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Rate independent processes in viscous solids at small strains

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Abstract: So-called generalized standard solids (of Halphen-Nguyen type) involving also activated typically rate-independent processes such as plasticity, damage, or phase transformations, are described as a system of a momentum equilibrium equation and a variational inequality for inelastic evolution of internal-parameter variables. Various definitions of solutions are examined, especially from the viewpoint of ability to combine rate-independent processes and other rate-dependent phenomena, as viscosity or also inertia. If those rate-dependent phenomena are suppressed, then the system becomes fully rate-independent. Illustrative examples are presented, too.

Keywords: energetic solution, weak solution, doubly-nonlinear variational inequalities, plasticity, damage, shape-memory alloys, magnetostriction, piezostriction.

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1 Introduction

The theory of rate-independent processes based on so-called energetic formulation by Mielke at al. [50, 51, 52] has been extensively developed and widely applied in [7, 18, 23, 37, 38, 39, 42, 45, 47, 46, 48]. It is well known that coupling rate-independent processes with some others that are rate dependent brings, in general, serious difficulties, cf. e.g. [20, 36]. In some cases when such processes are coupled rather indirectly, such combination is, however, well possible. For some special cases we refer to [1, 10, 11, 12, 13, 63]. The goal of this contribution is to illustrate this coupling on visco-elasto-inelastic solids at small strains. The rate-independent processes may involve plasticity with hardening, damage, or various phase transformations in shape-memory alloys or other ferroic materials. The rate dependent phenomena are viscosity and inertia.

After formulation of the problem in Section 2, we propose its discretization and derive basic a-priori estimates in Section 3. In Section 4 we introduce various concepts of weak solutions that, however, needs rather restrictive data qualification to prove existence. Therefore, in Section 5, we develop another concept that combines so-called energetic solutions for rate-independent processes with weak solutions for rate-dependent processes. Scaling time, we will show in Section 6 that very slow loading eventually yields in the limit indeed fully rate-independent response. Eventually, Section 7 presents various illustrative examples.

2 Activated processes in generalized standard materials

Using the ansatz of so-called *generalized standard solids* (due to Halphen and Nguyen [31]), we consider a momentum equilibrium equation involving a *viscous-like response* of the material in a Kelvin-Voigt-type rheology and inertia, combined with an inclusion for inelastic evolution of internal-parameter variables. We assume small strains and allow for a so-called *gradient theory* as far as the internal parameters concern. Altogether, we thus have in mind the following system

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\big(\sigma_{\rm vi}+\sigma_{\rm el}\big) = f, \qquad \sigma_{\rm el} = \varphi'_e\big(e(u),z,\nabla z\big), \qquad \sigma_{\rm vi} = \zeta'_2\big(e(\frac{\partial u}{\partial t})\big), \tag{2.1a}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in} \ni 0, \qquad \sigma_{\rm in} = \varphi'_z\big(e(u),z,\nabla z\big) - \mathrm{div}\,\varphi'_Z\big(e(u),z,\nabla z\big), \tag{2.1b}$$

where $u:\Omega\to\mathbb{R}^n$ is a displacement, $z:\Omega\to\mathbb{R}^m$ a vector of certain internal parameters, $\varphi:\mathbb{R}^{n\times n}_{\rm sym}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to[0,+\infty]$ with $\mathbb{R}^{n\times n}_{\rm sym} := \{A\in\mathbb{R}^{n\times n};\ A^\top = A\}$ is a stored energy being a function of small-strain tensor e and the vector z and its spatial gradient for which we will use $Z\in\mathbb{R}^{m\times n}$ in the position of a variable in $\varphi(e,z,Z)$. Further, $\varrho>0$ is a mass density, and $\zeta_2:\mathbb{R}^{n\times n}_{\rm sym}\to[0,+\infty)$ and $\zeta_1:\mathbb{R}^m\to[0,+\infty]$ are (pseudo)potentials of dissipative forces. From φ, one derives the "elastic" stress $\sigma_{\rm el}$ and an "inelastic" driving force $\sigma_{\rm in}$ as said in (2.1). Such z may involve plastic strain, hardening, damage, or volume fractions in various phase transformations, etc. We will assume each ζ_ℓ *homogeneous* of degree ℓ, i.e.

$$\forall\ell=1,2\ \forall r\ge 0\ \forall v:\qquad \zeta_\ell(rv) = r^\ell\zeta_\ell(v). \tag{2.2}$$

As to ζ_2, its homogeneity of degree 2 is just responsible for the viscous-like response and we will consider it just quadratic. Elementary calculus shows the formula for the directional derivative $\zeta'_\ell(a)a$, namely

$$\zeta'_\ell(v)v = \lim_{\varepsilon\to0+}\frac{\zeta_\ell(v+\varepsilon v)-\zeta_\ell(v)}{\varepsilon} = \lim_{\varepsilon\to0+}\frac{(1+\varepsilon)^\ell-1}{\varepsilon}\zeta_\ell(v) = \ell\,\zeta_\ell(v). \tag{2.3}$$

We will confine ourselves to ζ_2 quadratic. Then, without loss of generality, we may consider

$$\zeta_1(\dot z) := \delta^*_S(\dot z) \quad \text{with } S\subset\mathbb{R}^m \text{ convex closed, and} \qquad \zeta_2(\dot e) := \frac12\mathbb{D}\dot e{:}\dot e, \tag{2.4}$$

where δ^*_S is the Legendre-Fenchel conjugate function to the indicator function δ_S of S and $\mathbb{D}:\mathbb{R}^{n\times n}_{\rm sym}\to\mathbb{R}^{n\times n}_{\rm sym}$ is a 4th-order tensor (assumed positive definite and symmetric $\mathbb{D}_{ijkl}=\mathbb{D}_{jikl}=\mathbb{D}_{klij}$). This means $\delta^*_S(\dot z) := \sup_{z\in\mathbb{R}^m}\dot z\cdot z - \delta_S(z) = \sup_{z\in S}\dot z\cdot z$. Assuming S bounded (resp. containing 0 in its interior) makes ζ_1 coercive (resp. bounded). Also, $S=\partial\zeta_1(0)$. Nonsmoothness of ζ_1 at 0, which follows from its homogeneity of degree 1 (except the trivial case where ζ_1 is linear), may describe various *activated processes*, i.e. to trigger z evolving, the driving force $\varphi'_z(e(u),z,\nabla z)$ must exceed a certain activation threshold, namely the boundary of S. Using the identity $(\partial\delta^*_S)^{-1} = \partial\delta_S$, the evolution rule (2.1b) can be rewritten into the form of the so-called *sweeping process*

$$\frac{\partial z}{\partial t} \in \partial\delta_S\big(-\varphi'_z(e(u),z,\nabla z)\big). \tag{2.5}$$

Let us assume the body of the material described by (2.1) to occupy a bounded Lipschitz domain Ω. The problem is to be completed by boundary conditions. Let us consider $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ with Γ_0 and Γ_1 disjoint open sets and $\mathrm{meas}_{n-1}(\Gamma_2) = 0$, and denote $\Sigma_0 := (0,T)\times\Gamma_0$, $\Sigma_1 := (0,T)\times\Gamma_1$, and $\Sigma := (0,T)\times\partial\Omega$, and then consider the boundary conditions

$$u|_{\Sigma_0} = 0, \quad (\sigma_{\rm el} + \sigma_{\rm vi})|_{\Sigma_1}\cdot\nu = 0 \quad \text{and} \quad \varphi'_Z(e(u), z, \nabla z)|_{\Sigma}\cdot\nu = 0 \tag{2.6}$$

where ν denotes the outward normal to the boundary $\partial\Omega$ of Ω.

