The incompressible limit of the full Navier-Stokes-Fourier system on domains with rough boundaries

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1 Introduction

The incompressible limit approximation frequently adopted for instance in meteorological models anticipates that sizeable elastic perturbations cannot establish permanently in the atmosphere as the fast elastic waves rapidly redistribute the associated energy giving rise to en equilibrium distribution void of the acoustic modes (see Klein [21], [22]). Such a scenario requires hypothetical existence of an unbounded physical space with a dominating dispersion effect. However, any real physical as well as computational domain is necessarily bounded and the interaction of the acoustic waves with the boundary represents an inevitable serious problem.

A proper choice of the boundary conditions for the velocity $\mathbf{u}$ of a viscous fluid confined to a bounded domain $\Omega \subset R^3$ has been discussed by many prominent physicists and mathematicians over the last two centuries (see the survey paper by Priezjev and Troian [27]). As a result, three different possible scenarios emerged: **(a)** *the no-slip boundary condition* - the velocity of the fluid equals that of the adjacent solid wall, specifically, if the boundary is at rest,

$$\mathbf{u}|_{\partial\Omega} = 0; \tag{1.1}$$

(b) the fluid slips against a thin film immobilized by wall imperfections; **(c)** *Navier's boundary condition* - the fluid slips against the solid surface with a speed proportional to the tangential component of the normal viscous stress:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}]_{\tan} + \beta[\mathbf{u}]_{\tan}|_{\partial\Omega} = 0, \tag{1.2}$$

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where the symbol $\mathbb{S}$ stands for the viscous stress tensor, and $\mathbf{n}$ denotes the outer normal vector.

For a long time, the no-slip boundary conditions have been the most widely accepted for their tremendous success in reproducing the observed velocity profiles for macroscopic flows. Note that even the situation (b) described above is based on the same phenomenological principle. Indeed as observed in numerous numerical analyses, it is convenient to *approximate* the complicated topography of the real physical boundary by a smooth one endowed with a suitable *wall law* similar to (1.2) (see Jaeger and Mikelic [20], Mohammadi et al. [26], among others).

Still the no-slip boundary condition is not intuitively obvious. Recently developed technologies of micro and nano-fluidics have shown the slip of the fluid on the boundary to be relevant when the system size approaches the nanoscale. The same argument applies in the case when the shear rate is sufficiently strong in comparison with the characteristic length scale as in some meteorological models (see Priezjev and Troian [27]). As a matter of fact, an alternative microscopic explanation of the no-slip condition argues that because most real surfaces are *rough*, the viscous dissipation as the fluid passes the surface irregularities brings it to rest regardless the character of the intermolecular forces acting between the fluid and the solid wall. A rigorous mathematical evidence of this hypothesis has been provided in a series of papers by Amirat et al. [3], [4], Casado-Díaz et al. [9], or, more recently, by [7], [8]. Thus the roughness argument reconciles convincingly the ubiquitous success of the no-slip condition with the boundary behaviour of real fluids predicted by molecular dynamics (cf. Qian an Wang [28]).

The present paper develops further the idea of (partial) slip in the context of singular limits (see Klein et al. [23]). The results can be viewed as a synthesis of two rather independent research programmes, the former devoted to the aymptotic limits on domains with rough boundaries originated in [7], [8], the latter dealing with the low Mach number limits for the complete Navier-Stokes-Fourier system developed in [15], [16].

In order to fix ideas, we consider a very simple geometry of the underlying spatial domain, namely

$$\Omega_\varepsilon = \{x = (x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ \Phi_\varepsilon(x_1, x_2) < x_3 < 1\}, \tag{1.3}$$

where $\mathcal{T}^2 = \{[0,1]|_{\{0,1\}}\}^2$ is a two-dimensional torus, and $\{\Phi_\varepsilon\}_{\varepsilon>0}$ a family of scalar functions. In particular, all physical quantities are supposed to be spatially periodic with respect to the "horizontal" coordinates (x_1, x_2).

The motion of the fluid is governed by the full Navier-Stokes-Fourier system describing the time evolution of the *density* $\varrho = \varrho(t,x)$, the *velocity* field $\mathbf{u} = \mathbf{u}(t,x)$, and the absolute *temperature* $\vartheta = \vartheta(t,x)$ of a general compressible, viscous, and heat conducting fluid. In addition, we assume that the speed of sound dominates the characteristic speed of the flow, in other words, the Mach number is small. Moreover, the fluid is stratified in the vertical direction because of the gravity, meaning the Froude number is small (see Klein et al. [23]). The dimensionless form of the corresponding field equations reads (see Gallavotti [19]):

$$\partial_t \varrho + \mathrm{div}_x(\varrho, \mathbf{u}) = 0, \tag{1.4}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S} + \frac{1}{\varepsilon}\varrho\nabla_x F, \tag{1.5}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma_\varepsilon, \tag{1.6}$$

supplemented with conservative boundary conditions

$$\left\{ \begin{array}{c} \mathbf{u}\cdot\mathbf{n}|_{\{x_3=\Phi_\varepsilon(x_1,x_2)\}}=0, \quad (\mathbb{S}\mathbf{n})\times\mathbf{n}|_{\{x_3=\Phi_\varepsilon(x_1,x_2)\}}=0, \\ \\ \mathbf{u}|_{\{x_3=1\}}=0, \end{array} \right\} \tag{1.7}$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0, \tag{1.8}$$

and the total energy balance

$$\frac{\rm d}{{\rm d}t}\int_{\Omega_\varepsilon}\Big(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)-\varepsilon\varrho F\Big)\ {\rm d}x=0, \tag{1.9}$$

where the *pressure* $p=p(\varrho,\vartheta)$, the specific *entropy* $s=s(\varrho,\vartheta)$, and the specific *internal energy* $e=e(\varrho,\vartheta)$ are interrelated through Gibbs' equation

$$\vartheta Ds(\varrho,\vartheta)=De(\varrho,\vartheta)+p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big). \tag{1.10}$$

The equations (1.4 - 1.6) are satisfied in a weak sense specified below. Moreover, in the framework of the weak solutions, the *entropy production rate* σ_ε is assumed to be a non-negative measure satisfying

$$\sigma_\varepsilon\geq\frac{1}{\vartheta}\Big(\varepsilon^2\mathbb{S}:\nabla_x\mathbf{u}-\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big) \tag{1.11}$$

(cf. [13]).

Furthermore, we suppose the fluid is linearly viscous, that means, the viscous stress tensor obeys Newton's rheological law

$$\mathbb{S}=\mu(\vartheta)\Big(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}{\rm div}_x\mathbf{u}\mathbb{I}\Big)+\eta(\vartheta){\rm div}_x\mathbf{u}\mathbb{I}, \tag{1.12}$$

while the heat flux $\mathbf{q}$ is determined through Fourier's law

$$\mathbf{q}=-\kappa(\vartheta)\nabla_x\vartheta, \tag{1.13}$$

where the transport coefficients μ, η, and κ depend on the absolute temperature.

Finally, system (1.4-1.6) is supplemented with the initial conditions

$$\varrho(0,\cdot)=\varrho_{0,\varepsilon}=\overline{\varrho}+\varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot)=\mathbf{u}_{0,\varepsilon}.\ \vartheta(0,\cdot)=\vartheta_{0,\varepsilon}=\overline{\vartheta}+\varepsilon\vartheta^{(1)}_{0,\varepsilon}, \tag{1.14}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants.

Sending $\varepsilon\to 0$ in the momentum equation (1.5) penalizes volume changes and drives the motion towards incompressibility (the low Mach number regime). In addition, small variations of the density and temperature imposed through the initial conditions (1.14) rend the pressure almost constant, ϱ being in this case a function of the temperature only (the low Froude number regime). In such a case, a formal asymptotic expansion with respect to the parameter ε gives rise to a simple and frequently used model system termed the Oberbeck-Boussinesq approximation that can be written in the form

$${\rm div}_x\mathbf{U}=0, \tag{1.15}$$

$$\overline{\varrho}\Big(\partial_t\mathbf{U}+{\rm div}_x(\mathbf{U}\otimes\mathbf{U}\Big)+\nabla_x\Pi={\rm div}_x\Big(\mu(\overline{\vartheta})\Big(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{U}\Big)\Big)+r\nabla_xF, \tag{1.16}$$

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\Big(\partial_t\Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa(\overline{\vartheta})\nabla_x\Theta) = 0, \tag{1.17}$$

$$r + \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta = 0, \tag{1.18}$$

where

$$G = \overline{\varrho}\ \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})F,$$

and c_p denotes the specific heat at constant pressure evaluated by means of the standard thermodynamics relation

$$c_p(\overline{\varrho},\overline{\vartheta}) = \frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta} + \alpha(\overline{\varrho},\overline{\vartheta})\frac{\vartheta}{\varrho}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}, \tag{1.19}$$

with the coefficient of thermal expansion

$$\alpha(\overline{\varrho},\overline{\vartheta}) = \frac{1}{\varrho}\frac{\partial_\vartheta p}{\partial_\varrho p}(\overline{\varrho},\overline{\vartheta}) \tag{1.20}$$

(see the survey paper by Zeytounian [30]).

More precisely, denoting $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ a family of solutions to the *primitive* system (1.4 - 1.14) we expect to recover the velocity field $\mathbf{U}$ as a weak limit of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, while

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)},\ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} = \Theta \text{ weakly in } L^1$$

where $\varrho^{(1)}$, Θ are related to $\nabla_x F$ through formula

$$\nabla_x\Big(\frac{p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Theta\Big) = \overline{\varrho}\nabla_x F.$$

Moreover, under the basic assumption

$$\Phi_\varepsilon \to \Phi \text{ in } C(\mathcal{T}^2), \tag{1.21}$$

the limit physical domain can be identified as

$$\Omega = \{x = (x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ \Phi(x_1,x_2) < x_3 < 1\},$$

where the heat flux satisfies the homogeneous Neumann boundary conditions

$$\nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{1.22}$$

and

$$\mathbf{U}|_{\{x_3=1\}} = 0. \tag{1.23}$$

Determining the limit conditions for $\mathbf{U}$ on the "bottom" part $\{x_3 = \Phi(x_1,x_2)\}$ of the boundary of the target domain Ω is a more delicate task. Intuitively, the impermeability condition

$$\mathbf{U}\cdot\mathbf{n}|_{\{x_3=\Phi(x_1,x_2)\}} = 0$$

remains valid, while the behaviour of the tangential component of the velocity is open to discussion. In accordance with the results of Amirat et al. [3], [4], Casado-Díaz et al. [9], and, more recently, [7], [8], the complete slip boundary conditions (1.7) give rise to the no-slip condition

$$\mathbf{U}|_{\{x_3=\Phi(x_1,x_2)\}} = 0 \tag{1.24}$$

in the asymptotic limit as soon as the bottom boundaries oscillate as $\varepsilon \to 0$ with amplitude inversely proportional to the frequency.

In order to rigorously justify these formal arguments, three fundamental issues have to be addressed:

- Existence of global-in-time solutions to the primitive system.
- Uniform estimates of the quantities
$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon},\ \mathbf{u}_\varepsilon,\ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}$$
independent of $\varepsilon \to 0$.
- The influence of the shape of the boundary on the time oscillations of the acoustic waves - the gradient part of the velocity field $\mathbf{u}_\varepsilon$.

The existence theory for system (1.4 - 1.14) has been developed in a series of papers [13], [14]. The necessary uniform estimates can be deduced from the dissipation balance equation associated to (1.4 - 1.14) as a consequence of the so-called thermodynamics stability hypothesis
$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0,\ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \tag{1.25}$$
(see [15], [16], and Section 4 below). Consequently, our main aim here is to discuss the influence of the boundary roughness, expressed through oscillations of Φ_ε as $\varepsilon \to 0$, on propagation of the acoustic waves. Note that, in accordance with the standard terminology used in the theory of singular limits, we deal with global-in-time solutions emanating from *ill prepared* initial data, meaning $\{\varrho_{0,\varepsilon}^{(1)}\}_{\varepsilon>0}$, $\{\vartheta_{0,\varepsilon}^{(1)}\}_{\varepsilon>0}$ are merely bounded and converge (weakly) in the Lebesgue norm to some non-zero limit. A similar situation for the isentropic Navier-Stokes system was studied by Bresch et al. [6], Desjardins and Grenier [11], Desjardins et al. [12], P.-L.Lions and Masmoudi [24], [25], among others (cf. also Alazard [1], [2]).

To illustrate the role of the boundary, consider a model case
$$\Phi_\varepsilon(x_1, x_2) = \Phi(x_1, x_2) - \omega_\varepsilon(x_1, x_2), \tag{1.26}$$
where Φ is a smooth function bounded above away from 1, and
$$\omega_\varepsilon = \varepsilon\omega\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big),\ \omega \geq 0.$$
As shown by Casado-Díaz et al. [9], the complete slip boundary conditions satisfied by the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ give rise to the no-slip boundary condition (1.24) for the limit velocity profile $\mathbf{U}$ as soon as ω is non-degenerate, meaning non-constant in two linearly independent directions. On the other hand, as observed by Desjardins et al. [12], the no-slip boundary conditions may "damp out" the associated acoustic waves provided $\partial\Omega$ enjoys certain geometrical properties. Thus a natural conjecture asserts that fast oscillations of ω_ε mimicking the "rougness" of the target domain Ω should lead to the same effect of anihilation of the acoustic waves. Our main goal is to show that this is indeed the case provided the boundary oscillations are fast enough with respect to the Mach number.

