call a thread $\eta \in \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ p -nonconcentrating if there is a net $\{u_{\alpha}\}_{\alpha \in A} \subset B_{\varrho} \subset B([0, T]; L^p(\Omega; \mathbb{R}^m))$ attaining the thread η such that $\{|u_{\alpha}(t, \cdot)|^p; \alpha \in A, t \in [0, T]\}$ is relatively weakly compact in $L^1(\Omega)$. Analogously, a sequence of threads $\{\eta_k\}_{k \in \mathbb{N}} \subset \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ is called equi- p -nonconcentrating if there are nets $\{u_{\alpha}^k\}_{\alpha \in A^k} \subset B_{\varrho} \subset B([0, T]; L^p(\Omega; \mathbb{R}^m))$ attaining the particular η_k such that $\{|u_{\alpha}^k(t, \cdot)|^p; \alpha \in A^k, t \in [0, T], k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$.

Proposition 6 (Helly principle II) *Let the assumptions of Proposition 5 hold except (31b) and, instead,*

$$\begin{aligned} \forall \Pi \in \mathfrak{F}([0, T]) \quad \forall h_0 \in H_{\Pi} \quad \text{such that} \\ |h_0(x, s_1, \dots, s_{\#(\Pi)})| \leq a(x) + o(|s_1|^p + \dots + |s_{\#(\Pi)}|^p), \quad a \in L^1(\Omega) \\ \forall \epsilon > 0 \quad \exists h \in H_{\Pi} \quad \exists K \in \mathbb{R} : \quad \|h_0 - h\|_{H_{\Pi}} \leq \epsilon \quad \text{and,} \\ |h(x, s_1, \dots, s_{\#(\Pi)}) - h(x, \tilde{s}_1, \dots, \tilde{s}_{\#(\Pi)})| \leq K \sum_{i=1}^{\#(\Pi)} d(x, s_i, \tilde{s}_i), \end{aligned} \quad (36a)$$

and let the bounded-dissipation sequence $(\eta_k)_{k \in \mathbb{N}} \subset \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ be equi- p -nonconcentrating. Then there is a subsequence and a thread $\eta \in \mathcal{Y}_{\mathcal{H}, \rho}^p(I; L^p(\Omega; \mathbb{R}^m))$ such that $\text{Pr}_{\Pi} \eta_k \rightarrow \text{Pr}_{\Pi} \eta$ weakly* for any $\Pi \in \mathfrak{F}([0, T])$. Moreover, η is p -nonconcentrating.

Proof. First, let us realize that the attaining nets $\{u_{\alpha}^k\}_{\alpha \in A^k}$ may use the same directed index set $A = A^k$ without loss of generality, e.g. A as in the proof of Proposition 2. Then, realizing that $A \times \mathbb{N}$ is directed, $\{u_{\alpha}^k\}_{(\alpha, k) \in A \times \mathbb{N}}$ is an attaining net for η which is thus shown to be p -nonconcentrating.

Then just a slight modification in Step 3 of the proof of Proposition 5 is needed: The desired convergence $\eta_{\Pi_n} \rightarrow \eta_{\Pi}$ weakly* in H_{Π}^* now follows again by density argument and by the argument that the test by integrands h_0 of the growth less than p from (36a) suffices due to the p -nonconcentration of η . \square

The importance of the above modification of Proposition 5 is that, taking d from Example 4, (36a) can be satisfied for H from (12) and for H_{Π} from (24) even if $p > 1$, contrary to the stronger condition (31b). On the other hand, the p -nonconcentrating threads can be expected in problems exhibiting coercivity of polynomial of p -degree with $p > 1$.

The following historical notes are worth: Generalization of the classical Helly theorem [21] to vector-valued bounded-variation mappings is by [3], to metric-space-valued bounded-variation mappings is by [8, 9] (assuming, in addition, continuity over $[0, T]$) and

also [5]. A generalization to topological-space-valued mappings by using the general dissipation distance is by [28, Sec.3]. Variation of threads had been introduced in [10, Sect.8] in a special case that the distance $d(x, s, \tilde{s}) = |s - \tilde{s}|$. The assertion similar to Proposition 4 (except convexity) was proved in [10, Thm.8.11]. For a special case of Proposition 5 see [10, Thm.8.10] where, in addition, the left-continuity of the limit was required and thus the convergence was obtained only up to a countable set of time instances. Similar result as Proposition 5 for a more general case like in Example 3 but only for $\#(\Pi) = 1$ and with a different definition of the variation is in [26, Thm.2.7]. In the context of Example 2, using a special ordering of Young measures, Helly-type principle was also proved in [6, Thm.3.20].

7 Concepts of relaxation of evolution problems outlined

Correlated systems (threads) of generalized Young measures can be used to a certain relaxation of evolution problems governed by stored- and dissipation-energy functionals lacking lower-semicontinuity property in a compact topology that naturally arises from some coercivity structure, as typically occurs in systems exhibiting phase changes accompanied by occurrence of microstructure or/and various spatial-concentration phenomena. For relaxation of quasistatic evolution problems with nontrivial correlations in time, it has been used in [11, 12, 16, 17].