Let us define, for given spatial profiles e and z, the overall stored energy and the overall potential of the dissipation of z respectively as

$$V(e,z) := \int_\Omega \varphi\big(e, z, \nabla z\big)\,\mathrm{d}x \quad \text{and} \quad R_1(\dot z) := \int_\Omega \zeta_1(\dot z)\,\mathrm{d}x. \tag{2.7}$$

Note that $\sigma_{\rm in}$ is the functional derivative of $V(e,\cdot)$. The weak formulation of (2.1b) can rely on the potentiality and monotonicity of $\partial\zeta_1$, i.e. it reads as

$$\forall v: \qquad \int_0^T \Big\langle V'_z(e(u), z), v - \frac{\mathrm{d}z}{\mathrm{d}t}\Big\rangle + R_1(v)\,\mathrm{d}t \ge \int_0^T R_1\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)\,\mathrm{d}t \tag{2.8}$$

where V'_z denotes the Gâteaux derivative of $V(e,\cdot)$ and $T > 0$ a fixed time horizon. Of course, the test functions v are to be suitably qualified but from a sufficiently large set. The last integral $\int_0^T R_1(\frac{\mathrm{d}z}{\mathrm{d}t})\,\mathrm{d}t = \int_0^T\int_\Omega \zeta_1(\frac{\partial z}{\partial t})\,\mathrm{d}x\mathrm{d}t$ in (2.8) has a sense if $\frac{\partial z}{\partial t} \in L^1(Q;\mathbb{R}^m)$ and $\frac{\partial z}{\partial t} \in \mathrm{dom}(\zeta_1)$, i.e. $\zeta_1(\frac{\partial z}{\partial t})$ is finite, a.e. on $[0,T]\times\Omega$. Note that we admit ζ_1 taking values also $+\infty$, i.e. S unbounded. It then means the variation $\mathrm{Var}_S(z;0,T)$ of the process $z : [0,T] \to L^1(\Omega;\mathbb{R}^m)$ over the time interval $[0,T]$ with respect to the (possibly nonsymmetric) "norm" $\int_\Omega \delta^*_S(\cdot)\,\mathrm{d}x$, namely

$$\int_0^T R_1\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)\,\mathrm{d}t = \int_0^T\int_\Omega \zeta_1\Big(\frac{\partial z}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t = \mathrm{Var}_S(z;0,T) := \sup\sum_{i=1}^k \int_\Omega \delta^*_S\big(z(t_i,x) - z(t_{i-1},x)\big)\,\mathrm{d}x \tag{2.9}$$

where the supremum is taken over all partitions of the type $0 = t_0 < t_1 < \ldots < t_k = T$, $k \in \mathbb{N}$.

Energetics of (2.1) can be obtained by testing formally (2.1a) by $\frac{\partial u}{\partial t}$ and using Green's formula for (2.1) together with (2.6), and by testing (2.1b) by $\frac{\partial z}{\partial t}$. In sum, using also (4.2) below, we thus get

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \varphi\big(e(u), z, \nabla z\big)\mathrm{d}x}_{\substack{\text{kinetic and}\\ \text{stored energy}}} + \underbrace{\int_\Omega \xi\Big(e(\frac{\partial u}{\partial t}), \frac{\partial z}{\partial t}\Big)\mathrm{d}x}_{\substack{\text{rate of}\\ \text{dissipation}}} = \underbrace{\int_\Omega f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x}_{\substack{\text{power of}\\ \text{external force}}} \tag{2.10}$$

where, in view of (2.4),

$$\xi\Big(e(\frac{\partial u}{\partial t}), \frac{\partial z}{\partial t}\Big) := \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e(\frac{\partial u}{\partial t}) + \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)\frac{\partial z}{\partial t} = 2\zeta_2\Big(e(\frac{\partial u}{\partial t})\Big) + \zeta_1\Big(\frac{\partial z}{\partial t}\Big). \tag{2.11}$$

The last equality uses (2.3); note that the different factors in front of ζ's reflect their different degree of homogeneity as specified in (2.2). The equality (2.10) itself uses also the equality $\int_Q \zeta_1(\frac{\partial z}{\partial t})\,\mathrm{d}x\mathrm{d}t = -\int_Q \sigma_{\rm in}\cdot\frac{\partial z}{\partial t}\,\mathrm{d}x\mathrm{d}t$ which follows from (2.8) with $v=0$ and from the arguments around the formula (5.9) below, assuming non-concentration of $\frac{\partial z}{\partial t}$. In general, we will assume the coercivity of the specific stored and the dissipative energies:

$$\exists c_0>0\ \forall e\in\mathbb{R}^{n\times n}_{\rm sym},\ z\in\mathbb{R}^m, Z\in\mathbb{R}^{m\times n}:\ \varphi(e,z,Z)\ge c_0|e|^p+c_0|Z|^q, \tag{2.12a}$$

$$\exists c_1>0\ \forall \dot z\in\mathbb{R}^m:\qquad \zeta_1(\dot z)\ge c_1|\dot z|, \tag{2.12b}$$

$$\exists c_2>0\ \forall \dot e\in\mathbb{R}^{n\times n}_{\rm sym}:\qquad \zeta_2(\dot e)\ge c_2|\dot e|^2 \tag{2.12c}$$

for some $p,q>1$, though often we will confine ourselves to $p=2=q$. In view of (2.4), we can see that (2.12b) means that S contains a neighbourhood of zero while (2.12c) means that $\mathbb{D}$ is positive definite.

3 Time-discretization of the system (2.1)

The conceptual numerical approach to the system (2.1) with (2.4) as well as its theoretical analysis we perform in Sections 4–5 can be based on the semi-discretization in time by a time step $\tau>0$ with a possible “vanishing-viscous” regularization of ζ_1 (if $\gamma>0$) and the recursive increment formula

$$\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}-{\rm div}\,\sigma_\tau^k=f_\tau^k,\qquad \sigma_\tau^k:=\mathbb{D}e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})+\varphi_e'(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k), \tag{3.1a}$$

$$\partial\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\gamma\frac{z_\tau^k-z_\tau^{k-1}}{\sqrt{\tau}}+\varphi_z'\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)-{\rm div}\,\varphi_Z'\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)\ \ni 0, \tag{3.1b}$$

$$u_\tau^k|_{\Gamma_0}=0,\quad \sigma_\tau^k|_{\Gamma_1}\cdot\nu=0\quad\text{and}\quad \varphi_Z'(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k)|_{\partial\Omega}\cdot\nu=0, \tag{3.1c}$$

$$\text{where}\ \ u_\tau^0=u_0,\qquad u_\tau^{-1}=u_0-\tau\dot u_0,\qquad z_\tau^0=z_0, \tag{3.1d}$$

where $k=1,...,T/\tau$. Here $f_\tau^k=\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}f(t)\,\mathrm{d}t$.