As already pointed out, the present work relies on the existence theory for the complete Navier-Stokes-Fourier system rewiewed, together with the necessary preliminaries, in Section 2. The main results are stated in Section 3. The uniform estimates based on the total dissipation balance are summarized in Section 4. The most essential ingredient of the analysis are uniform bounds on the *rate of convergence of the traces* of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ towards zero on the boundary of the target domain deduced in Section 5. This piece of information is then combined with the analysis of the boundary layer by means of the technique developed by Vishik and Ljusternik [29] and later adapted by Desjardins et al. [12] (see Section 7).

2 Preliminaries, weak solutions

2.1 Hypotheses, constitutive relations

The existence result as well as the uniform estimates obtained in Section 4 are conditioned by a number of technical assumptions imposed on the constitutive equations. The reader may consult [14] for the physical background as well as possible generalizations.

The state equation for the pressure takes the form

$$p(\varrho,\vartheta) = \underbrace{p_M(\varrho,\vartheta)}_{\text{molecular pressure}} + \underbrace{p_R(\vartheta)}_{\text{radiation pressure}}, \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta^4,\ a>0, \tag{2.1}$$

while the integral energy reads

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho}P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta^4}{\varrho}, \tag{2.2}$$

and, in accordance with Gibbs' relation (1.10),

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{2.3}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z>0. \tag{2.4}$$

The standard thermodynamics stability hypothesis (1.25) reformulated in terms of the structural properties of P reads

$$P \in C^1[0,\infty)\cap C^2(0,\infty),\ P(0)=0,\ P'(Z)>0 \text{ for all } Z\geq 0, \tag{2.5}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0}\frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty. \tag{2.6}$$

Furthermore, it follows from (2.6) that $P(Z)/Z^{5/3}$ is a decreasing function of Z, and we assume that

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{2.7}$$

The above hypotheses should be viewed as a compromise between a physically relevant constitutive theory and the need to have suitable *a priori* estimates required by the available mathematical tools. Note that if a is small and P linear for moderate values of the "degeneracy" argument $\varrho/\vartheta^{3/2}$, the state equation for the pressure reduces to that of a perfect gas $p \approx \mathrm{R}\varrho, \vartheta$.

The transport coefficients μ, η, and κ are continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left.\begin{aligned} 0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \\ 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta) \end{aligned}\right\} \text{for all } \vartheta \geq 0, \tag{2.8}$$

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta\geq 0, \tag{2.9}$$

where $\underline{\mu}$, $\overline{\mu}$, $\overline{\eta}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

2.2 Weak solutions - global existence

The following global existence result may be viewed as a corollary of the general theory developed in [13], [14].

Theorem 2.1 *Let $\Omega_\varepsilon \subset R^3$ be given through (1.3), with the bottom part determined by a function $\Phi_\varepsilon \in C^{2+\nu}(\mathcal{T}^2)$, $\sup \Phi_\varepsilon < 1$. Assume that p, e, s satisfy hypotheses (2.1 - 2.7), and the transport coefficients μ, η, and κ meet the growth restrictions (2.8), (2.9). Finally, let the initial data be determined through (1.14), where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants, $\varrho^{(1)}_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ are bounded measurable functions, and let $F \in W^{1,\infty}(\Omega)$ be a given function.*

Then for any $\varepsilon > 0$, small enough for the initial data $\varrho_{0,\varepsilon}$ and $\vartheta_{0,\varepsilon}$ to be strictly positive, there exists a weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ to the Navier-Stokes-Fourier system (1.4 - 1.14). More specifically, we have:

- *$\varrho_\varepsilon \geq 0$, $\varrho_\varepsilon \in L^\infty(0,T;L^{5/3}(\Omega_\varepsilon))$, $\mathbf{u}_\varepsilon \in L^2(0,T;W^{1,2}(\Omega_\varepsilon;R^3))$, and the integral identity*
$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big) \ {\rm d}x \ {\rm d}t \tag{2.10}$$
$$= \int_0^T \int_{\Omega_\varepsilon} b(\varrho_\varepsilon) {\rm div}_x \mathbf{u}_\varepsilon \varphi \ {\rm d}x \ {\rm d}t - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} B(\varrho_{0,\varepsilon}) \varphi(0,\cdot) \ {\rm d}x$$
holds for any b , B,
$$b \in C[0,\infty) \cap L^\infty(0,\infty), \ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2} \ {\rm d}z, \tag{2.11}$$
and any test function $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}_\varepsilon)$;

- *$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon;R^3))$, $p \in L^q((0,T) \times \Omega_\varepsilon)$, $\mathbb{S}_\varepsilon \in L^q((0,T) \times \Omega_\varepsilon; R^{3\times 3})$ for a certain $q > 1$,*
$$\mathbf{u}_\varepsilon|_{\{x_3 = 1\}} = 0, \ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\{x_3 = \Phi_\varepsilon(x_1,x_2)\}} = 0, \tag{2.12}$$
and
$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho_\varepsilon, \vartheta_\varepsilon) {\rm div}_x \varphi \Big) \ {\rm d}x \ {\rm d}t = \tag{2.13}$$
$$= \int_0^T \int_{\Omega_\varepsilon} \Big(\mathbb{S}_\varepsilon : \nabla_x \varphi - \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F \cdot \varphi \Big) \ {\rm d}x \ {\rm d}t - \int_{\Omega_\varepsilon} (\varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}) \cdot \varphi \ {\rm d}x$$
for any test function
$$\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}_\varepsilon); \ \varphi|_{\{x_3=1\}} = 0, \ \varphi \cdot \mathbf{n}|_{\{x_3 = \Phi_\varepsilon(x_1,x_2)\}} = 0;$$

- *$\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \in L^q((0,T) \times \Omega_\varepsilon)$ for a certain $q > 1$, and*
$$\int_{\Omega_\varepsilon} \Big(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F \Big)(t) \ {\rm d}x \tag{2.14}$$
$$= \int_{\Omega_\varepsilon} \Big(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_{0,\varepsilon} F \Big) \ {\rm d}x \ \textit{for a.a. } t \in (0,T);$$

- *$\vartheta_\varepsilon \in L^\infty(0,T;L^4(\Omega_\varepsilon)) \cap L^2(0,T;W^{1,2}(\Omega_\varepsilon))$, $\log(\vartheta_\varepsilon) \in L^2(0,T;W^{1,2}(\Omega_\varepsilon))$, and the integral identity*

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{2.15}$$

$$+ < \sigma_\varepsilon, \varphi > = - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \varphi(0,\cdot) \ \mathrm{d}x$$

holds for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}_\varepsilon)$, where $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega}_\varepsilon)$ is a non-negative measure satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_\varepsilon\Big), \tag{2.16}$$

with

$$\mathbb{S}_\varepsilon = \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \ \mathbb{I}\Big) + \eta(\vartheta_\varepsilon) \mathrm{div}_x \mathbf{u}_\varepsilon \ \mathbb{I},$$

and

$$\mathbf{q}_\varepsilon = -\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon.$$

3 Main result

3.1 Geometry of the physical space

Motivated by possible applications in meteorological models, the shape of the bottom part of the domain Ω_ε is determined by the graph of a scalar function

$$\Phi_\varepsilon(x_1,x_2) = \Phi(x_1,x_2) - \varepsilon^k \omega_\varepsilon\Big(\frac{x_1}{\varepsilon^k}, \frac{x_2}{\varepsilon^k}\Big), \tag{3.1}$$

where $k > 0$ is a positive parameter, and $\Phi \in C^{2+\nu}(\mathcal{T}^2)$ satisfies

$$\Phi \not\equiv \text{const}, \ \Phi(x_1,x_2) < 1 \text{ for all } (x_1,x_2) \in \mathcal{T}^2. \tag{3.2}$$

In addition, we assume that $\omega_\varepsilon \in C^{2+\nu}(R^2)$ are periodic in the horizontal variables (x_1,x_2) with a period $1/\varepsilon^k$, and, moreover,

$$0 \leq \omega_\varepsilon(x_1,x_2) \leq \sup_{(y_1,y_2) \in R^2} \omega_\varepsilon(y_1,y_2) \leq \overline{\omega}, \quad \sup_{(x_1,x_2) \in R^2} |\nabla \omega_\varepsilon(x_1,x_2)| \leq L, \tag{3.3}$$

in other words, the family $\{\omega_\varepsilon\}_{\varepsilon>0}$ is equi-bounded and equi-Lipschitz.

Finally, we suppose that the family $\{\omega_\varepsilon\}_{\varepsilon>0}$ is *non-degenerate* in the sense that there exists $h > 0$ independent of ε such that each unit square $Q_1 = [a,a+1] \times [b,b+1] \in R^2$ contains a ball $B_r(x_1,x_2) = \{[y_1,y_2] \mid |[y_1,y_2] - [x_1,x_2]| < r\}$ such that

$$\sup_{[y_1,y_2] \in B_r(x_1,x_2)} \omega_\varepsilon(y_1,y_2) \geq \sup_{(y_1,y_2) \in \partial B_r(x_1,x_2)} \omega_\varepsilon(y_1,y_2) + h. \tag{3.4}$$

3.2 Asymptotic limit - the main result

Having introduced the necessary preliminary material we are in a position to state our main result.

Theorem 3.1 *Let $\Omega_\varepsilon \subset R^3$, $\varepsilon \to 0$ be a family of domains determined by (1.3), where Φ_ε satisfies (3.1 - 3.4), with $k > 1$. Suppose that the functions p, e, s as well as the transport coefficients μ, η, κ satisfy the hypotheses of Theorem 2.1. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions of the Navier-Stokes-Fourier system (1.4 - 1.14) in the sense specified in Theorem 2.1 emanating from the initial data*

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}.\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \int_{\Omega_\varepsilon} \varrho^{(1)}_{0,\varepsilon}\ \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0 \textit{ for all } \varepsilon > 0, \tag{3.5}$$

and

$$\left\{ \begin{array}{c} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \textit{ weakly-(*) in } L^\infty(R^3), \\ \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \textit{ weakly-(*) in } L^\infty(R^3;R^3), \\ \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \textit{ weakly-(*) in } L^\infty(R^3). \end{array} \right\}$$

Finally, let $F \in W^{1,\infty}(R^3)$ be given such that

$$\int_\Omega F\ \mathrm{d}x = 0, \tag{3.6}$$

where

$$\Omega = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ \Phi(x_1,x_2) < x_3 < 1\}. \tag{3.7}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon c,$$

and, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)), \textit{ and (strongly) in } L^2((0,T)\times\Omega;R^3), \tag{3.8}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta^{(1)}_\varepsilon \to \Theta \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$

where

$$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3)),$$

$$\Theta \in W^{1,q}(0,T;L^q(\Omega)) \cap L^q(0,T;W^{2,q}(\Omega)) \textit{ for a certain } q > 1$$

solve the Oberbeck-Boussinesq approximation (1.15 - 1.18) on the set $(0,T)\times\Omega$, endowed with the initial data

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{u}_0],\ \Theta(0,\cdot) = \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\overline{\vartheta})}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}_0 + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}_0 + \alpha(\overline{\varrho},\overline{\vartheta})F\Big), \tag{3.9}$$

where the symbol $\mathbf{H}$ stands for the Helmholtz projection onto the space of solenoidal functions. Specifically, we have

- $$\mathrm{div}_x \mathbf{U} = 0 \ \textit{a.a. on } (0,T)\times\Omega, \ \mathbf{U}|_{\partial\Omega} = 0 \ \textit{in the sense of traces}; \tag{3.10}$$

- $$\left\{ \begin{array}{c} \overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})\Big(\partial_t \Theta + \mathbf{U}\cdot\nabla_x\Theta\Big) - \mathrm{div}_x(\kappa(\overline{\vartheta})\nabla_x\Theta) \\ \\ = \overline{\varrho}\ \overline{\vartheta}\alpha(\overline{\varrho}, \overline{\vartheta})\mathbf{U}\cdot\nabla_x F \ \textit{a.a. in } (0,T)\times\Omega, \\ \\ \nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0; \end{array} \right\} \tag{3.11}$$

- $$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}\mathbf{U}\otimes\mathbf{U} : \nabla_x\varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{3.12}$$

$$= \int_0^T \int_\Omega \Big(\mu(\overline{\vartheta})[\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}] : \nabla_x\varphi - r\nabla_x F\cdot\varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega (\overline{\varrho}\mathbf{U}_0)\cdot\varphi \ \mathrm{d}x$$

for any test function

$$\varphi \in \mathcal{D}([0,T)\times\Omega); R^3), \ \mathrm{div}_x\varphi = 0 \ \textit{in } \Omega;$$

- $$r + \overline{\varrho}\alpha(\overline{\varrho}, \overline{\vartheta})\Theta = 0. \tag{3.13}$$

The rest of the paper will be devoted to the proof of Theorem 3.1. The main novelty with respect to the previous results established in [15] is the strong convergence of the velocity fields claimed in (3.8). In particular, we show that

$$\mathbf{H}^\perp[\mathbf{u}_\varepsilon] \to 0 \text{ strongly in } L^2((0,T)\times\Omega; R^3), \tag{3.14}$$

where

$$\mathbf{H}^\perp[\mathbf{v}] = \nabla_x\Psi, \ \Delta\Psi = \mathrm{div}_x\mathbf{v}, \ \nabla_x\Psi\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{v}\cdot\mathbf{n}, \ \int_\Omega \Psi \ \mathrm{d}x = 0, \tag{3.15}$$

denotes the gradient part of the Helmholtz decomposition on the target domain Ω.