Let us illustrate it on a rather academical example involving the stored-energy functional $E(t, z) = \int_{\Omega} \varphi(t, x, z(x)) dx$ and the dissipation functional $D(z, \tilde{z}) = \int_{\Omega} r(x, z(x) - \tilde{z}(x)) dx$ with $r(x, \cdot)$ positively 1-homogeneous and convex. We consider a quasistatic evolution that would be formally governed by the doubly-nonlinear inclusion

$$\partial_z r\left(x, \frac{\partial z}{\partial t}\right) + \varphi'_z(t, x, z) \ni 0. \quad (37)$$

Modern theory of such inclusions exploits 0-homogeneity of the subdifferential $\partial_z r(x, \cdot)$ and a concept of energetic solutions devised by Mielke & collaborators [30, 33, 34]. Here, in the spirit of [12, Definition 4.6] or [16, Definition 6.14] it might lead to the definition of $\eta \in \mathcal{Y}_{\mathcal{H}, \rho}^p([0, T]; L^p(\Omega))$ as a (generalized relaxed) energetic solution if it satisfies *stability*:

$$\forall t \in [0, T], z \in L^p(\Omega) : \quad \langle \eta_{\{t\}}, \varphi(t, \cdot, \cdot) \rangle \leq \langle \eta_{\{t\}}, \varphi(t, \cdot, \cdot + z(\cdot)) \rangle + \int_{\Omega} r(z) dx \quad (38a)$$

with $\varphi(t, \cdot, \cdot + z(\cdot)) : (x, s) \mapsto \varphi(t, x, s + z(x))$, and *energy inequality*

$$\langle \eta_{\{t\}}, \varphi(t, \cdot, \cdot) \rangle + \text{Diss}_d(\eta; [0, t]) \leq \langle \eta_{\{0\}}, \varphi(0, \cdot, \cdot) \rangle + \int_0^t \langle \eta_{\{\tau\}}, \varphi'_t(\tau, \cdot, \cdot) \rangle d\tau \quad (38b)$$

with $\text{Diss}_d(\eta; [0, t])$ from (27) with $d(x, s_1, s_2) = r(x, s_2 - s_1)$. In fact, [12, Definition 4.6] devised (38b) as an equality. Of course, data qualification and a suitable choice of the collection $\mathcal{H} = (H_{\Pi})_{\mathfrak{F}([0, T])}$ should ensure $\varphi(t, \cdot, \cdot + z(\cdot)), \varphi'_t(t, \cdot, \cdot) \in H_{\{t\}}$ for any $z \in L^p(\Omega)$ and also (26). Interestingly, the set of solutions to (38) (possibly in a variant of (38b) as equality) is convex; here convexity of (15) and also Proposition 4 come into use. Also, it is important that this definition is consistent in the sense that, if $\eta = i(u)$ for some $u \in \mathcal{B}([0, T]; L^p(\Omega))$, then u is an energetic solution in the standard sense [30, 33, 34].

Equivalently, (38a) might also be written as

$$\forall t \in [0, T], z \in L^p(\Omega) : \quad \langle \eta_{\{t\}}, \varphi(t, \cdot, \cdot) \rangle \leq \langle [\eta + z]_{\{t\}}, \varphi(t, \cdot, \cdot) \rangle + \int_{\Omega} r(z) dx \quad (39)$$

with the “shifted” thread $\eta+z \in \mathcal{Y}_{\mathcal{H},\rho+\|z\|_{L^p}}^p([0,T];L^p(\Omega))$ being defined as induced by a net $\{u_\alpha(t,x)+z(x)\}_{\alpha \in A}$ for $\{u_\alpha\}_{\alpha \in A} \subset B([0,T];L^p(\Omega))$ a net attaining η .

Some other definitions might be considered, too. E.g. [12, Lemma 4.13] might suggest to consider

$$\forall 0 < t_1 \leq t_2 \leq T : \quad \langle \eta_{\{t_1\}}, \varphi(t_1, \cdot, \cdot) \rangle \leq \langle \eta_{\{t_2\}}, \varphi(t_2, \cdot, \cdot) \rangle + \langle \eta_{\{t_1, t_2\}}, d \rangle. \quad (40)$$

Again, convexity of the set of solutions would be preserved but consistency would be lost.

The convexity structure allows for various finer investigations as optimality conditions leading to maximum principles and benefits in possible numerical implementation. Existence of such kind of solutions is relatively easy to prove by some approximation (e.g. implicit time discretisation [12, 16] or various regularizations like [11, 17]) and the limit passage by using Propositions 4 and 5. Here, e.g., one can think about a regularization $E_\epsilon(t, z) = \int_\Omega \varphi(t, x, z(x)) + \epsilon |\nabla z(x)|^2 dx$ which would allow us to obtain energetic solution $z_\epsilon \in \text{BV}([0, T]; L^1(\Omega)) \cap L^\infty(0, T; W^{1,2}(\Omega))$ in the conventional sense of [30, 33, 34]. This, however, indicates that such definitions are only little selective in the sense that, roughly speaking, they might admit a lot of solutions lacking a desired sense, cf. [38] for more detailed discussion of this phenomenon. Therefore, more sophisticated concepts are likely to be devised in particular cases.