We consider evolution on time interval $I:=(0,T)$ with a fixed time horizon $T>0$ and denote $Q:=(0,T)\times\Omega$, $\Sigma:=(0,T)\times\partial\Omega$, and $\bar I:=[0,T]$. We will use standard notation for function space, namely spaces of continuous $\mathbb{R}^k$-valued functions $C(\bar\Omega;\mathbb{R}^k)$, continuously differentiable functions $C^1(\bar\Omega;\mathbb{R}^k)$, Lebesgue spaces $L^p(\Omega;\mathbb{R}^k)$ and Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^k)$, Bochner spaces of X-valued functions $L^p(I;X)$. For Dirichlet boundary condition (2.6), we also introduce the Banach space

$$W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n):=\big\{v\in W^{1,p}(\Omega;\mathbb{R}^n);\ v|_{\Gamma_0}=0\big\}. \tag{3.2}$$

Moreover, we denote by $\mathrm{B}(\bar I;X)$, $\mathrm{BV}(\bar I;X)$, or $C_{\rm w}(\bar I;X)$ the Banach space of the functions $\bar I\to X$ that are bounded measurable, have a bounded variation, or are weakly continuous, respectively; note that all these functions are defined everywhere on $\bar I$. We will use the notation $q'=q/(q{-}1)$ and $q^*=nq/(n{-}q)$ (with $q^*<+\infty$ for $q\ge n$) for the conjugate and the Sobolev critical exponents to q, respectively. Instead of $u(t,\cdot)$ or $z(t,\cdot)$, we will write briefly $u(t)$ or $z(t)$, respectively.

We define the piecewise constant interpolant $(\bar u_\tau,\bar z_\tau)$ by $\bar u_\tau|_{((k-1)\tau,k\tau]}=u_\tau^k$ and $\bar z_\tau|_{((k-1)\tau,k\tau]}=z_\tau^k$ for $k=0,...,T/\tau$, and the piecewise affine interpolant (u_τ,z_τ) by $u_\tau(t)=\frac{t-(k-1)\tau}{\tau}u_\tau^k+$

$\frac{k\tau-t}{\tau}u_\tau^{k-1}$ and $z_\tau(t)=\frac{t-(k-1)\tau}{\tau}z_\tau^k+\frac{k\tau-t}{\tau}z_\tau^{k-1}$ for $t\in[(k{-}1)\tau,k\tau]$ with $k=0,...,T/\tau$. Also, we define $\bar f_\tau$ by $\bar f_\tau|_{((k-1)\tau,k\tau]}=f_\tau^k$. Moreover, let us define $\bar\sigma_{\mathrm{in},\tau}^k\in W^{1,q}(\Omega;\mathbb{R}^m)^*$ by

$$\big\langle\sigma_{\mathrm{in},\tau}^k,v\big\rangle:=\int_\Omega\varphi_z'\big(e(\bar u_\tau^k),\bar z_\tau^k,\nabla\bar z_\tau^k\big)\cdot v+\varphi_Z'\big(e(\bar u_\tau^k),\bar z_\tau^k,\nabla\bar z_\tau^k\big):\nabla v\,\mathrm{d}x \tag{3.3}$$

for $v\in W^{1,q}(\Omega;\mathbb{R}^m)$. Then $\bar\sigma_{\mathrm{in},\tau}$ again denotes the corresponding piecewise constant interpolant in time.

It is not surprising that convexity of φ facilitates a lot of useful consequences but, on the other hand, excludes some interesting applications. This is why we will use weakened properties. Let us remind that, standardly, a function is called semiconvex if it becomes convex after adding a square power of the norm with a sufficiently large weight. Here we will need compromise convexity and semiconvexity. Namely, we call φ to be:

$$e\text{-semiconvex if:}\qquad \exists\ell\ge0:\quad (e,z,Z)\mapsto\varphi(e,z,Z)+\ell|e|^2\quad\text{is convex,} \tag{3.4}$$

$$(e,z)\text{-semiconvex if:}\qquad \exists\ell\ge0:\quad (e,z,Z)\mapsto\varphi(e,z,Z)+\ell|e|^2+\ell|z|^2\text{is convex.} \tag{3.5}$$

Analogously, we could also define (e,z,Z)-semiconvexity, which would then coincide with the mentioned standard semiconvexity. Obviously, convex of φ implies (e)-semiconvexity which further implies (e,z)-semiconvexity which eventually implies the mentioned standard semiconvexity. Note also that (e)-semiconvex or (e,z)-semiconvex functions must be convex in (z,Z) or in Z, respectively. Furthermore, it makes sense to call φ *strictly* (e)-*semiconvex* if "convexity" in (3.4) is replaced by strict convexity. Analogously we define strict (e,z)-semiconvexity by replacing "convexity" in (3.5) by strict convexity.

A nontrivial example for $n=1=m$ is $\varphi(e,z,Z)=ez+\epsilon z^2+\varepsilon Z^2$ with $\epsilon,\varepsilon>0$ which is nonconvex but strictly (e)-semiconvex, satisfying (3.4) for $\ell>\frac{1}{4\epsilon}$, because then just the Jacobian of the mapping from (3.4) is positive definite; note that this Jacobian is constant and equals $\begin{pmatrix}2\ell&1&0\\1&2\epsilon&0\\0&0&2\varepsilon\end{pmatrix}$. Another example of a strictly (e)-semiconvex function is $\varphi(e,z,Z)=\phi(e,z)+\alpha z^2+\varepsilon Z^2$ with $\alpha,\varepsilon>0$ and with ϕ having bounded second derivatives and satisfying $|\phi_{zz}''|\le2\alpha-\epsilon$ with some $\epsilon>0$. Also, the function $\varphi(e,z,Z)=ze^2+\epsilon(e^6{+}z^2{+}Z^2)$ with $\epsilon>0$ is strictly (e)-semiconvex on the domain $\{(e,z,Z);\ z\ge0\}$.