4 Total dissipation balance - uniform estimates

Our goal is to derive uniform estimates on the family of solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ for $\varepsilon \to 0$. Since $\Omega \subset \Omega_\varepsilon$ for any ε, these estimates yield automatically uniform bounds on the target domain. This rather technical part relies on the so-called total dissipation balance introduced in [16]. Consequently, we delineate the principal ideas only and give a list of the relevant uniform bounds, referring to [15], [16] for technicalities.

4.1 Conservation of the total mass

In accordance with hypothesis (3.5), the total mass of the fluid is a constant of motion independent of ε. In particular, we can deduce from the equation of continuity (2.10) that

$$\int_{\Omega_\varepsilon} (\varrho_\varepsilon(t) - \overline{\varrho}) \ \mathrm{d}x = 0 \tag{4.1}$$

for all $t \in [0,T]$, and all $\varepsilon > 0$.

4.2 Dissipation equality and related estimates

The total energy balance (2.14) can be combined with the enrtopy equation (2.15) in order to obtain the *total dissipation balance* in the form

$$\int_{\Omega_\varepsilon} \Big(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F \Big)(t) \ \mathrm{d}x + \overline{\vartheta} \sigma_\varepsilon \Big[[0,t] \times \overline{\Omega} \Big] \tag{4.2}$$

$$= \int_{\Omega_\varepsilon} \Big(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_{0,\varepsilon} F \Big) \ \mathrm{d}x$$

for a.a. $t \in (0,T)$, where we have introduced the "free" energy function

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta). \tag{4.3}$$

In addition, by virtue of (4.1), relation (4.2) can be written in the form

$$\int_{\Omega_\varepsilon} \Big(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 - \frac{1}{\varepsilon} (\varrho_\varepsilon - \overline{\varrho}) F \Big)(t) \ \mathrm{d}x \tag{4.4}$$

$$+ \int_{\Omega_\varepsilon} \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big)(t) \ \mathrm{d}x$$

$$+ \frac{\overline{\vartheta}}{\varepsilon^2} \sigma_\varepsilon \Big[[0,t] \times \overline{\Omega}_\varepsilon \Big]$$

$$= \int_{\Omega_\varepsilon} \Big(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 - \frac{1}{\varepsilon} (\varrho_{0,\varepsilon} - \overline{\varrho}) F \Big) \ \mathrm{d}x +$$

$$\int_{\Omega_\varepsilon} \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \ \mathrm{d}x.$$

Similarly to [15], it seems convenient to introduce the following notation:

$$\mathcal{M}_{\mathrm{ess}} = \{ (\varrho, \vartheta) \in R^2 \mid \overline{\varrho}/2 < \varrho < 2\overline{\varrho}, \ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta} \}, \tag{4.5}$$

$$\mathcal{M}_{\mathrm{res}} = \{ (\varrho, \vartheta) \in [0, \infty)^2 \mid (\varrho, \vartheta) \notin \mathcal{M}_{\mathrm{ess}} \}. \tag{4.6}$$

The "essential" set $\mathcal{M}_{\mathrm{ess}}$ contains all points in R^2 belonging to an open neighborhood of $(\overline{\varrho}, \overline{\vartheta})$, the "residual" set $\mathcal{M}_{\mathrm{res}}$ being its complement in $[0,\infty)^2$.

Similarly, we define the "essential" part of a function $h_\varepsilon \in L^1((0,T) \times \Omega_\varepsilon)$ as

$$[h]_{\mathrm{ess}} = h \ 1_{\{(t,x) \in (0,T) \times \Omega_\varepsilon) \ \mid \ (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x)) \in \mathcal{M}_{\mathrm{ess}} \}}, \tag{4.7}$$

and

$$[h]_{\mathrm{res}} = h - [h]_{\mathrm{ess}} = h \ 1_{\{(t,x) \in (0,T) \times \Omega_\varepsilon) \ \mid \ (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x)) \in \mathcal{M}_{\mathrm{res}} \}}. \tag{4.8}$$

Unlike the sets $\mathcal{M}_{\mathrm{ess}}$, $\mathcal{M}_{\mathrm{res}}$ determined uniquely by $(\overline{\varrho}, \overline{\vartheta})$, the projections $[\cdot]_{\mathrm{ess}}$, $[\cdot]_{\mathrm{res}}$ depend on the values of the state variables ϱ_ε, ϑ_ε, in particular, they vary with $\varepsilon \to 0$.

The function $H_{\overline{\vartheta}}$, reminiscent of the Helmholtz free energy, enjoys remarkable coercivity properties, namely

$$c_1 \Big(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \Big) \le H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \tag{4.9}$$

$$\le c_2\Big(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\Big) \text{ for all } (\varrho, \vartheta) \in \mathcal{M}_{\rm ess},$$

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \tag{4.10}$$

$$\ge \inf_{(r,\Theta)\in\partial\mathcal{M}_{\rm ess}} \Big\{ H_{\overline{\vartheta}}(r, \Theta) - (r - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big\} > 0 \text{ for all } (\varrho, \vartheta) \in \mathcal{M}_{\rm res},$$

and

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \tag{4.11}$$

$$\ge c\Big(\varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)|\Big) \text{ for all } (\varrho, \vartheta) \in \mathcal{M}_{\rm res}$$

(see Lemma 2.1 in [15]).

In accordance with the hypotheses imposed on the initial data, we easily observe that the quantity on the right hand side of (4.4) is bounded, uniformly for $\varepsilon \to 0$. Consequently, making use of the structural properties of the function $H_{\overline{\vartheta}}$, together with hypotheses (2.4 - 2.7), we deduce the following list of estimates (see Section 2 in [15]):

$$\text{ess sup}_{t\in(0,T)} \Big\| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm ess}(t) \Big\|_{L^2(\Omega_\varepsilon)} \le c, \tag{4.12}$$

$$\text{ess sup}_{t\in(0,T)} \Big\| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}(t) \Big\|_{L^2(\Omega_\varepsilon)} \le c, \tag{4.13}$$

$$\text{ess sup}_{t\in(0,T)} \| [\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \le \varepsilon^2 c, \tag{4.14}$$

$$\text{ess sup}_{t\in(0,T)} \| [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \le \varepsilon^2 c. \tag{4.15}$$

$$\text{ess} \sup_{t\in(0,T)} \int_{\Omega_\varepsilon} \Big([\varrho_\varepsilon]_{\rm res}^{\frac{5}{3}} + [\vartheta_\varepsilon]^4_{\rm res} \Big)(t) \ {\rm d}x \le \varepsilon^2 c, \tag{4.16}$$

and, as a direct consequence of (4.10),

$$\text{ess} \sup_{t\in(0,T)} \Big| \Big\{ x \in \Omega_\varepsilon \ \Big| \ (\varrho_\varepsilon, \vartheta_\varepsilon)(t, x) \in \mathcal{M}_{\rm res} \Big\} \Big| \le \varepsilon^2 c. \tag{4.17}$$

In addition, we get

$$\text{ess} \sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)} \le c, \tag{4.18}$$

and

$$\sigma_\varepsilon \Big[[0, T] \times \overline{\Omega} \Big] \le \varepsilon^2 c, \tag{4.19}$$

in particular, by virtue of hypotheses (2.8), (2.9),

$$\|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x \mathbf{u}_\varepsilon \mathbb{I}\|^2_{L^2((0,T)\times\Omega_\varepsilon;R^{3\times3})} \le c \tag{4.20}$$

and

$$\|\nabla_x \vartheta_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon;R^3} + \|\nabla_x \log(\vartheta_\varepsilon)\|_{L^2((0,T)\times\Omega_\varepsilon;R^3)} \le \varepsilon c. \tag{4.21}$$

Relation (4.20) yields a uniform bound on the velocity gradients. In order to see that we need a generalized version of Korn's inequality that may be of independent interest.

4.3 A generalized Korn's inequality

To begin with, consider an auxiliary problem:

Given

$$g \in \mathcal{D}(\Omega_\varepsilon), \ \int_{\Omega_\varepsilon} g \ \mathrm{d}\mathbf{x} = 0, \tag{4.22}$$

find a vector field $\mathbf{v} = \mathcal{B}_\varepsilon[g]$ *such that*

$$\mathbf{v} \in \mathcal{D}(\Omega_\varepsilon; R^3), \ \mathrm{div}_x \mathbf{v} = g \ \textit{in} \ \Omega_\varepsilon. \tag{4.23}$$

Obviously, problem (4.22), (4.23) admits many solutions. Here we use the construction due to Bogovskii [5]. Specifically, we report the following result (see Galdi [18, Chapter III.3]).

Lemma 4.1 *For each* $\varepsilon > 0$ *there is a (linear) solution operator* $\mathcal{B}_\varepsilon$ *associated to problem (4.22), (4.23) such that*

$$\| \ \mathcal{B}_\varepsilon[g] \ \|_{W^{m+1,q}(\Omega_\varepsilon;R^3)} \leq c(m,q)\|g\|_{W^{m,q}(\Omega_\varepsilon)}, \tag{4.24}$$

in particular, the norm of $\mathcal{B}_\varepsilon$ *is independent of* ε.

Remark:

Since the functions Φ_ε are equi-Lipschitz, the norm of $\mathcal{B}_\varepsilon$ is independent of $\varepsilon \to 0$ in accordance with Theorem 3.1 and Remark 3.2 in Galdi's book [18].

Following the arguments of Dain [10] we show the following result.

Proposition 4.1 *Let* r *be a non-negative scalar function on* Ω_ε *such that*

$$0 < m \leq \int_{\Omega_\varepsilon} r \ \mathrm{d}x, \ \int_{\Omega_\varepsilon} r^\gamma \ \mathrm{d}x \leq K \tag{4.25}$$

for a certain $\gamma > \max\{1, 3p/(4p-3)\}, \ p \in (1,\infty)$.

Then

$$\|\mathbf{v}\|_{W^{1,p}(\Omega_\varepsilon;R^3)} \leq c(m,K,p)\Big(\Big\|\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v} - \frac{2}{3}\mathrm{div}_x\mathbf{v}\mathbb{I}\Big\|_{L^p(\Omega_\varepsilon;R^{3\times3})} + \int_{\Omega_\varepsilon} r|\mathbf{v}| \ \mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega_\varepsilon; R^3)$. *In particular, the constant* c *is independent of* $\varepsilon \to 0$.

Proof:

Step 1. Clearly, it is enough to consider smooth functions $\mathbf{v}$. Denoting

$$\mathbb{D} = \frac{1}{2}\Big(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\Big) - \frac{1}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}$$

we easily compute

$$\partial_{x_i}(\Delta v_j) = \partial_{x_i}\partial_{x_j}D_{i,j} + \Delta D_{i,j} - \partial_{x_j}\partial_{x_k}D_{i,k} - \frac{1}{2}\delta_{i,j}\partial_{x_k}\partial_{x_n}D_{k,n}$$

for any fixed $i,j=1,\dots,3$. In particular, we have

$$\Big|\int_{\Omega_\varepsilon} \Delta v_j \mathrm{div}_x\varphi \ \mathrm{d}x\Big| \leq c\|\mathbb{D}\|_{L^p(\Omega;R^3)}\|\varphi\|_{W^{2,p'}(\Omega_\varepsilon;R^3)} \text{ for all } \varphi\in\mathcal{D}(\Omega_\varepsilon),$$

where $1/p+1/p'=1$.

Thus a direct application of Lemma 4.1 yields

$$\Big|\int_{\Omega_\varepsilon} \Delta v_j \varphi \ \mathrm{d}x\Big| \leq c\|\mathbb{D}\|_{L^p(\Omega;R^3)}\|\varphi\|_{W^{1,p'}(\Omega_\varepsilon)} \text{ for all } \varphi\in\mathcal{D}(\Omega), \int_{\Omega_\varepsilon}\varphi \ \mathrm{d}x=0$$

$j=1,2,3$.

Finally, we can write a function $\varphi\in\mathcal{D}(\Omega_\varepsilon)$ as

$$\varphi=\Big(\varphi-\psi\int_{\Omega_\varepsilon}\varphi \ \mathrm{d}x\Big)+\psi\int_{\Omega_\varepsilon}\varphi \ \mathrm{d}x \tag{4.26}$$

for a fixed $\psi\in\mathcal{D}(\Omega_\varepsilon)$, $\int_{\Omega_\varepsilon}\psi \ \mathrm{d}x=1$ in order to conclude that

$$\Big|\int_{\Omega_\varepsilon} \Delta v_j \varphi \ \mathrm{d}x\Big| \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;R^3)}+\int_{\Omega_\varepsilon}|\mathbf{v}| \ \mathrm{d}x\Big)\|\varphi\|_{W^{1,p'}(\Omega_\varepsilon)} \text{ for all } \varphi\in\mathcal{D}(\Omega), \tag{4.27}$$

$j=1,2,3$.