In applications, the simple scenario (37), used above just for an illustration, must be made more complicated, however. In mechanics, for example, relying on the concept of the generalized standard materials [20], one would rather require a deformation gradient ∇y and possibly a vector of internal parameters λ in the position of mere z . Thus, because of that spatial gradient, Ω is to be further qualified as a bounded domain in, say \mathbb{R}^m , and one would rather consider the stored-energy functional $E(t, y, \lambda) = \int_\Omega \varphi(t, x, y(x), \nabla y(x), \lambda(x)) dx$ and the dissipation functional $D(\lambda, \tilde{\lambda}) = \int_\Omega r(x, \lambda(x) - \tilde{\lambda}(x)) dx$. Then a quasistatic evolution is governed, instead of (37), by the system of the doubly-nonlinear equation/inclusion

$$\varphi'_y(t, x, y, \nabla y, \lambda) - \text{div}(\varphi'_{\nabla y}(t, x, y, \nabla y, \lambda)) = 0, \quad (41a)$$

$$\partial_\lambda r\left(x, \frac{\partial \lambda}{\partial t}\right) - \text{div}(\varphi'_\lambda(t, x, y, \nabla y, \lambda)) \ni 0. \quad (41b)$$

Problems of such type allow us to describe quasistatic evolution in various isothermal inelastic processes that may simultaneously develop microstructure, as applicable typically in phase-change problems, and has been considered in [16, 17, 22, 24, 25, 29, 31, 32]. The above regularization might be considered as $E_{\epsilon_1, \epsilon_2}(t, z) = \int_\Omega \varphi(t, x, y, \nabla y, \lambda) + \epsilon_1 |\nabla^2 y|^2 + \epsilon_2 |\nabla \lambda|^2 dx$ and would allow us to obtain energetic solution $(y_{\epsilon_1, \epsilon_2}, \lambda_{\epsilon_1, \epsilon_2}) \in L^\infty(0, T; W^{2,2}(\Omega; \mathbb{R}^m)) \times L^\infty(0, T; W^{1,2}(\Omega; \mathbb{R}^n))$ with $\lambda_{\epsilon_1, \epsilon_2} \in \text{BV}([0, T]; L^1(\Omega; \mathbb{R}^n))$ in the above mentioned conventional sense; here one assumes y being \mathbb{R}^m -valued and λ being \mathbb{R}^n -valued. If one assumes, for simplicity, $\varphi : (t, y, \cdot, \cdot) : \mathbb{R}^{m \times m} \times \mathbb{R}^n \rightarrow \mathbb{R}$ having the same p -growth in all variables (as in [16, 17] for $p = 2$), the thread η representing an energetic solution in the spirit of (38) is to be considered as $\eta \in \mathcal{Y}_{\mathcal{H},\rho}^p([0, T]; L^p(\Omega; \mathbb{R}^{m \times m + n}))$ attainable by gradients, i.e. η should be the weak* limit (in the topology used in (15)) of a bounded sequence $\{(\nabla y_k, \lambda_k)\}_{k \in \mathbb{N}}$ when imbeded into $\prod_{\Pi \in \mathfrak{F}(I)} H_\Pi^*$ by $i := (e_{H_\Pi})_{\Pi \in \mathfrak{F}(I)}$ as used in (15). However, like it is well known for the conventional gradient Young measures, cf. [36], such a subset of “gradient threads” does not form a convex set in $\mathcal{Y}_{\mathcal{H},\rho}^p([0, T]; L^p(\Omega; \mathbb{R}^{m \times m + n}))$ except some trivial choices of \mathcal{H} or $m = 1$. In fact, in [22, 32], also a holonomic constraint linking ∇y with λ was considered and thus (41) would be even more complicated as φ is not smooth. Moreover, [22, 24, 25, 32, 31] used a “half” regularization and kept a gradient theory for λ , i.e. $\epsilon_1 = 0$ and $\epsilon_2 > 0$, which causes “compactness” in terms of λ 's and makes possible to use more conventional non-correlated relaxation as in Example 2 in terms of ∇y only. The theory of correlated threads presented above does not need such

compactness in terms of λ 's and allow us to relax the problem in terms of $(\nabla y, \lambda)$'s and to avoid usage of a (bit artificial) gradient-concept for λ . On the other hand, it should be emphasized that the concept of such relaxed energetic solutions is likely not to be much selective and further refinements would be desired.

In [11, 12], the correlated relaxation has been applied to a special case of linearized plasticity at small strains where $\lambda = (\pi, \zeta)$ consists of a plastic-strain tensor and a hardening/softening-like internal parameter. Then $\varphi(t, x, y, A, \lambda) = \frac{1}{2}\mathbb{C}(e(A) - \pi) : (e(A) - \pi) + f(\zeta)$ with $e(A) = \frac{1}{2}(A^\top + A)$ and \mathbb{C} a 4th-order tensor of elastic moduli.

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