Lemma 3.1 *Let (2.4), (2.12), and let*

$$u_0\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n),\quad \dot u_0\in L^2(\Omega;\mathbb{R}^n),\quad z_0\in W^{1,q}(\Omega;\mathbb{R}^m) \tag{3.6}$$

be such that $\int_\Omega\varphi(u_0,z_0,\nabla z_0)\,\mathrm{d}x<+\infty$, $f\in L^1(I;L^2(\Omega;\mathbb{R}^n))$, φ lower semicontinuous, and let one of the two following conditions holds:

(i) *φ is strictly (e)-semiconvex, $\gamma=0$ in (3.1b), and $\tau\le c_2^2/\ell^2$ with c_2 from (2.12c) and ℓ from (3.4) (or just $\tau>0$ arbitrary if φ is convex and thus $\ell=0$), or*

(ii) *φ is strictly (e,z)-semiconvex, $\gamma>0$ in (3.1b), and $\tau\le\min(c_2^2,\frac14\gamma^2)/\ell^2$ with ℓ from (3.5).*

Then there exists an (even unique) solution $(u_\tau,\bar z_\tau)$ of the recursive scheme (3.1) that *satisfies the estimates*

$$\big\|u_\tau\big\|_{L^\infty(I;W^{1,p}(\Omega;\mathbb{R}^n))\cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))} \le C_1, \tag{3.7a}$$

$$\Big\|\frac{\partial u_\tau}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))} \le C_2 \tag{3.7b}$$

$$\big\|\bar z_\tau\big\|_{L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))\cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))} \le C_3, \tag{3.7c}$$

$$\big\|\bar\sigma_{\mathrm{in},\tau}\big\|_{L^2(Q;\mathbb{R}^m)} \le C_4 \qquad \text{if } S \text{ is bounded}, \tag{3.7d}$$

$$\Big\|\frac{\partial z_\tau}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^m)} \le \frac{C_5}{\tau^{1/4}} \qquad \text{if } \gamma>0, \tag{3.7e}$$

where $C_1,...,C_5 < +\infty$ *and* $\bar\sigma_{\mathrm{in},\tau}$ *defined by means of* $(\bar u_\tau, \bar z_\tau)$ *through (3.3); note that (3.7d) is nontrivial only if* S *is bounded. Also, the following discrete analog of (2.10) integrated over* I *holds:*

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2 + \varphi\big(e(u_\tau(T)), z_\tau(T), \nabla z_\tau(T)\big)\,\mathrm{d}x + \int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big) + \big(2-\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x + \int_Q \bar f_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Moreover, if also

$$\big|\varphi'_e(e,z,Z)\big| \le C\big(1+|e|^p+|z|^{q^*}+|Z|^q\big) \tag{3.9}$$

with some C *and with* q^* *the Sobolev exponent to* q*, then we still have the "dual" estimate of* $\frac{\partial^2}{\partial t^2}u_\tau$ *as a measure, namely*

$$\Big\|\frac{\partial u_\tau}{\partial t}\Big\|_{\mathrm{BV}(\bar I;W^{1,\infty}(\Omega;\mathbb{R}^n)^*)} \le C_6. \tag{3.10}$$

Proof. Due to the assumed modes of convexity and coercivity of φ in both cases (i) and (ii), the existence of the discrete solution (u_τ^k, z_τ^k) can be based on the direct method, i.e. (u_τ^k, z_τ^k) can be taken as a solution to the minimization problem:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\int_\Omega \frac{\tau^2\varrho}{2}\Big|\frac{u-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big|^2 + \tau\zeta_1\Big(\frac{z-z_\tau^{k-1}}{\tau}\Big) + \tau^{3/2}\frac{\gamma}{2}\Big|\frac{z-z_\tau^{k-1}}{\tau}\Big|^2 \\ & \displaystyle\qquad + \tau\zeta_2\Big(e\big(\frac{u-u_\tau^{k-1}}{\tau}\big)\Big) + \varphi\big(e(u),z,\nabla z\big) - f_\tau^k\cdot u\,\mathrm{d}x \\ \text{subject to} & (u,z)\in W^{1,p}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m), \qquad u|_{\Gamma_0}=0.\end{array}\right\} \tag{3.11}$$

Knowing already u_τ^k, let us still consider a modified minimization problem, namely

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\int_\Omega \frac{\varrho}{2}\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\cdot u + \tau\zeta_1\Big(\frac{z-z_\tau^{k-1}}{\tau}\Big) + \tau^{3/2}\frac{\gamma}{2}\Big|\frac{z-z_\tau^{k-1}}{\tau}\Big|^2 \\ & \displaystyle\qquad + \big(1-\sqrt{\tau}\big)\mathbb{D}e\big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\big):e(u) + \tau^{3/2}\zeta_2\Big(e\big(\frac{u-u_\tau^{k-1}}{\tau}\big)\Big) \\ & \displaystyle\qquad + \varphi\big(e(u),z,\nabla z\big) - f_\tau^k\cdot u\,\mathrm{d}x \\ \text{subject to} & (u,z)\in W^{1,p}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m), \qquad u|_{\Gamma_0}=0.\end{array}\right\} \tag{3.12}$$

If τ is small as specified, by (2.12c) and strict (3.4) or (3.5), the functionals in both (3.11) and (3.12) are (even strictly) convex. Clearly, the first-order optimality condition for (3.11) is just (3.1a-c). Denoting by $(\tilde u_\tau^k,\tilde z_\tau^k)$ the (unique) solution to (3.12), writing optimality conditions for it gives

$$\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}-\mathrm{div}\Big(\big(1{-}\sqrt{\tau}\big)\mathbb{D}e\big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\big)\\ +\sqrt{\tau}\mathbb{D}e\big(\frac{\tilde u_\tau^k{-}u_\tau^{k-1}}{\tau}\big)+\varphi_e'\big(e(\tilde u_\tau^k),\tilde z_\tau^k,\nabla\tilde z_\tau^k\big)\Big)=f_\tau^k, \tag{3.13a}$$

$$\partial\zeta_1\Big(\frac{\tilde z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\gamma\frac{\tilde z_\tau^k-z_\tau^{k-1}}{\sqrt{\tau}}+\varphi_z'\big(e(\tilde u_\tau^k),\tilde z_\tau^k,\nabla\tilde z_\tau^k\big)-\mathrm{div}\varphi_Z'\big(e(\tilde u_\tau^k),\tilde z_\tau^k,\nabla\tilde z_\tau^k\big)\ni 0, \tag{3.13b}$$

with the boundary conditions (3.1c) now for $(\tilde u_\tau^k,\tilde z_\tau^k)$ and with $(1{-}\sqrt{\tau})\mathbb{D}e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})+\sqrt{\tau}\mathbb{D}e(\frac{\tilde u_\tau^k-u_\tau^{k-1}}{\tau})+\varphi_e'(e(\tilde u_\tau^k),\tilde z_\tau^k,\nabla\tilde z_\tau^k)$ in place of σ_τ^k, of course. Now, testing the difference of (3.1a) and (3.13a) by $u_\tau^k-\tilde u_\tau^k$ and the difference of (3.1b) and (3.13b) by $z_\tau^k-\tilde z_\tau^k$, in the sum we can see that $z_\tau^k=\tilde z_\tau^k$ and $u_\tau^k=\tilde u_\tau^k$ when taken into account the strict convexity of the underlying potential $(e,z,Z)\mapsto\int_\Omega\varphi(e,z,Z)+\zeta_1(z{-}z_\tau^{k-1})+(\zeta_2(e)+\frac{\gamma}{2}|z|^2)/\sqrt{\tau}\,\mathrm{d}x$ if $\tau>0$ is small as specified above. Then the functional in (3.12) must have bigger or equal value on $(u_\tau^{k-1},z_\tau^{k-1})$ than on (u_τ^k,z_τ^k), which gives