Step 2.

Using (4.27) together with the identity

$$\partial_{x_j}D_{i,j}=\frac{1}{2}\Delta u_i+\frac{1}{6}\partial_{x_i}\mathrm{div}_x\mathbf{v}, \ i=1,2,3$$

we obtain

$$\Big|\int_{\Omega_\varepsilon} \mathrm{div}_x\mathbf{v}\mathrm{div}_x\varphi \ \mathrm{d}x\Big| \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;R^3)}+\int_{\Omega_\varepsilon}|\mathbf{v}| \ \mathrm{d}x\Big)\|\varphi\|_{W^{1,p'}(\Omega_\varepsilon)} \text{ for all } \varphi\in\mathcal{D}(\Omega).$$

Thus, by virtue of the same argument as in the previous step, we conclude that

$$\Big|\int_{\Omega_\varepsilon} \mathrm{div}_x\mathbf{v}\varphi \ \mathrm{d}x\Big| \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;R^3)}+\int_{\Omega_\varepsilon}|\mathbf{v}| \ \mathrm{d}x\Big)\|\varphi\|_{L^{p'}(\Omega_\varepsilon)} \text{ for all } \varphi\in\mathcal{D}(\Omega);$$

whence

$$\|\mathrm{div}_x\mathbf{v}\|_{L^p(\Omega_\varepsilon)} \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;R^3)}+\int_{\Omega_\varepsilon}|\mathbf{v}| \ \mathrm{d}x\Big), \tag{4.28}$$

where the constant is independent of $\varepsilon\to 0$.

Step 3.

Seeing that

$$\partial_{x_k}\partial_{x_j}v_i=\partial_{x_j}D_{i,k}+\partial_{x_k}D_{i,j}-\partial_{x_i}D_{j,k}+\frac{1}{3}\Big(\delta_{j,k}\partial_{x_i}\mathrm{div}_x\mathbf{v}-\delta_{i,k}\partial_{x_j}\mathrm{div}_x\mathbf{v}-\delta_{i,j}\partial_{x_k}\mathrm{div}_x\mathbf{v}\Big)$$

we can follow step by step the previous arguments in order to conclude that

$$\|\nabla_x\mathbf{v}\|_{L^p(\Omega_\varepsilon;R^3)} \leq c(p)\Big(\Big\|\nabla_x\mathbf{v}+\nabla_x^t\mathbf{v}-\frac{2}{3}\mathrm{div}_x\mathbf{v}\mathbb{I}\Big\|_{L^p(\Omega_\varepsilon;R^{3\times 3})}+\int_{\Omega_\varepsilon}|\mathbf{v}| \ \mathrm{d}x\Big) \tag{4.29}$$

uniformly for $\varepsilon \to 0$.

Step 4.
Finally, arguing by contradiction, we construct (i) a sequence of domains Ω_n, where either $\Omega_n = \Omega_\varepsilon$ for all n large enough or $\Omega_n \to \Omega$ in the sense that

$$\Omega \subset \Omega_n \text{ for all } n, \ \Omega_n \subset \{x \in R^3 \ \| \ \mathrm{dist}[x, \Omega] < 1/n\};$$

(ii) a sequence $\{r_n\}_{n=1}^\infty$

$$0 < m \leq \int_{\Omega_n} r_n \ \mathrm{d}\mathbf{x}, \ \int_{\Omega_n} r_n^\gamma \ \mathrm{d}\mathbf{x} \leq K;$$

and (iii) a sequence $\{\mathbf{v}_n\}_{n=1}^\infty$ such that

$$\left(\left\| \nabla_x \mathbf{v}_n + \nabla_x^t \mathbf{v}_n - \frac{2}{3} \mathrm{div}_x \mathbf{v}_n \mathbb{I} \right\|_{L^p(\Omega_n; R^{3\times 3})} + \int_{\Omega_n} r_n |\mathbf{v}_n| \ \mathrm{d}\mathbf{x} \right) \leq \frac{1}{n} \tag{4.30}$$

$$\|\mathbf{v}_n\|_{W^{1,p}(\Omega_n; R^3)} = 1. \tag{4.31}$$

Combining (4.29) with (4.30), (4.31) we conclude that

$$\mathbf{v}_n \to \mathbf{v} \text{ strongly in } W^{1,p}(\Omega; R^3),$$

where the limit function satisfies

$$\|\mathbf{v}\|_{W^{1,p}(\Omega; R^3)} = 1, \ \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3} \mathrm{div}_x \mathbf{v} \mathbb{I} = 0, \ \int_\Omega r |\mathbf{v}| \ \mathrm{d}x = 0 \tag{4.32}$$

for a certain r satisfying (4.25). Thus (4.32) leads to contradiction as $\mathbf{v}$ is necessarily a conformal Killing vector that cannot vanish on a set of positive measure (see Dain [10]).

q.e.d.

4.4 Space-time estimates based on energy dissipation

Since $\mathbf{u}_\varepsilon$ admits the bounds established in (4.18), (4.19), we can apply Proposition 4.1 to obtain

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon; R^3)} \ \mathrm{d}t \leq c \tag{4.33}$$

uniformly for $\varepsilon \to 0$. Similarly, using a “scalar” version (Poincare’s inequality) of Proposition 4.1, we deduce from (4.13), (4.17), and (4.21) that

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t \leq c, \tag{4.34}$$

and

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t \leq c. \tag{4.35}$$

5 Rate of convergence of the traces

In this section, we determine the rate of convergence towards zero of the traces of functions belonging to the space $W^{1,2}(\Omega_\varepsilon; R^3)$. The main result reads as follows.

Proposition 5.1 *Let $\Omega_\varepsilon \subset R^3$ be a family of domains defined through (1.3), with Φ_ε satisfying (3.1 - 3.4).*

Then there exists $\varepsilon_0 > 0$, and $c > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$, such that

$$\int_{\{x_3 = \Phi(x_1,x_2)\}} |\mathbf{v}|^2 \ \mathrm{d}\sigma \leq \varepsilon^k \ c \|\nabla_x \mathbf{v}\|^2_{L^2(\Omega_\varepsilon; R^{3\times 3})} \ \textit{for all } 0 < \varepsilon < \varepsilon_0 \tag{5.1}$$

for any $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; R^3)$),

$$\mathbf{v}|_{\{x_3=1\}} = 0, \ \mathbf{v} \cdot \mathbf{n}|_{\{x_3 = \Phi_\varepsilon(x_1.x_2)\}} = 0.$$

Proof:

First of all, observe that it is enough to prove the result for $k = 1$. Moreover, it suffices to show (5.1) on each cell

$$C_{n_1,n_2} = \Big\{ (x_1, x_2, x_2) \ \Big| \ (x_1, x_2) \in (n_1\varepsilon, n_1\varepsilon + \varepsilon) \times (n_2\varepsilon, n_2\varepsilon + \varepsilon),$$

$$\Phi(x_1, x_2) - \varepsilon\omega_\varepsilon\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big) < x_3 < \Phi(n_1\varepsilon, n_2\varepsilon) + 2\varepsilon L_\Phi \Big\}$$

for n_1, n_2 integers, $0 \leq n_1, n_2 < \frac{1}{\varepsilon}$, $L_\Phi = \sup_{\mathcal{T}^2} |\nabla\Phi|$, where the constant c in (5.1) must be independent of n_1, n_2.

Thus, after changing the variables

$$x_1 \approx x_1 - n_1\varepsilon, \ x_2 \approx x_2 - n_2\varepsilon, x_3 \approx x_3 - \Phi(n_1\varepsilon, n_2\varepsilon),$$

the problem reduces to showing

$$\int_{\{x_3 = \tilde{\Phi}_\varepsilon(x_1,x_2)\}} |\mathbf{v}|^2 \ \mathrm{d}\sigma \leq \varepsilon \ c \|\nabla_x \mathbf{v}\|^2_{L^2(C_\varepsilon; R^3)}$$

for any $\mathbf{v} \in W^{1,2}(C_\varepsilon; R^3)$),

$$\mathbf{v} \cdot \mathbf{n}|_{\{x_3 = \tilde{\Phi}_\varepsilon(x_1.x_2)\}} = 0,$$

where

$$\tilde{\Phi}_\varepsilon(x_1, x_2) = \tilde{\Phi}(x_1, x_2) - \varepsilon\tilde{\omega}_\varepsilon\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big),$$

where

$$\tilde{\Phi}(0, 0) = 0, \ |\nabla\tilde{\Phi}| \leq L_\Phi,$$

the functions

$$\tilde{\omega}_\varepsilon\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) = \omega_\varepsilon\Big(\frac{x_1 + n_1\varepsilon}{\varepsilon}, \frac{x_2 + n_2\varepsilon}{\varepsilon}\Big)$$

enjoy the properties (3.3), (3.4) with the same constants $\overline{\omega}$, L, h, and

$$C_\varepsilon = \Big\{ (x_1, x_2, x_2) \ \Big| \ (x_1, x_2) \in (0, \varepsilon)^2, \ \tilde{\Phi}(x_1, x_2) - \varepsilon\tilde{\omega}_\varepsilon\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big) < x_3 < 2\varepsilon L_\Phi \Big\}.$$

Finally, introducing the scaling $x \approx x/\varepsilon$ we arrive at

$$\int_{\{x_3=\Psi_\varepsilon(x_1,x_2)\}} |\mathbf{v}|^2 \ \mathrm{d}\sigma \leq c\|\nabla_x \mathbf{v}\|^2_{L^2(C_\varepsilon;R^3)} \tag{5.2}$$

for any $\mathbf{v} \in W^{1,2}(\tilde{C}_\varepsilon;R^3))$,

$$\mathbf{v}\cdot\mathbf{n}|_{\{x_3=\Psi_\varepsilon(x_1.x_2)\}} = 0,$$

where

$$\Psi_\varepsilon(x_1,x_2) = \frac{1}{\varepsilon}\tilde{\Phi}(\varepsilon x_1, \varepsilon x_2) - \tilde{\omega}_\varepsilon(x_1,x_2),\ \tilde{\Phi}(0,0)=0,\ |\nabla\tilde{\Phi}| \leq L_\Phi, \tag{5.3}$$

and

$$\tilde{C}_\varepsilon = \Big\{(x_1,x_2,x_2)\ \Big|\ (x_1,x_2)\in(0,1)^2,\ \frac{1}{\varepsilon}\tilde{\Phi}(\varepsilon x_1,\varepsilon x_2) - \tilde{\omega}_\varepsilon(x_1,x_2) < x_3 < 2L\Big\}. \tag{5.4}$$

Now, we show (5.2) arguing by contradiction. Accordingly, we suppose there is a sequence $\{\mathbf{v}_n\}_{n=1}^\infty$ such that

$$\|\nabla_x \mathbf{v}_n\|^2_{L^2(C_n;R^3)} \leq \frac{1}{n},\ \int_{\{x_3=\Psi_n(x_1,x_2)\}} |\mathbf{v}_n|^2 \ \mathrm{d}\sigma = 1, \tag{5.5}$$

$$\mathbf{v}_n\cdot\mathbf{n}|_{\{x_3=\Psi_n(x_1,x_2)\}} = 0,$$

where

$$C_n = \Big\{(x_1,x_2,x_2)\ \Big|\ (x_1,x_2)\in(0,1)^2,\ \Psi_n(x_1,x_2) < x_3 < 2L\Big\},$$

and, by virtue of the non-degeneracy hypotheses (3.3), (3.4),

$$\Psi_n \to \Psi \text{ in } C[0,1]^2,$$

and there exists a vector $\mathbf{M}\in R^2$ such that

$$\Psi \in W^{1,\infty}[0,1]^2,\ \Psi - \mathbf{M}\cdot x \text{ possesses a strong local maximum in } C_n. \tag{5.6}$$

Since the sequence of domains $\{C_n\}_{n=1}^\infty$ is equi-Lipschitz, the functions $\mathbf{v}_n$ can be extended to a larger set, say,

$$B = \{(x_1,x_2,x_3)\ |\ (x_1,x_2)\in(0,1)^2,\ -K < x_3 < 2L$$

in such a way that, in accordance with (5.5),

$$\mathbf{v_n} \to \mathbf{v} \text{ weakly in } W^{1,2}(B;R^3),$$

where

$$\nabla_x\mathbf{v} \equiv 0 \text{ on } C,$$

$$C = \Big\{(x_1,x_2,x_2)\ \Big|\ (x_1,x_2)\in(0,1)^2,\ \Psi(x_1,x_2) < x_3 < 2L\Big\}.$$

Moreover, using compactness of the trace operator together with (5.5), (5.6) we obtain

$$\int_{\{x_3=\Psi(x_1,x_2)\}} |\mathbf{v}|^2 \ \mathrm{d}\sigma = 1;$$

whence

$$\mathbf{v} \equiv \overline{\mathbf{v}} \not\equiv 0 - \text{ a constant vector on } C. \tag{5.7}$$

On the other hand, since the normal trace of $\mathbf{v}_n$ vanishes on the "bottom" part of the boundary $\{x_3 = \Psi_n(x_1, x_2)\}$, we have

$$\int_{C_n} \Big(\mathrm{div}_x \mathbf{v}_n \varphi + \mathbf{v}_n \nabla_x \cdot \varphi \Big) \, \mathrm{d}\mathbf{x} = 0 \tag{5.8}$$

for any test function $\varphi \in C^1(R^3)$,

$$\mathrm{supp} \varphi \subset \{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2, \ x_3 < 2L\}.$$

Letting $n \to \infty$ in (5.8) we infer that

$$\int_C \Big(\mathrm{div}_x \mathbf{v} \varphi + \mathbf{v} \nabla_x \cdot \varphi \Big) \, \mathrm{d}\mathbf{x} = 0$$

for the same class of test functions. Thus, using the standard "weak" definition of the normal trace we conclude that

$$\mathbf{v} \cdot \mathbf{n}|_{\{x_3 = \Psi(x_1, x_2)\}} = 0$$

in contrast with (5.7) since Ψ satisfies the non-degeneracy condition (5.6).

q.e.d.

6 Convergence

Our goal is to exploit the estimates obtained in the previous two sections in order to let $\varepsilon \to 0$ in the system of field equations.

6.1 Incompressibility

Taking advantage of the fact that $\Omega \subset \Omega_\varepsilon$ for all $\varepsilon > 0$ we can use estimate (4.33) to deduce that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)) \tag{6.1}$$

passing to a subsequence as the case may be.

In addition, seeing that ϱ_ε obeys (4.12), (4.16) we have

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly in } L^\infty(0,T; L^{5/3}(\Omega)), \tag{6.2}$$

where $\varrho^{(1)}$ belongs to the space $L^\infty(0,T; L^2(\Omega))$. In particular,

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T; L^{5/3}(\Omega),$$

therefore letting $\varepsilon \to 0$ in the equation of continuity (2.10) yields

$$\int_\Omega \overline{\varrho} \mathbf{U} \cdot \nabla_x \varphi \, \mathrm{d}x = 0 \text{ for any } \varphi \in \mathcal{D}((0,T) \times \Omega),$$

in other words

$$\mathrm{div}_x \mathbf{U} = 0 \text{ a.a. on } (0,T) \times \Omega. \tag{6.3}$$

Finally, by virtue of Proposition 5.1,

$$\mathbf{U}(t,\cdot)|_{\partial\Omega} = 0 \text{ for a.a. } t \in (0,T). \tag{6.4}$$

Indeed due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\partial\Omega)$ and (5.1), (6.1) we can show that

$$\eta * U(t,\cdot)|_{\partial\Omega} = 0 \text{ for any } t \in (0,T)$$

where η is a mollifying kernel in the *time* variable. Letting $\eta \to \delta$ (the Dirac mass) yields (6.4).

6.2 Entropy balance yielding the heat equation

We show that the entropy balance expressed through (2.15) gives rise to equation (3.11) provided, in accordance with (4.34),

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{6.5}$$

To this end, we re-write (2.15) by means of (2.10) in the form

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big) \Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.6}$$

$$- \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \Big(\frac{\vartheta_\varepsilon}{\varepsilon} \Big) \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon} < \sigma_\varepsilon, \varphi >= - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \Big(\frac{s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big) \varphi(0,\cdot) \ \mathrm{d}x$$

satisfied for any test function $\varphi \in \mathcal{D}([0,T) \times R^3)$.

In order to identify the asymptotic limit of (6.6), we proceed by several steps.

Step 1. Writing

$$\varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big)$$

$$= [\varrho_\varepsilon]_{\rm ess} \frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}$$

$$+ \Big[\frac{\varrho_\varepsilon}{\varepsilon} \Big]_{\rm res} \Big([s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta}) \Big) + \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm res}$$

we can use (4.12), (4.13) in order to deduce that

$$\Big\| [\varrho_\varepsilon]_{\rm ess} \frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \le c, \tag{6.7}$$

$$[\varrho_\varepsilon]_{\rm ess} \frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \to \overline{\varrho} \Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} \Theta \Big) \tag{6.8}$$

weakly-(*) in $L^\infty(0,T;L^2(\Omega))$. Furthermore, in accordance with (4.16),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \Big\| \Big[\frac{\varrho_\varepsilon}{\varepsilon} \Big]_{\rm res} \Big([s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta}) \Big) \Big\|_{L^{5/3}(\Omega_\varepsilon)} \to 0. \tag{6.9}$$

Finally, by virtue of (4.15),

$$\int_0^T \int_{\Omega_\varepsilon} \Big| \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm res} \Big| \ {\rm d}x \ {\rm d}t \leq \varepsilon c. \tag{6.10}$$

Thus we conclude, in agreement with (6.7 - 6.10), that

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \partial_t \varphi \ {\rm d}x \ {\rm d}t \tag{6.11}$$

$$\to \int_0^T \int_{\Omega} \overline{\varrho} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \Big) \partial_t \varphi \ {\rm d}x \ {\rm d}t$$

for any $\varphi \in \mathcal{D}([0,T) \times R^3)$.

Step 2. It follows from the structural hypotheses (2.3 - 2.7) that

$$|\varrho s(\varrho, \vartheta)| \leq c(1 + \varrho \log(\varrho) + \varrho \log(\vartheta) + \vartheta^3); \tag{6.12}$$

whence we can use the uniform bounds (4.16), together with (4.33 - 4.35), in order to obtain

$$\Big\| \Big[\frac{\varrho_\varepsilon}{\varepsilon} \Big]_{\rm res} \Big([s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho}, \overline{\vartheta}) \Big) \mathbf{u}_\varepsilon \Big\|_{L^q((0,T) \times \Omega_\varepsilon; R^3)} \to 0, \tag{6.13}$$

and

$$\Big\| \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm res} \mathbf{u}_\varepsilon \Big\|_{L^q((0,T) \times \Omega_\varepsilon; R^3)} \to 0 \tag{6.14}$$

for a certain $q > 1$. In addition,

$$\Big\| [\varrho_\varepsilon]_{\rm ess} \frac{[s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \mathbf{u}_\varepsilon \Big\|_{L^q((0,T) \times \Omega_\varepsilon; R^3))} \leq c \text{ for a certain } q > 1, \tag{6.15}$$

and we conclude

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t \tag{6.16}$$

$$\to \int_0^T \int_{\Omega} \mathcal{F} \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t$$

for all $\varphi \in C^1([0,T) \times R^3)$, where $\mathcal{F}$ is identified through

$$[\varrho_\varepsilon]_{\rm ess} \frac{[s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \mathbf{u}_\varepsilon \to \mathcal{F} \text{ weakly in } L^q((0,T) \times \Omega; R^3). \tag{6.17}$$

As a matter of fact, it can be shown (see [15]) that

$$\mathcal{F} = \overline{\varrho} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \Big) \mathbf{U}, \tag{6.18}$$

however, such a relation will follow immediately from (6.8), (6.16) as soon as we establish the strong (pointwise) convergence of the velocity fields claimed in (3.8).

Step 3. As a direct consequence of (4.19), we get

$$\frac{1}{\varepsilon} < \sigma_\varepsilon, \varphi > \to 0 \text{ for any fixed } \varphi \in C([0,T] \times R^3). \tag{6.19}$$

Step 4.
Finally, writing

$$\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big) = \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big) + \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big),$$

we deduce from (4.34) that

$$\Big\|\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\Big\|_{L^2((0,T)\times\Omega_\varepsilon;R^3)} \leq c, \tag{6.20}$$

$$\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big) \to \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\Theta \text{ weakly in } L^2(0,T;L^2(\Omega;R^3)), \tag{6.21}$$

while, by virtue of (4.16), (4.19),

$$\Big\|\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\Big\|_{L^q((0,T)\times\Omega_\varepsilon;R^3))} \to 0 \text{ for a certain } q > 1. \tag{6.22}$$

Thus summing up (6.11) with (6.17 - 6.19), (6.22) we can let $\varepsilon \to 0$ in (6.6) in order to conclude that

$$\int_0^T\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Theta\Big)\Big(\partial_t\varphi + \vec{U}\cdot\nabla_x\varphi\Big)\ {\rm d}x\ {\rm d}t \tag{6.23}$$

$$-\int_0^T\int_\Omega \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\Theta\cdot\nabla_x\varphi\ {\rm d}x{\rm d}t$$

$$= -\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\Big)\varphi(0,\cdot)\ {\rm d}x$$

for any $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$ as soon as we show strong convergence of the velocities. In particular, by virtue of hypothesis (3.5),

$$\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Theta\Big)(t)\ {\rm d}x$$

$$= \int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\Big)\ {\rm d}x = 0 \text{ for a.a. } t \in (0,T),$$

and, consequently,

$$\int_\Omega \Theta(t,\cdot)\ {\rm d}x = 0 \text{ for a.a. } t \in (0,T). \tag{6.24}$$

The relation between $\varrho^{(1)}$ and Θ will become clear in the next section.

6.3 Momentum equation

Since we have already shown in (6.4) that the limit velocity field $\mathbf{U}$ satisfies the homogeneous Dirichlet boundary conditions on the whole boundary $\partial\Omega$ of the target domain, and $\Omega \subset \Omega_\varepsilon$ for all $\varepsilon > 0$, it is enough to consider the momentum equation (2.13) with the test functions

$$\varphi \in \mathcal{D}([0,T)\times\Omega;R^3),\ {\rm div}_x\varphi = 0.$$

Consequently, in view of the uniform bounds established in Section 4, it is easy to conclude that

$$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U} \cdot \partial_t \varphi + \overline{\varrho[\mathbf{U} \otimes \mathbf{U}]} : \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = \tag{6.25}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \Big(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}) : \nabla_x \varphi - \varrho^{(1)} \nabla_x F \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_{\Omega_\varepsilon} \overline{\varrho} \mathbf{U}_0 \cdot \varphi \ \mathrm{d}x$$

for any $\varphi \in \mathcal{D}([0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$. Here, similarly to (6.17), we have

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{U} \otimes \mathbf{U}} \text{ weakly in } L^q((0,T) \times \Omega; R^{3\times 3}) \text{ for a certain } q > 1,$$

where

$$\overline{\varrho \mathbf{U} \otimes \mathbf{U}} = \varrho \mathbf{U} \otimes \mathbf{U}$$

as soon as we show strong convergence of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in $L^2((0,T) \times \Omega; R^3)$.

Focusing now on the pressure, we write

$$p(\varrho_\varepsilon, \vartheta_\varepsilon) = [p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} + [p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}},$$

where, in accordance with hypotheses (2.1), (2.7),

$$0 \le \frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}}}{\varepsilon} \le c \Big(\Big[\frac{1}{\varepsilon} \Big]_{\mathrm{res}} + \Big[\frac{\varrho_\varepsilon^{\frac{5}{3}}}{\varepsilon} \Big]_{\mathrm{res}} + \Big[\frac{\vartheta_\varepsilon^4}{\varepsilon} \Big]_{\mathrm{res}} \Big) \tag{6.26}$$

Consequently, the uniform estimates (4.16), (4.17) imply that

$$\mathrm{ess} \sup_{t \in (0,T)} \Big\| \Big[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\mathrm{res}} \Big\|_{L^1(\Omega_\varepsilon)} \le \varepsilon c. \tag{6.27}$$

Thus multiplying (2.13) on ε and letting $\varepsilon \to 0$ we obtain

$$\int_0^T \int_\Omega \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta - \overline{\varrho} F \Big) \mathrm{div}_x \vec{\varphi} \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{6.28}$$

for all $\vec{\varphi} \in \mathcal{D}((0,T) \times \Omega; R^3)$, which, together with hypothesis (3.6) and (6.24), yields the desired relation

$$\varrho^{(1)} = -\frac{\partial_\vartheta p}{\partial_\varrho p}(\overline{\varrho}, \overline{\vartheta})\Theta + \frac{\overline{\varrho}}{p_\varrho(\overline{\varrho}, \overline{\vartheta})} F. \tag{6.29}$$

Expressing $\varrho^{(1)}$ in (6.23) by means of (6.29) gives rise to

$$\int_0^T \int_\Omega \overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Theta \Big(\partial_t \varphi + \vec{U} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.30}$$

$$- \int_0^T \int_\Omega \Big(\overline{\varrho} \ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F \vec{U} \cdot \nabla_x \varphi + \kappa(\overline{\vartheta}) \nabla_x \Theta \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t =$$

$$- \int_\Omega \overline{\varrho} \ \overline{\vartheta} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + \alpha(\overline{\varrho}, \overline{\vartheta}) F \Big) \varphi(0, \cdot) \ \mathrm{d}x$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$, where the physical constants c_p, α are determined through (1.19), (1.20). Note that relation (6.30) is nothing other than a weak formulation of equation (3.11) endowed with the homogeneous Neumann boundary conditions.

Now, since

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\mathbf{U} \text{ weakly in } L^\infty(0,T;L^2(\Omega;R^3)),$$

we have

$$\vec{U} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3)), \tag{6.31}$$

and, consequently,

$$\mathrm{div}_x(\vec{U}\Theta) = \vec{U}\cdot\nabla_x\Theta \in L^q((0,T)\times\Omega) \text{ for a certain } q>1.$$

Thus by means of the standard linear theory of parabolic equations we conclude that Θ satisfies (3.11) supplemented with the initial conditions (3.9).