$$\int_\Omega\frac{\varrho}{2}\Big|\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big|^2+\tau\zeta_1\Big(\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big)+\tau\big(2{-}\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\big)\Big)\\ +\tau^{3/2}\frac{\gamma}{2}\Big|\frac{z_\tau^k{-}z_\tau^{k-1}}{\tau}\Big|^2+\varphi\big(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k\big)\,\mathrm{d}x\\ \le\int_\Omega\frac{\varrho}{2}\Big|\frac{u_\tau^{k-1}{-}u_\tau^{k-2}}{\tau}\Big|^2+\varphi\big(e(u_\tau^{k-1}),z_\tau^{k-1},\nabla z_\tau^{k-1}\big)+\tau f_\tau^k\cdot\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\,\mathrm{d}x \tag{3.14}$$

when employing also the algebraic inequality $(u_\tau^k{-}2u_\tau^{k-1}{+}u_\tau^{k-2})\cdot(u_\tau^k{-}u_\tau^{k-1})\ge\frac12|u_\tau^k{-}u_\tau^{k-1}|^2-\frac12|u_\tau^{k-1}{-}u_\tau^{k-2}|^2$. From (3.14) and by coercivity of both ζ_1 and ζ_2 and of φ, one gets (3.7a-c,e) by Hölder's, Young's, Korn's and (discrete) Gronwall's inequalities when using also the qualification of the initial conditions.

To prove the estimate (3.7d), we realize that z_τ^k minimizes also $z\mapsto\int_\Omega\zeta_1(z{-}z_\tau^{k-1})+\frac12\tau^{-1/2}\gamma|z{-}z_\tau^{k-1}|^2+\varphi(e(u_\tau^k),z,\nabla z)\,\mathrm{d}x$ so that $\sigma_{\mathrm{in},\tau}^k+\sqrt{\tau}\gamma\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\in-\partial\zeta_1(z_\tau^k-z_\tau^{k-1})\subset-\partial\zeta_1(0)=-S$; note that $\tau\zeta_1(\frac{z-z_\tau^{k-1}}{\tau})$ is just $\zeta_1(z-z_\tau^{k-1})$ because of the degree-1 homogeneity of ζ_1. If S is bounded and $\gamma=0$, we have $\bar\sigma_{\mathrm{in},\tau}$ bounded even in $L^\infty(Q;\mathbb{R}^m)$ just by $\max_{s\in S}|s|$. If $\gamma>0$, we can use (3.7e) to have $\sqrt{\tau}\gamma\frac{\partial}{\partial t}z_\tau$ bounded in $L^2(Q;\mathbb{R}^m)$ and thus also $\bar\sigma_{\mathrm{in},\tau}$, as claimed in (3.7d).

Moreover, summing (3.14) for $k=1,...,T/\tau$ and forgetting the γ-term, one gets (3.8).

Eventually, by the test by $v\in L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^n))$, we get the estimate (3.10) by Hölder's inequality, namely

$$\Big\|\frac{\partial u_\tau}{\partial t}\Big\|_{\mathrm{BV}(\bar I;W^{1,\infty}(\Omega;\mathbb{R}^n)^*)}\le\sup_{\|v\|_{L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^n))}\le1}\frac{1}{\varrho}\int_Q\mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big):e(v)\\ +\varphi_e'\big(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau\big):e(v)-\bar f_\tau\cdot v\,\mathrm{d}x\mathrm{d}t\\ \le C\Big(\Big\|\nabla\frac{\partial u_\tau}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^{n\times n})}+\big\|\varphi_e'\big(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau\big)\big\|_{L^1(Q)}+\big\|\bar f_\tau\big\|_{L^1(Q)}\Big).$$

□

Remark 3.2 (*Convex stored energy.*) For φ convex, we can use $\gamma = 0$ (as in the (e)-semiconvex case) and we can derive (3.8) even without the factor $1-\sqrt{\tau}$ simply by imitating (2.10)–(2.11), i.e. by testing directly the optimality conditions (3.1) for (3.11) without using (3.12), namely (3.1a) by $\frac{u_\tau^k - u_\tau^{k-1}}{\tau}$ and simultaneously (3.1b) by $\frac{z_\tau^k - z_\tau^{k-1}}{\tau}$. By using the assumed convexity of φ which implies $\int_\Omega \varphi(e(u_\tau^k), z_\tau^k, \nabla z_\tau^k) - \varphi(e(u_\tau^{k-1}), z_\tau^{k-1}, \nabla z_\tau^{k-1})\,\mathrm{d}x \le \int_\Omega \varphi_e'(e(u_\tau^k), z_\tau^k, \nabla z_\tau^k) : e(u_\tau^k - u_\tau^{k-1})\,\mathrm{d}x + \langle \sigma_{\mathrm{in},\tau}^k, z_\tau^k - z_\tau^{k-1}\rangle$, it gives directly (3.14) without the factor $1-\sqrt{\tau}$.

Remark 3.3 (*Nonsmooth stored energy.*) In fact, (3.11) and (3.14) work even for φ nonsmooth and then (3.1) is to be understand rather in a generalized sense as inclusion $\partial R_1(\frac{\partial z_\tau}{\partial t}) + \sqrt{\tau}\gamma\frac{\partial z_\tau}{\partial t} + \bar\sigma_{\mathrm{in},\tau} \ni 0$ with some $\bar\sigma_{\mathrm{in},\tau} \in \partial_z V(e(\bar u_\tau), \bar z_\tau)$.

4 Weak solutions to the system (2.1)

Due to the degree-1 homogeneity of ζ_1, we may expect only L^1-bounds for $\frac{\partial z}{\partial t}$, and then some space-time "L^∞-compactness" of $\sigma_{\mathrm{in}} := V_z'$ would be needed for a direct limit passage (2.8), even if $\frac{\partial z}{\partial t}$ would be understood in a generalized sense as a finitely-additive measure. This is, however, too ambitious except possibly some special cases (see [46] for $L^\infty(I; L^1(\Omega;\mathbb{R}^m))$-estimates of $\frac{\partial z}{\partial t}$ in case of a uniformly convex potential φ and sufficiently small dissipation ζ_1). Hence, eligible notions of generalized solutions seem necessarily to substitute the term $\langle V_z'(e(u), z), \frac{\partial z}{\partial t}\rangle$. Here, however, some problems begin and restrictions are originated.

Let us consider an initial-value problem with the initial conditions

$$u(0) = u_0, \qquad \frac{\partial u}{\partial t}(0) = \dot u_0, \qquad z(0) = z_0. \tag{4.1}$$

As φ does not depend on time, at least formally we can use the identity

$$\int_Q \sigma_{\mathrm{in}}\cdot\frac{\partial z}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_Q \varphi_z'(e(u), z, \nabla z)\cdot\frac{\partial z}{\partial t} + \varphi_Z'(e(u), z, \nabla z) : \nabla\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t \tag{4.2a}$$

$$= \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\Big)\mathrm{d}t - \int_Q \varphi_e'\big(e(u), z, \nabla z\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \tag{4.2b}$$

$$= \int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big) - \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x - \int_Q \sigma_{\mathrm{el}} : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t. \tag{4.2c}$$

Note that the last boundary condition (2.6) has been used for (4.2a). Testing (2.1a) by $\frac{\partial u}{\partial t}$ and using also (2.3) for ζ_2 give

$$-\int_Q \sigma_{\mathrm{el}} : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t = \int_Q 2\zeta_2\Big(e\Big(\frac{\partial u}{\partial t}\Big)\Big) - f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 - \frac{\varrho}{2}|\dot u_0|^2\mathrm{d}x\,. \tag{4.3}$$

Combination of (4.2) and (4.3) then gives

$$\int_Q \sigma_{\mathrm{in}}\cdot\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = \int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi\big(e(u(T)), z(T), \nabla z(T)\big)\,\mathrm{d}x$$
$$- \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x - \int_Q f\cdot\frac{\partial u}{\partial t} + 2\zeta_2\Big(e\Big(\frac{\partial u}{\partial t}\Big)\Big)\,\mathrm{d}x\mathrm{d}t. \tag{4.4}$$

Using (4.4) for (2.8), we obtain

$$\int_Q \sigma_{\rm in}\cdot v + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \ge \mathrm{Var}_S(z;0,T) + \int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi\big(e(u(T)),z(T),\nabla z(T)\big)\,\mathrm{d}x$$
$$+2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t - \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x - \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t \quad (4.5)$$

for all v, where $\sigma_{\rm in} = \varphi'_z(e,z,\nabla z) - \mathrm{div}\,\varphi'_Z(e(u),z,\nabla z)$ as in (2.1b) and where $\mathrm{Var}_S(z;0,T)$ is from (2.9). Note that, for $v=0$, yields an energy balance (as an inequality), which is related to the homogeneity of ζ_1 (cf. (6.19) below where this does not work).

As (4.5) is equivalent to the inclusion (2.1b) if u and z are smooth and satisfy (2.1a), we can base the definition of a weak solution to (2.1) on the identity (4.5) and on a conventional weak formulation of (2.1a):

Definition 4.1 *Let φ be smooth with*

$$\forall\,(e,z,Z)\in\mathbb{R}^{n\times n}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}:\quad \big|\varphi(e,z,Z)\big| \le C\big(1+|e|^p+|z|^{q^*}+|Z|^q\big), \quad (4.6a)$$
$$\big|\varphi'(e,z,Z)\big| \le C\big(1+|e|^p+|z|^{q^*}+|Z|^q\big), \quad (4.6b)$$

cf. also (3.9). We call a couple (u,z) with

$$u\in C_{\rm w}(\bar I; W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)) \cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n)), \quad (4.7a)$$
$$\frac{\partial u}{\partial t}\in C_{\rm w}(\bar I;L^2(\Omega;\mathbb{R}^n)), \quad (4.7b)$$
$$z\in \mathrm{B}(\bar I;W^{1,q}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m)) \quad (4.7c)$$

a weak solution to (2.1) with the initial/boundary conditions (4.1) and (2.6) if

(i) *the (weakly formulated) momentum equilibrium (2.1a), namely*

$$\int_Q \Big(\mathbb{D}e(\frac{\partial u}{\partial t}) + \varphi'_e(e(u),z,\nabla z)\Big):e(v) - \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial t} - f\cdot v\,\mathrm{d}x\mathrm{d}t = \int_\Omega \varrho\dot u_0\cdot v(0)\,\mathrm{d}x \quad (4.8)$$

holds for any $v\in C^1(\bar Q;\mathbb{R}^n)$ such that $v(T)=0$ and $v|_{\Gamma_0}=0$,

(ii) *(4.5) holds for any $v\in C^1(\bar Q;\mathbb{R}^m)$ in the weak sense, i.e.*

$$\int_Q \varphi'_z\big(e(u),z,\nabla z\big)\cdot v + \varphi'_Z\big(e(u),z,\nabla z\big) : \nabla v + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t$$
$$\ge \mathrm{Var}_S(z;0,T) + \int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi\big(e(u(T)),z(T),\nabla z(T)\big)\,\mathrm{d}x$$
$$+\,2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t - \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x - \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t, \quad (4.9)$$

where $\mathrm{Var}_S(z;0,T)$ is from (2.9), and

(iii) *$u(0)=u_0$ and $z(0)=z_0$.*

Remark 4.2 Note that $z\in\mathrm{B}(\bar I;W^{1,q}(\Omega;\mathbb{R}^m))$ makes sense both $\nabla z(T)$ and $z(0)=z_0$, used in (4.9). Also (4.7b) makes sense $\frac{\partial u}{\partial t}(T)$ in (4.9) if $\varrho>0$ while for $\varrho=0$ this term just disappears from (4.9). Also not that (4.6) makes all integrands integrable.

Proposition 4.3 *Let (4.6) and the assumptions of Lemma 3.1 hold, and let*

$$\begin{aligned}&\varphi \text{ satisfy (2.12a) with } p=2=q, \text{ and have the form}\\ &\varphi(e,z,Z)=\phi_0(z)\cdot\phi_1(e,Z)+\phi_2(e,z,Z) \text{ with } \phi_1 \text{ affine, } \phi_2 \text{ quadratic, and}\\ &|\phi_0(z)|\le C(1+|z|^{q^*-\epsilon}) \quad\text{and}\quad |\phi_0'(z)|\le C(1+|z|^{q^*/2-\epsilon}) \text{ for some } \epsilon>0.\end{aligned} \tag{4.10}$$

Then there is a weak solution due to Definition 4.1.