Finally, setting

$$r = \varrho^{(1)} - \frac{\overline{\varrho}}{p_\varrho(\overline{\varrho},\overline{\vartheta})}F$$

we recover the Boussinesq relation (3.13).

7 Boundary layer analysis

In view of the previous discussion, the only missing point in the proof of Theorem 3.1 is the strong convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in $L^2((0,T)\times\Omega;R^3)$ claimed in (3.8). This will be achieved by means of a refined analysis of the so-called *acoustic equation* governing the evolution of the gradient part $\mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ of the Helmholtz decomposition introduced in (3.15). For the sake of simplicity, we shall assume $\eta\equiv 0$.

7.1 Acoustic equation

We start rewriting the equation of continuity (2.10) in the form

$$\int_0^T\int_{\Omega_\varepsilon}\Big(\varepsilon\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\partial_t\varphi + \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = -\int_{\Omega_\varepsilon}\varepsilon\frac{\varrho_{0,\varepsilon}-\overline{\varrho}}{\varepsilon}\ \mathrm{d}x \tag{7.1}$$

for any $\varphi\in\mathcal{D}([0,T)\times\overline{\Omega}_\varepsilon)$.

Similarly, the momentum equation (2.13) gives rise to

$$\int_0^T\int_\Omega \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{7.2}$$

$$+\int_0^T\int_\Omega\Big(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\mathrm{ess}} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\mathrm{ess}} - \overline{\varrho}F\Big)\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^T\int_\Omega\varepsilon\mathbb{S}_\varepsilon:\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t = -\varepsilon\int_\Omega\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\varphi\ \mathrm{d}x$$

$$+\varepsilon\int_0^T\int_\Omega\mathbb{G}^1_\varepsilon:\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \varepsilon\int_0^T\int_\Omega\mathbf{G}^2_\varepsilon\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^T\int_\Omega\Big(G^3_\varepsilon+G^4_\varepsilon\Big)\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t,$$

for any $\varphi\in\mathcal{D}([0,T)\times\Omega;R^3)$, where we have set

$$\mathbb{G}^1_\varepsilon = -\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon,\ \mathbf{G}^2_\varepsilon = \frac{\overline{\varrho}-\varrho_\varepsilon}{\varepsilon}\nabla_xF, \tag{7.3}$$

$$G^3_\varepsilon = - \frac{p_{\rm res}(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon}, \tag{7.4}$$

and

$$G^4_\varepsilon = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} - \Big(\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big). \tag{7.5}$$

Moreover, we can use the standard Lebesgue convergence theorem in order to extend validity of (7.2) to a more general class of test functions, specifically,

$$\varphi \in W^{1,\infty}((0,T) \times \Omega), \ \varphi|_{\partial \Omega} = 0, \ \varphi(T, \cdot) = 0. \tag{7.6}$$

Now, in accordance with (7.6),

$$\int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \varphi \ {\rm d}x \ {\rm d}t = - \varepsilon \int_0^T \int_\Omega \frac{2 \mu(\overline{\vartheta})}{\overline{\varrho}} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \varphi]] \ {\rm d}x \ {\rm d}t \tag{7.7}$$

$$+ 2 \varepsilon \int_0^T \int_{\partial \Omega} \mu(\overline{\vartheta}) ([[\nabla_x \varphi]] \mathbf{u}_\varepsilon) \cdot \mathbf{n} \ {\rm d}\sigma + \int_0^T \int_\Omega \frac{2 \varepsilon \mu(\overline{\vartheta})}{\overline{\varrho}} (\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \varphi]] \ {\rm d}x \ {\rm d}t$$

$$+ \int_0^T \int_\Omega \varepsilon \Big(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}) \Big) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) : \nabla_x \varphi \ {\rm d}x \ {\rm d}t,$$

for all

$$\varphi \in W^{2,\infty}((0,T) \times \Omega), \ \varphi|_{\partial \Omega} = 0, \ \varphi(T, \cdot) = 0,$$

where we have introduced the notation

$$[[\mathbb{M}]] = \frac{1}{2} \Big[\mathbb{M} + \mathbb{M}^t - \frac{2}{3} {\rm trace}[\mathbb{M}] \ \mathbb{I} \Big].$$

In a similar way, the entropy balance equation (2.15) can be written in the form

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon \Big(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \partial_t \varphi \ {\rm d}x \ {\rm d}t \tag{7.8}$$

$$= - \int_{\Omega_\varepsilon} \varepsilon \Big(\frac{\varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon} \vartheta_{0,\varepsilon}) - \varrho_{0,\varepsilon} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \partial_t \varphi \ {\rm d}x$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \Big(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon \Big) \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t - < \sigma_\varepsilon, \varphi >$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}_\varepsilon)$.

Thus a suitable linear combination of (7.1), (7.2), (7.8) yields

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t \tag{7.9}$$

$$= - \int_{\Omega_\varepsilon} \varepsilon r_{0,\varepsilon} \varphi(0, \cdot) \ {\rm d}x + \frac{\Lambda}{\omega} \Big(\int_0^T \int_{\Omega_\varepsilon} \mathbf{G}^\varepsilon_5 \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t - < \sigma_\varepsilon, \varphi > \Big)$$

for any $\varphi \in W^{1,\infty}((0,T) \times R^3)$, $\varphi(T, \cdot) = 0$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega r_\varepsilon {\rm div}_x \varphi + \varepsilon D \mathbf{V}_\varepsilon \cdot {\rm div}_x [[\nabla_x \varphi]] \Big) \ {\rm d}x \ {\rm d}t \tag{7.10}$$

$$= - \int_\Omega \varepsilon \mathbf{V}_{0,\varepsilon} \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

$$+ \int_0^T \int_\Omega \Big(\mathbf{G}_6^\varepsilon \cdot \mathrm{div}_x [[\nabla_x \varphi]] + \mathbb{G}_7^\varepsilon : \nabla_x \varphi + G_8^\varepsilon \mathrm{div}_x \varphi + \mathbf{G}_9^\varepsilon \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ 2\varepsilon \int_0^T \int_{\partial\Omega} \mu(\overline{\vartheta}) ([[\nabla_x \varphi]] \mathbf{u}_\varepsilon) \cdot \mathbf{n} \ \mathrm{d}\sigma$$

for any $\varphi \in W^{2,\infty}((0,T) \times \Omega; R^3)$, $\varphi|_{\partial\Omega} = 0$, $\varphi(T,\cdot) = 0$, where we have set

$$r_\varepsilon = \frac{1}{\omega} \Big(\omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \Lambda \varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F \Big), \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon, \tag{7.11}$$

$$r_{0,\varepsilon} = \frac{1}{\omega} \Big(\omega \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} + \Lambda \varrho_{0,\varepsilon} \frac{s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F \Big), \mathbf{V}_\varepsilon = \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}, \tag{7.12}$$

with

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \ \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}, \ \Lambda = \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\overline{\varrho} \ \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}, \ D = \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}}, \tag{7.13}$$

and

$$\mathbf{G}_5^\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon, \tag{7.14}$$

$$\mathbf{G}_6^\varepsilon = \varepsilon D (\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon, \tag{7.15}$$

$$\mathbb{G}_7^\varepsilon = \frac{\varepsilon}{2} (\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta})) [[\nabla_x \mathbf{u}_\varepsilon]] - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{7.16}$$

$$G_8^\varepsilon = \Lambda \varrho_\varepsilon \Big[\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\mathrm{res}} - \Big[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\mathrm{res}} \tag{7.17}$$

$$+ \Lambda \Big\{ \Big[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\mathrm{ess}}$$

$$- \overline{\varrho} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} \Big) \Big\}$$

$$+ \Big\{ \frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} \Big) \Big\}$$

$$+ \omega \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{res}},$$

$$\mathbf{G}_9^\varepsilon = (\overline{\varrho} - \varrho_\varepsilon) \nabla_x F. \tag{7.18}$$

Note that the left-hand side of (7.9), (7.10) can be formally interpreted as a distributional form of the wave operator

$$\begin{bmatrix} r \\ \mathbf{V} \end{bmatrix} \mapsto \varepsilon \partial_t \begin{bmatrix} r \\ \mathbf{V} \end{bmatrix} + \begin{bmatrix} \mathrm{div}_x \mathbf{V} \\ \omega \nabla_x r + \varepsilon D \mathrm{div}_x [[\nabla_x \mathbf{V}]] \end{bmatrix},$$

however, equations (7.9), (7.10) are defined on *different* spatial domains, namely Ω_ε, Ω, respectively.

7.2 Spectral analysis of the acoustic operator

We examine the spectral properties of the linear differential operator associated to problem (7.9), (7.10), specifically we consider the operator

$$\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} \mapsto \mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} + \varepsilon \mathcal{B}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix}, \tag{7.19}$$

with

$$\mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \omega \mathrm{div}_x \mathbf{w} \\ \nabla_x v \end{bmatrix}, \ \mathcal{B}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ D\mathrm{div}_x[[\nabla_x \mathbf{w}]] \end{bmatrix}$$

that can be viewed as the adjoint of the "elliptic" part in (7.9), (7.10). The operator will be considered on the target domain Ω and supplemented with the homogeneous Dirichlet boundary condition for $\mathbf{w}$,

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{7.20}$$

Let us start with the unperturbed problem

$$\mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} v \\ \mathbf{w} \end{bmatrix}, \ \text{meaning} \ \left\{ \begin{array}{c} \omega \mathrm{div}_x \mathbf{w} = \lambda v \\ \\ \nabla_x v = \lambda \mathbf{w} \end{array} \right\}, \tag{7.21}$$

which can be *equivalently* reformulated as

$$\Delta v = \xi v, \ \xi = \frac{\lambda^2}{\omega}, \tag{7.22}$$

where the boundary condition (7.20) reads

$$\nabla_x v|_{\partial\Omega} = 0, \tag{7.23}$$

in particular,

$$\nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.24}$$

As is well-known, the *Neumann problem* (7.22), (7.24) admits a countable set of real eigenvalues $\{\xi_n\}_{n=0}^{\infty}$,

$$0 = \xi_0 > \xi_1 \geq \xi_2 \ldots$$

with the associated family of real eigenfunctions $\{v_n\}_{n=0}^{\infty}$, $v_0 = 1/\sqrt{|\Omega|}$, forming an orthonormal basis of the Hilbert space $L^2(\Omega)$.

On the other hand the *overdetermined problem* (7.22), (7.23) admits only the trivial solution $v = \text{const}$, $\xi = 0$ as soon as Ω is given by (3.7), where Φ is a non-constant periodic function of the horizontal variables (x_1, x_2). Indeed if v is a non-constant solution to (7.22), (7.23) on the periodic strip given by (3.7), then

$$v(x_1, x_2, 1) = v_0 = \text{const}$$

and the function

$$V(x) = v(x_1, x_2, x_3) - v_0 \cos(\sqrt{|\xi|}(x_3 - 1))$$

solves (7.22), where, in addition, $V(x_1, x_2, 1) = 0$. By means of the unique continuation property for elliptic equations, we conclude $V \equiv 0$ in Ω. Then, however, v

cannot be constant on the bottom part of the boundary $\{x_3 = \Phi(x_1, x_2)\}$ unless Φ is a constant function.

From this perspective, the component $\varepsilon\mathcal{B}$, supplemented with the Dirichlet boundary condition (7.20), may be viewed as a *singular perturbation* of the operator $\mathcal{A}$.

For the perturbed operator, we consider an "approximate" eigenvalue problem in the form

$$\mathcal{A}\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \varepsilon\mathcal{B}\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} = \lambda_\varepsilon \begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \varepsilon \begin{bmatrix} s^1_\varepsilon \\ \mathbf{s}^2_\varepsilon \end{bmatrix},$$

that means,

$$\left\{ \begin{array}{c} \omega \mathrm{div}_x \mathbf{w}_\varepsilon = \lambda_\varepsilon v_\varepsilon + \varepsilon s^1_\varepsilon \\ \\ \nabla_x v_\varepsilon + \varepsilon D \mathrm{div}_x[[\nabla_x \mathbf{w}_\varepsilon]] = \lambda_\varepsilon \mathbf{w}_\varepsilon + \varepsilon \mathbf{s}^2_\varepsilon, \end{array} \right\} \tag{7.25}$$

supplemented with the homogeneous Dirichlet boundary condition

$$\mathbf{w}_\varepsilon|_{\partial\Omega} = 0. \tag{7.26}$$

There is a large amount of literature, in particular in applied mathematics, devoted to formal asymptotic analysis of singularly perturbed problems based on the so-called WKB (Wentzel-Kramers-Brilbuin) expansions for boundary layers using sophisticated multiple layer analysis and matched asymptotics. The spectral analysis of singularly perturbed operators was developed by Vishik and Ljusternik in [29] and later adapted by Desjardins et al. [12] to the present setting. Here, we report the following result (see Desjardins et al. [12, Proposition 2] or [17, Proposition 5.2]).