Proof. Testing (3.1a) by some v^k and using the discrete "by-part" summation

$$\sum_{k=1}^{T/\tau}(u^k-2u^{k-1}+u^{k-2})v^k=\big(u^{T/\tau}-u^{T/\tau-1}\big)v^{T/\tau}-(u^0-u^{-1})v^1-\sum_{k=2}^{T/\tau}(u^{k-1}-u^{k-2})(v^k-v^{k-1}), \tag{4.11}$$

we obtain the discrete variant of (4.8), namely

$$\begin{aligned}&\int_Q \mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big):e(\bar v_\tau)+\varphi_e'\big(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau\big):e(\bar v_\tau)-\bar f_\tau\cdot\bar v_\tau\,\mathrm{d}x\mathrm{d}t\\ &-\int_\tau^T\!\!\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(\cdot-\tau)\cdot\frac{\partial v_\tau}{\partial t}\mathrm{d}x\mathrm{d}t+\int_\Omega\varrho\frac{\partial u_\tau}{\partial t}(T)\cdot v_\tau(T)\,\mathrm{d}x=\int_\Omega\varrho\dot u_0\cdot v_\tau(\tau)\,\mathrm{d}x\end{aligned} \tag{4.12}$$

where $\bar v_\tau$ and v_τ denote respectively the piecewise constant and the piecewise affine interpolants of the $\{v^k\}_{k=0}^{T/\tau}$ on the equidistant partition of $[0,T]$ with the time step τ. Also, in view of (3.1b), we obtain the discrete variant of (2.8):

$$\begin{aligned}\int_Q \varphi_z'(\bar e_\tau,\bar z_\tau,\nabla\bar z_\tau)\cdot\Big(v-\frac{\partial z_\tau}{\partial t}\Big)+\varphi_Z'(\bar e_\tau,\bar z_\tau,\nabla\bar z_\tau):\nabla\Big(v-\frac{\partial z_\tau}{\partial t}\Big)&\\ +\zeta_1(v)+\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t}\cdot v\,\mathrm{d}x\mathrm{d}t\ge\int_Q\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t&\end{aligned} \tag{4.13}$$

for all v. We obtain also the discrete variant of (4.9) in the sense that

$$\begin{aligned}&\int_Q\varphi_z'\big(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau\big)\cdot v+\varphi_Z'\big(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau\big):\nabla v+\zeta_1(v)+\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t}\cdot v\,\mathrm{d}x\mathrm{d}t\\ &\qquad\ge\int_Q\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+\big(2-\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\ &\qquad+\int_\Omega\frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2+\varphi\big(e(u_\tau(T)),z_\tau(T),\nabla z_\tau(T)\big)\,\mathrm{d}x\\ &\qquad-\int_\Omega\frac{\varrho}{2}|\dot u_0|^2+\varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x-\int_Q\bar f_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.14}$$

for all v. This follows by realizing that the right-hand side of (4.14) is nonpositive by (3.8) while the left-hand side of (4.14) is non-negative. The last fact is just $\langle\bar\sigma_{\mathrm{in},\tau}+\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t},v\rangle+\int_Q\zeta_1(v)\,\mathrm{d}x\mathrm{d}t\ge0$ where $\bar\sigma_{\mathrm{in},\tau}$ is determined through (3.3), which holds because $\bar\sigma_{\mathrm{in},\tau}+\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t}\in-\partial\zeta_1(\frac{\partial u_\tau}{\partial t})\subset-\partial\zeta_1(0)$; here (4.13) and the degree-1 homogeneity of ζ_1 are exploited.

For convergence both in (4.12) and in (4.14), we choose subsequences converging weakly* in topologies indicated in (3.7)–(3.10). Also, for (4.12), we consider a test function v as in (4.8) and the corresponding interpolants $\bar v_\tau$ and v_τ. Thus we have $v_\tau\to v$ strongly in $C^1(\bar Q;\mathbb{R}^n)$ and $\bar v_\tau\to v$ strongly in $L^\infty(I;C^1(\bar\Omega;\mathbb{R}^n))$. Also, we need

$\varrho\frac{\partial u_\tau}{\partial t}(\cdot-\tau) \to \varrho\frac{\partial u}{\partial t}$ weakly in $L^2(Q;\mathbb{R}^n)$, which is seen from (3.7b) when using also (3.10) which yields $\|\varrho\frac{\partial u_\tau}{\partial t}(\cdot-\tau) - \varrho\frac{\partial u_\tau}{\partial t}\|_{L^1(\tau,T;W^{1,\infty}(\Omega;\mathbb{R}^n)^*)} \le \tau\|\varrho\frac{\partial u_\tau}{\partial t}\|_{\mathrm{BV}(\bar I;W^{1,\infty}(\Omega;\mathbb{R}^n)^*)} \to 0$. By (3.7) and Aubin-Lions' theorem (suitably generalized as [56, Cor.7.9]), we have $\bar z_\tau \to z$ in $L^{r_1}(I;L^{r_2}(\Omega;\mathbb{R}^m))$ for any $r_1 < +\infty$ and $r_2 < 2n/(n-2)$ (or just $r_2 < +\infty$ for $n \ge 2$); hence $\bar z_\tau \to z$ in $L^{q^*-\epsilon}(Q;\mathbb{R}^m)$ for $\epsilon > 0$ and here $q = 2$. Due to our simplifying assumptions (4.10), the equation (2.1a) is semilinear and weak convergence suffices for the limit passage in (4.12) to obtain (4.8) is easy; note that $v_\tau(T) = 0$ is considered and the growth (4.10) of ϕ_0 is used for $\varphi'_e(e,z,Z) = \phi_0(z)\cdot[\phi_1]'_e + [\phi_2]'_e(e,z,Z)$.

Let us now pass to the limit in (4.14). As to (3.7b), in fact, we have more than a mere $L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))$ bound: even $z_\tau(t)$ is bounded in $W^{1,q}(\Omega;\mathbb{R}^m)$ uniformly for $t \in [0,T]$ and $\tau > 0$. Due to the BV-bound, by Helly's principle (generalized for infinite-dimensional spaces, see [48, Theorem A.1], in particular here using that bounded sets in $W^{1,q}(\Omega;\mathbb{R}^m)$ are weakly sequentially compact), we can consider $z_\tau(t) \to z(t)$ weakly in $W^{1,q}(\Omega;\mathbb{R}^m)$ for all $t \in [0,T]$ and $z \in \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))$. Similarly, using (3.10) and Helly's principle once again, we can also see that $\frac{\partial u_\tau}{\partial t}(t)$ converges to $\frac{\partial u}{\partial t}(t)$ weakly in $L^2(\Omega;\mathbb{R}^n)$ for all $t \in [0,T]$.