Lemma 7.1 *Let $\Omega \subset R^3$ by determined through (3.7), with Φ satisfying (3.2). Assume that v, $\mathbf{w}$ and $\lambda \neq 0$ is a solution of the unperturbed problem (7.21), (7.24) normalized so that*

$$\int_\Omega |v|^2 \, \mathrm{d}x = 1.$$

Then, for any $\varepsilon > 0$ small enough, the perturbed problem (7.25), (7.26) admits a solution v_ε, $\mathbf{w}_\varepsilon$, λ_ε, s^1_ε, s^2_ε such that

$$\|v_\varepsilon\|_{L^\infty(\Omega;C)},\ \|\mathbf{w}_\varepsilon\|_{L^\infty(\Omega;C^3)} \leq c, \tag{7.27}$$

$$\|\nabla_x v_\varepsilon\|_{L^\infty(\Omega;C^3)},\ \|\sqrt{\varepsilon}\nabla_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega;C^{3\times 3})} \leq C, \tag{7.28}$$

$$\|s^1_\varepsilon\|_{L^\infty(\Omega;C)},\ \|\mathbf{s}^2_\varepsilon\|_{L^\infty(\Omega;C^3)} \leq c, \tag{7.29}$$

and

$$\lambda_\varepsilon \to \lambda,\ \limsup_{\varepsilon\to 0} \frac{\mathrm{Re}[\lambda_\varepsilon]}{\sqrt{\varepsilon}} < -c < 0 \tag{7.30}$$

$$v_\varepsilon \to v,\ \mathbf{w}_\varepsilon \to \mathbf{w} \text{ a.a. in } \Omega. \tag{7.31}$$

7.3 Reduction to a finite number of modes

To begin, let us decompose the velocity field as

$$\mathbf{u}_\varepsilon = \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\mathbf{u}_\varepsilon],$$

where $\mathbf{H}$ denotes the Helmholtz projection defined on the *target* domain Ω through (3.15).

Expressing the time derivative $\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon)$ by means of the momentum equation (2.13) and using the uniform estimates obtained in Section 4 we easily obtain that the family of functions

$$t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot)\cdot\varphi \ \mathrm{d}x \text{ is precompact in } C[0,T] \text{ for any } \varphi \in \mathcal{D}(\Omega;R^3),\ \mathrm{div}_x\varphi = 0,$$

from which we immediately conclude that

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho}\mathbf{H}[\mathbf{U}] \text{ in } C_{\mathrm{weak}}([0,T];L^{5/4}(\Omega;R^3)). \tag{7.32}$$

Consequently, by virtue of (6.1), compactness of the imbedding $W^{1,2}(\Omega;R^3) \hookrightarrow\hookrightarrow L^5(\Omega;R^3)$, and the standard Lions-Aubin argument, we get

$$\int_0^T\int_\Omega \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon]\cdot\mathbf{u}_\varepsilon \ \mathrm{d}x\ \mathrm{d}t \to \overline{\varrho}\int_0^T\int_\Omega \mathbf{H}[\mathbf{U}]\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t = \overline{\varrho}\int_0^T\int_\Omega |\mathbf{H}[\mathbf{U}]|^2\ \mathrm{d}x\ \mathrm{d}t. \tag{7.33}$$

On the other hand, we have

$$\int_0^T\int_\Omega \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon]\cdot\mathbf{u}_\varepsilon \ \mathrm{d}x\ \mathrm{d}t = \varepsilon\int_0^T\int_\Omega \mathbf{H}\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]\cdot\mathbf{u}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t + \overline{\varrho}\int_0^T\int_\Omega |\mathbf{H}[\mathbf{u}_\varepsilon]|^2\ \mathrm{d}x\ \mathrm{d}t, \tag{7.34}$$

where, in accordance with (6.1), (6.2), the former integral on the right-hand side tends to zero for $\varepsilon \to 0$. Thus relations (7.33), (7.34) give rise to

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{U}] \text{ (strongly) in } L^2((0,T)\times\Omega;R^3). \tag{7.35}$$

Accordingly the proof of the strong convergence of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ reduces to showing

$$\mathbf{H}^\perp[\mathbf{u}_\varepsilon] \to 0 \text{ in } L^2((0,T)\times\Omega;R^3).$$

Moreover, as the space $W^{1,2}(\Omega;R^3)$ is compactly embedded into $L^2(\Omega;R^3)$, it suffices to show

$$\Big[t \mapsto \int_\Omega \mathbf{u}_\varepsilon\cdot\mathbf{w}\ \mathrm{d}x\Big] \to 0 \text{ in } L^2(0,T),$$

for any fixed $\mathbf{w} = \frac{1}{\lambda}\nabla_x v$, where $[v,\mathbf{w}]$ is a solution of the eigenvalue problem (7.21), (7.24). In addition, since the solutions of (7.21), (7.24) form symmetric pairs $[v,\mathbf{w},\lambda]$, $[v,-\mathbf{w},-\lambda]$, it is enough to show

$$\Big[t \mapsto \int_\Omega \Big(r_\varepsilon v + \mathbf{V}_\varepsilon\cdot\mathbf{w}\Big)\ \mathrm{d}x\Big] \to 0 \text{ in } L^2(0,T), \tag{7.36}$$

where r_ε, $\mathbf{V}_\varepsilon$ satisfy the acoustic equation (7.9), (7.10).

Finally, in view of (7.31), relation (7.36) may be replaced by

$$\Big[t \mapsto \int_\Omega \Big(r_\varepsilon v_\varepsilon + \mathbf{V}_\varepsilon\cdot\mathbf{w}_\varepsilon\Big)\ \mathrm{d}x\Big] \to 0 \text{ in } L^2(0,T), \tag{7.37}$$

where $[v_\varepsilon,\mathbf{w}_\varepsilon]$ are the "approximate" eigenfunctions constructed in Lemma 7.1.

7.4 Strong convergence of the gradient components $\mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$

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In the section we complete the proof of Theorem 3.1 by showing (7.37). To this end, we use of the specific form of the acoustic equation (7.9), (7.10), together with the properties of the approximate eigenfucntions $[v_\varepsilon, \mathbf{w}_\varepsilon]$ established in Lemma 7.1.

A natural idea is, of course, to take v_ε, $\mathbf{w}_\varepsilon$ as test functions in (7.9), (7.10), respectively. Unfortunately, however, the integral identity (7.9) is defined on the larger domain Ω_ε; whence the function v_ε must be *extended* as $\tilde{v}_\varepsilon$ to Ω_ε in such a way that

$$\tilde{v}_\varepsilon \in W^{1,\infty}(R^3),\ \tilde{v}_\varepsilon|_\Omega = v.\ \|\tilde{v}_\varepsilon\|_{W^{1,\infty}(R^3)} \le c\|v_\varepsilon\|_{W^{1,\infty}(\Omega)}, \tag{7.38}$$

where the constant is independent of ε. Accordingly, instead of (7.37), we show

$$\Big[t \mapsto \Big(\int_{\Omega_\varepsilon} r_\varepsilon \tilde{v}_\varepsilon \ \mathrm{d}x + \int_\Omega \mathbf{V}_\varepsilon \cdot \mathbf{w}_\varepsilon \ \mathrm{d}x\Big)\Big] \to 0 \text{ in } L^2(0,T). \tag{7.39}$$

Taking the quantity $\psi(t)\tilde{v}_\varepsilon$, $\psi(t)\mathbf{w}_\varepsilon$, with $\psi \in \mathcal{D}[0,T)$, as a test function in (7.9), (7.10), respectively, we get

$$\int_0^T \Big(\varepsilon\chi_\varepsilon\partial_t\psi + \lambda_\varepsilon\chi_\varepsilon\psi\Big)\ \mathrm{d}t = -\psi(0)\chi_{0,\varepsilon} + \sum_{m=1}^{11} I_m^\varepsilon, \tag{7.40}$$

where we have set

$$\chi_\varepsilon(t) = \int_{\Omega_\varepsilon} r_\varepsilon(t,\cdot)\tilde{v}_\varepsilon \ \mathrm{d}x + \int_\Omega \mathbf{V}_\varepsilon \cdot \mathbf{w}_\varepsilon \ \mathrm{d}x,$$

$$\chi_{0,\varepsilon}(t) = \int_{\Omega_\varepsilon} r_{0,\varepsilon}(t,\cdot)\tilde{v}_\varepsilon \ \mathrm{d}x + \int_\Omega \mathbf{V}_{0,\varepsilon} \cdot \mathbf{w}_\varepsilon \ \mathrm{d}x,$$

and

$$I_1^\varepsilon = \frac{\Lambda}{\omega}\int_0^T \psi \int_{\Omega_\varepsilon} \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon\Big] \cdot \nabla_x\tilde{v}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_2^\varepsilon = -\frac{\Lambda}{\omega} < \sigma_\varepsilon, \psi\tilde{v}_\varepsilon >,$$

$$I_3^\varepsilon = D\int_0^T \psi \int_\Omega \varepsilon^2\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \mathrm{div}_x[[\nabla_x\mathbf{w}_\varepsilon]] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_4^\varepsilon = \int_0^T \psi \int_\Omega \varepsilon^2\Big(\frac{\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta})}{\varepsilon}\Big)[[\nabla_x\mathbf{u}_\varepsilon]] : \nabla_x\vec{w}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_5^\varepsilon = -\int_0^T \psi \int_\Omega \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\mathbf{w}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_6^\varepsilon = \int_0^T \psi \int_\Omega \varepsilon\Big(\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon}\Big)\nabla_x F \cdot \mathbf{w}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_7^\varepsilon = \int_0^T \psi \int_\Omega G_8^\varepsilon \ \mathrm{div}_x\mathbf{w}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_8^\varepsilon = 2\varepsilon\int_0^T \psi \int_{\partial\Omega} \mu(\overline{\vartheta})([[\nabla_x\mathbf{w}_\varepsilon]]\mathbf{u}_\varepsilon) \cdot \mathbf{n} \ \mathrm{d}\sigma$$

$$I_9^\varepsilon = - \int_0^T \psi \int_{\Omega_\varepsilon \setminus \Omega} \mathbf{V}_\varepsilon \cdot \nabla_x \tilde{v}_\varepsilon \ \mathrm{d}\mathbf{x} \ \mathrm{d}t$$

$$I_{10}^\varepsilon = \lambda_\varepsilon \int_0^T \psi \int_{\Omega_\varepsilon \setminus \Omega} r_\varepsilon \tilde{v}_\varepsilon \ \mathrm{d}\mathbf{x} \ \mathrm{d}t,$$

$$I_{11}^\varepsilon = \varepsilon \int_0^T \psi \int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \Big) \ \mathrm{d}x \ \mathrm{d}t$$

where G_8^ε is given by (7.17).

Our next goal is to derive uniform estimates for the integrals I_m^ε, $m = 1, \dots, 11$.

(i) By virtue of Hölder's inequality, we have

$$\Big| \int_{\Omega_\varepsilon} \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x \tilde{v}_\varepsilon \ \mathrm{d}x \Big| \tag{7.41}$$

$$\leq \varepsilon \|\tilde{v}_\varepsilon\|_{W^{1,\infty}(R^3)} \Big[\Big| \int_{\Omega_\varepsilon} \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \Big]_{\rm ess} \Big| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \Big| \ \mathrm{d}x \Big| + \Big| \int_{\Omega_\varepsilon} \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \Big]_{\rm res} \Big| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \Big| \ \mathrm{d}x \Big| \Big]$$

$$= \varepsilon \gamma_{1,1}^\varepsilon, \ \text{with } \{\gamma_1^\varepsilon\}_{\varepsilon > 1} \text{ bounded in } L^q(0,T) \text{ for a certain } q > 1,$$

where we have used estimates (6.20), (6.22), together with the uniform bound on $\nabla_x \tilde{v}_\varepsilon$ established in (7.27), (7.28), and (7.38).

Similarly,

$$\Big| \int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon \cdot \nabla_x \tilde{v}_\varepsilon \ \mathrm{d}x \Big| \tag{7.42}$$

$$\leq \varepsilon \|\tilde{v}_\varepsilon\|_{W^{1,\infty}(R^3)} \Big[\int_{\Omega_\varepsilon} \Big| \Big[\frac{\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm ess} \Big| |\mathbf{u}_\varepsilon| \ \mathrm{d}x$$

$$+ \int_{\Omega_\varepsilon} \Big| \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm res} \mathbf{u}_\varepsilon \Big| \ \mathrm{d}x \ \mathrm{d}t + |s(\overline{\varrho}, \overline{\vartheta})| \int_{\Omega_\varepsilon} \Big[\frac{\varrho_\varepsilon}{\varepsilon} \Big]_{\rm res} |\mathbf{u}_\varepsilon| \ \mathrm{d}x \Big] = \varepsilon \gamma_{1,2}^\varepsilon.$$

Consequently, the uniform estimates (6.13 - 6.15) give rise to

$$\{\gamma_{1,2}^\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q > 1.$$

Summing up (7.41), (7.42) we infer that

$$I_1^\varepsilon = \varepsilon \int_0^T \psi(t) \gamma_1^\varepsilon(t) \ \mathrm{d}t, \text{ with } \{\gamma_1^\varepsilon\}_{\varepsilon > 0} \text{ bounded in } L^q(0,T) \text{ for a certain } q > 1. \tag{7.43}$$

(ii) As a straightforward consequence of (4.19), we get

$$I_2^\varepsilon = \varepsilon^2 < \Gamma_2^\varepsilon, \psi >, \ \text{where } \{\Gamma_2^\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } \mathcal{M}^+[0,T]. \tag{7.44}$$

(iii) It follows from the second equation in (7.25) and the uniform estimates (7.27 - 7.29) that

$$\|\varepsilon \mathrm{div}_x [[\nabla_x \vec{w}_\varepsilon]] \ \|_{L^\infty(\Omega; R^3)} \leq c$$

uniformly for $\varepsilon \to 0$. Thus combining (6.1), (6.2) with continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$I_3^\varepsilon = \varepsilon \int_0^T \psi(t)\gamma_3^\varepsilon(t) \ {\rm d}t, \tag{7.45}$$

where

$$\{\gamma_3^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

(iv) As a consequence of (4.33), (4.34), and (7.28), we have

$$I_4^\varepsilon = \varepsilon^{3/2} \int_0^T \psi(t)\Gamma_4^\varepsilon(t) \ {\rm d}t, \tag{7.46}$$

where

$$\{\Gamma_4^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T).$$

(v) In order to handle I_5^ε, we write

$$\int_0^T \psi \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t$$

$$\int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t + \overline{\varrho} \int_0^T \int_\Omega \varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x,$$

where, by virtue of (4.12), (4.16), and (7.28),

$$\int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t = \varepsilon^{3/2} \int_0^T \psi(t)\Gamma_{5,1}^\varepsilon(t) \ {\rm d}t, \tag{7.47}$$

with

$$\{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^1(0,T).$$

On the other hand, since $\mathbf{w}_\varepsilon$ satisfies the homogeneous Dirichlet boundary conditions on Ω, we get

$$\int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x = - \int_\Omega {\rm div}_x \mathbf{u}_\varepsilon \ \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \ {\rm d}x - \int_\Omega (\nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{w}_\varepsilon \ {\rm d}x. \tag{7.48}$$

In addition, we have

$$\int_\Omega {\rm div}_x \mathbf{u}_\varepsilon \ \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \ {\rm d}x = \int_\Omega {\rm div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\rm ess} \cdot \mathbf{w}_\varepsilon \ {\rm d}x + \int_\Omega {\rm div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\rm res} \cdot \mathbf{w}_\varepsilon \ {\rm d}x,$$

where, by virtue of the uniform bounds (4.18), (4.33),

$$\{{\rm div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\rm ess}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^1(\Omega;R^3)). \tag{7.49}$$

Furthermore,

$$\|{\rm div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\rm res}\|_{L^1(0,T;L^1(\Omega;R^3))}$$
$$\leq c\varepsilon^{2/3} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega;R^{3\times 3}))} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(\Omega;R^3))},$$

where we combined Hölder's inequality with (4.33), the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, and the bound on the measure of the "residual set" established in (4.17).

Finally, applying the same treatment to the last integral on the right-hand side of (7.48), we conclude that

$$I_5^\varepsilon = \varepsilon^{3/2} \int_0^T \psi(t) \Gamma_{5,1}^\varepsilon \ \mathrm{d}t + \varepsilon \int_0^T \psi(t) \gamma_5^\varepsilon(t) \ \mathrm{d}t + \varepsilon^{5/3} \int_0^T \psi(t) \Gamma_{5,2}^\varepsilon \ \mathrm{d}t, \tag{7.50}$$

where

$$\{\gamma_5^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T), \text{ and } \{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0}, \{\Gamma_{5,2}^\varepsilon\}_{\varepsilon>0} \text{ are bounded in } L^1(0,T).$$

(vi) Estimate (6.2) yields immediately

$$I_6^\varepsilon = \varepsilon \int_0^T \psi(t) \gamma_6^\varepsilon(t) \ \mathrm{d}t, \tag{7.51}$$

with

$$\{\gamma_6^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T).$$

(vii) In accordance with the first equation in (7.25) and (7.27), (7.29), we have

$$\|\mathrm{div}_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega)} \leq c.$$

Combining this fact with the uniform estimates established in (4.12 - 4.16) we deduce

$$I_7^\varepsilon = \varepsilon \int_0^T \psi(t) \gamma_7^\varepsilon(t) \ \mathrm{d}t, \tag{7.52}$$

where

$$\{\gamma_7^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

(viii) In order to control the boundary integral I_8^ε, we employ the bounds on the trace of the function $\mathbf{u}_\varepsilon$ established in Proposition 5.1 in order to obtain

$$\Big| \int_{\partial\Omega} ([[\nabla_x \mathbf{w}_\varepsilon]] \mathbf{u}_\varepsilon) \cdot \mathbf{n} \ \mathrm{d}\sigma \Big|$$

$$\leq \varepsilon^{(k-1)/2} \|\mathbf{u}_\varepsilon(t,\cdot)\|_{W^{1,2}(\Omega_\varepsilon;R^3)} \|\sqrt{\varepsilon} \nabla_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega)}.$$

Thus we conclude, by help of (4.33), (7.28), that

$$I_8^\varepsilon = \varepsilon^{1+\frac{k-1}{2}} \int_0^T \psi(t) \gamma_8^\varepsilon, \tag{7.53}$$

where

$$\{\gamma_8^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

(ix) As for I_9^ε, we get

$$\Big| \int_{\Omega_\varepsilon \setminus \Omega} \mathbf{V}_\varepsilon \cdot \nabla_x \tilde{v}_\varepsilon \ \mathrm{d}\mathbf{x} \Big|$$

$$\leq \|\nabla_x \tilde{v}_\varepsilon\|_{L^\infty(R^3)} \Big(\varepsilon \int_{\Omega_\varepsilon} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big| |\mathbf{u}_\varepsilon| \ \mathrm{d}x + \overline{\varrho} \int_{\Omega_\varepsilon \setminus \Omega} |\mathbf{u}_\varepsilon| \ \mathrm{d}\mathbf{x} \Big).$$

Furthermore, by means of Hölder's inequality

$$\int_{\Omega_\varepsilon \setminus \Omega} |\mathbf{u}_\varepsilon| \ \mathrm{d}\mathbf{x} \leq c\varepsilon^{5k/6} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega;R^3)}.$$

Thus, making use of (4.33), (6.2), (7.28), and (7.38) we infer that

$$I_9^\varepsilon = \varepsilon^{\min\{5k/6,1\}} \int_0^T \psi(t) \gamma_9^\varepsilon \ \mathrm{d}t, \tag{7.54}$$

where

$$\{\gamma_9^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

(x) In order to control the integral I_{10}^ε, we write

$$\Big| \int_{\Omega_\varepsilon \setminus \Omega} r_\varepsilon \tilde{v}_\varepsilon \ \mathrm{d}\mathbf{x} \Big|$$

$$\leq c \|\tilde{v}_\varepsilon\|_{L^\infty(R^3)} \Big(\int_{\Omega_\varepsilon \setminus \Omega} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big| \ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon \setminus \Omega} \varrho_\varepsilon \Big| \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big| \ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon \setminus \Omega} \overline{\varrho} |F| \ \mathrm{d}\mathbf{x} \Big),$$

where, furthermore,

$$\int_{\Omega_\varepsilon \setminus \Omega} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big| \ \mathrm{d}\mathbf{x}$$

$$\leq \int_{\Omega_\varepsilon \setminus \Omega} \Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} \Big| \ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} \Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{res}} \Big| \ \mathrm{d}x.,$$

and, similarly,

$$\int_{\Omega_\varepsilon \setminus \Omega} \varrho_\varepsilon \Big| \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big| \ \mathrm{d}\mathbf{x}$$

$$\leq c \Big(\int_{\Omega_\varepsilon \setminus \Omega} \Big(\Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} \Big| + \Big| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} \Big| \ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big| \Big[\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\mathrm{res}} \Big| \ \mathrm{d}x \Big).$$

Thus the uniform bounds (4.12 - 4.16) can be used in order to conclude

$$I_{10}^\varepsilon = \varepsilon^{\min\{k/2,1\}} \int_0^T \psi(t) \gamma_{10}^\varepsilon \ \mathrm{d}t, \tag{7.55}$$

where

$$\{\gamma_{10}^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

(xi)

Finaly, as a direct consequence of (7.29),

$$I_{11}^\varepsilon = \varepsilon \int_0^T \psi(t) \gamma_{11}^\varepsilon \ \mathrm{d}t, \tag{7.56}$$

where

$$\{\gamma_{11}^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

Summing up the previous estimates, we obtain

$$\int_0^T \Big(\varepsilon\chi_\varepsilon\partial_t\psi + \lambda_\varepsilon\chi_\varepsilon\psi\Big)\ \mathrm{d}t \tag{7.57}$$

$$= \varepsilon\int_0^T \beta_1^\varepsilon\psi\ \mathrm{d}t + \varepsilon^2 < \beta_\varepsilon^2, \psi > + \varepsilon^{3/2}\int_0^T \beta_3^\varepsilon\psi\ \mathrm{d}t$$

$$+\varepsilon^{\min\{5k/6,1\}}\int_0^T \beta_4^\varepsilon\psi\ \mathrm{d}t + \varepsilon^{\min\{k/2,1\}}\int_0^T \beta_5^\varepsilon\psi\ \mathrm{d}t,$$

where

$$\{\beta_1^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q>1,$$
$$\{\beta_2^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } \mathcal{M}[0,T],$$
$$\{\beta_3^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T),$$
$$\{\beta_4^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T),$$

and

$$\{\beta_5^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T)$$

for any $\psi \in \mathcal{D}(0,T)$.

Having established all the necessary estimates we are now in a position to show (7.39). To this end, we introduce a family of regularizing kernels

$$\psi_\delta(t) = \tfrac{1}{\delta}\Psi\Big(\tfrac{t}{\delta}\Big),\ \delta\to 0,$$

$$\Psi\in\mathcal{D}(-1,1),\ \Psi\geq 0,\ \textstyle\int_{-1}^1 \Psi(t)\ \mathrm{d}t = 1.$$

Taking ψ_δ as a test function in (7.57), and denoting

$$\chi_{\varepsilon,\delta}(t) = \int_R \chi_\varepsilon(t-s)\psi_\delta(s)\ \mathrm{d}s,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_{\varepsilon,\delta}(t) - \frac{\lambda_\varepsilon}{\varepsilon}\chi_{\varepsilon,\delta}(t) = g^{\varepsilon,\delta}(t), \quad \text{for } t\in(\delta,T-\delta) \tag{7.58}$$

where

$$g^{\varepsilon,\delta} = \beta_1^{\varepsilon,\delta} + \varepsilon\beta_2^{\varepsilon,\delta} + \varepsilon^{1/2}\beta_3^{\varepsilon,\delta} + \varepsilon^{\min\{5k/6-1,0\}}\beta_4^{\varepsilon,\delta} + \varepsilon^{\min\{k/2-1,0\}}\beta_5^{\varepsilon,\delta}$$

and

$$\beta_m^{\varepsilon,\delta}(t) = \int_R \beta_m^\varepsilon(t-s)\psi_\delta(s)\ \mathrm{d}s,\ m=1,\dots,m$$

satisfy the same estimates as β_m^ε, $m=1,\dots,11$, uniformly for $\delta\to 0$.

It follows from (7.58) that

$$\chi_{\varepsilon,\delta}(t) = \exp\Big(\frac{\lambda_\varepsilon}{\varepsilon}(t-\delta)\Big)\chi_{\varepsilon,\delta}(\delta) + \int_\delta^t \exp\Big(\frac{\lambda_\varepsilon}{\varepsilon}(t-s)\Big)g^{\varepsilon,\delta}(s)\ \mathrm{d}s \text{ for } t\in(\delta,T-\delta).$$

Consequently, letting $\delta\to 0$ we deduce

$$|\chi_\varepsilon(\tau)| \leq \exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big]\tau\Big)\mathrm{ess}\sup_{t\in(0,T)}|\chi_\varepsilon(t)| + \sqrt{\varepsilon}c \tag{7.59}$$

$$
\begin{aligned}
&+\int_0^\tau \exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big](\tau-s)\Big)|\beta_1^\varepsilon(s)|\ \mathrm{d}s\\
&+\varepsilon^{\min\{5k/6-1,0\}}\int_0^\tau \exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big](\tau-s)\Big)|\beta_4^\varepsilon(s)|\ \mathrm{d}s\\
&+\varepsilon^{\min\{k/2-1,0\}}\int_0^\tau \exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big](\tau-s)\Big)|\beta_5^\varepsilon(s)|\ \mathrm{d}s
\end{aligned}
$$

Since the eigenvalues $\lambda_e p$ satisfy (7.30) and $\{\beta_1^\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L^q(0,T)$ for some $q>1$, it is easy to check that

$$
\int_0^\tau \exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big](\tau-s)\Big)|\beta_1^\varepsilon(s)|\ \mathrm{d}s \to 0 \text{ as } \varepsilon\to 0
$$

uniformly for $\tau\in[0,T]$.

On the other hand, we check easily that

$$
\tau\mapsto \frac{1}{\sqrt{\varepsilon}}\exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big]\tau\Big) \text{ is bounded in } L^1(0,T).
$$

while

$$
\tau\mapsto \frac{1}{\varepsilon^{1/4}}\exp\Big(\mathrm{Re}\Big[\frac{\lambda_\varepsilon}{\varepsilon}\Big]\tau\Big) \text{ is bounded in } L^2(0,T).
$$

Consequently, as $k>1$, relation (7.39) follows.

Having shown (7.39) we have completed the proof of Theorem 3.1.

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