By (4.10), also $\varphi'_z(e,z,Z) = \phi'_0(z)\cdot\phi_1(e,Z) + [\phi_2]'_z(e,z,Z)$ is affine in terms of (e,Z) and $\varphi'_z(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau)$ thus converges weakly in $L^1(Q;\mathbb{R}^m)$ to $\varphi'_z(e(u),z,\nabla z)$ because of the growth of ϕ'_0 in (4.10). Also $\varphi'_Z(e,z,Z) = \phi_0(z)\cdot[\phi_1]'_Z + [\phi_2]'_Z(e,z,Z)$ is affine in terms of (e,Z) and $\varphi'_Z(e(\bar u_\tau),\bar z_\tau,\nabla\bar z_\tau)$ thus converges weakly in $L^1(Q;\mathbb{R}^{m\times n})$ to $\varphi'_Z(e(u),z,\nabla z)$. We also showed that $\frac{\partial u_\tau}{\partial t}(T)$ converges to $\frac{\partial u}{\partial t}(T)$ weakly in $L^2(\Omega;\mathbb{R}^n)$, and we thus can use weak lower-semicontinuity of the functional $\int_\Omega \frac{\varrho}{2}|\cdot|^2\mathrm{d}x$. Further, we use convexity of both ζ_1 and ζ_2 and then the weak* lower-semicontinuous argument. In particular,

$$\liminf_{\tau\to0}\int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)\mathrm{d}x\mathrm{d}t \ge \mathrm{Var}_S(z;0,T). \tag{4.15}$$

If $\gamma > 0$, we must still prove $\int_Q \gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t}\cdot v\,\mathrm{d}x\mathrm{d}t \to 0$, which is however easily seen from (3.7e) which makes this term $\mathcal{O}(\tau^{1/4})$ if v is smooth. Also, we use $\bar f_\tau \to f$ in $L^1(I;L^2(\Omega;\mathbb{R}^n))$. The limit passage in (4.14) to (4.9) is then proved. □

Alternatively, for $\varphi(e,\cdot,\cdot)$ nonsmooth, following [17], we can use (4.5) with $\sigma_{\mathrm{in}} \in \partial_z V(e(u),z)$ with V from (2.7), and then, assuming convexity of $V(e,\cdot)$, formulate the inclusion $\sigma_{\mathrm{in}} \in \partial_z V(e(u),z)$ weakly, see (4.17) below. Thus we come to the alternative definition:

Definition 4.4 *Let φ satisfy (3.9), $q^* \ge 2$, $\int_\Omega \varphi(e(u_0),z_0,\nabla z_0)\,\mathrm{d}x < +\infty$, and $\varphi(e,\cdot,\cdot)$ be convex. We call a triple $(u,z,\sigma_{\mathrm{in}})$ with*

$$u \in C_{\mathrm{w}}(\bar I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)) \cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n)), \tag{4.16a}$$

$$\frac{\partial u}{\partial t} \in C_{\mathrm{w}}(\bar I;L^2(\Omega;\mathbb{R}^n)), \tag{4.16b}$$

$$z \in \mathrm{B}(\bar I;W^{1,q}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m)), \tag{4.16c}$$

$$\sigma_{\mathrm{in}} \in L^2(Q;\mathbb{R}^m), \tag{4.16d}$$

a weak solution to (2.1) with the initial/boundary conditions (4.1) and (2.6) if

(i) *(4.8) holds for any $v \in C^1(\bar Q;\mathbb{R}^n)$ such that $v(T) = 0$ and $v|_{\Gamma_0} = 0$,*

(ii) *(4.5) holds for any $v \in C^1(\bar Q;\mathbb{R}^m)$,*

(iii) *for any* $v \in C^1(\bar{Q};\mathbb{R}^m)$,

$$\int_Q \sigma_{\rm in}\cdot(z-v)+\varphi\big(e(u),v,\nabla v\big)\,\mathrm{d}x\mathrm{d}t \geq \int_Q \varphi\big(e(u),z,\nabla z\big)\,\mathrm{d}x\mathrm{d}t, \tag{4.17}$$

(iv) $u(0)=u_0$ *and* $z(0)=z_0$.

Because of the term $\varphi(e(u),v,\nabla v)$ on the left-hand side of (4.17), we now have more benefit from a possible strong convergence of $e(\bar{u}_\tau)$ than before. Hence, we further modify also the technique of the existence proof, which now will mostly be based on strong convergence of $e(\bar{u}_\tau)$. To this aim, we qualify $\varphi(\cdot,z,Z)$ as smooth function with a "p-strongly monotone" gradient in the sense, with some $\alpha>0$ (independent of z and Z),

$$\forall e,\tilde{e}\in\mathbb{R}^{n\times n}_{\rm sym}:\quad \alpha\big(|e|^{p-2}e-|\tilde{e}|^{p-2}\tilde{e}\big):(e-\tilde{e}) \leq \big(\varphi'_e(e,z,Z)-\varphi'_e(\tilde{e},z,Z)\big):(e-\tilde{e}). \tag{4.18}$$

An example $\varphi(e,z)=\phi(z)|e|^p$ with $p>1$ satisfies (4.18) with $\alpha:=p\inf_{\mathbb{R}^m}\phi(\cdot)>0$.

Proposition 4.5 *Let the assumptions of Lemma 3.1 and in Definition 4.4 hold, let*

$$\begin{aligned}&\varphi \text{ be of the form } \varphi(e,z,Z)=\phi_1(e,z)+\phi_2(z,Z), \text{ and}\\ &\quad\text{satisfy (4.18) and (2.12a) with } p>1 \text{ and } q>1,\\ &\quad\text{and } \big|[\phi_1]'_e(e,z)\big| \leq C\big(1+|e|^{p-1}+|z|^{q^*/p'-\epsilon}\big) \text{ with some } \epsilon>0, \qquad (4.19a)\\ &\zeta_1:\mathbb{R}^m\to[0,+\infty) \text{ be continuous.} \qquad (4.19b)\end{aligned}$$

Then there is a weak solution due to Definition 4.4.

Let us note that (4.19a) admits ϕ_1 valued in $\mathbb{R}\cup\{+\infty\}$, containing e.g. the terms $a(z)+\delta_K(z)$ with $a:\mathbb{R}^m\to\mathbb{R}$ convex continuous and δ_K the indicator function of a nonempty closed set $K\subset\mathbb{R}^m$, cf. Examples 7.2, 7.4, and 7.5 below. Note that these terms do not affect φ'_e so that (4.19a) can still be satisfied.

Proof of Proposition 4.5. Again like in the proof of Proposition 4.3, we obtain an analog of (4.14) in the sense

$$\begin{aligned}\int_Q \bar{\sigma}_{{\rm in},\tau}\cdot v+\zeta_1(v)+\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t}\cdot v\,\mathrm{d}x\mathrm{d}t \geq \int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+\big(2-\sqrt{\tau}\big)\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)-\bar{f}_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t\\ +\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2+\varphi\big(e(u_\tau(T)),z_\tau(T),\nabla z_\tau(T)\big)-\frac{\varrho}{2}|\dot{u}_0|^2-\varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.20}$$

where $\bar{\sigma}_{{\rm in},\tau}\in L^2(Q;\mathbb{R}^m)$ satisfies the discrete analog of (4.17), i.e.

$$\int_Q \bar{\sigma}_{{\rm in},\tau}\cdot(v-\bar{z}_\tau)+\varphi\big(e(\bar{u}_\tau),v,\nabla v\big)\,\mathrm{d}x\mathrm{d}t \geq \int_Q \varphi\big(e(\bar{u}_\tau),\bar{z}_\tau,\nabla\bar{z}_\tau\big)\,\mathrm{d}x\mathrm{d}t. \tag{4.21}$$

Here $\sigma_{{\rm in},\tau}$ just exists in $\partial_z V(e(\bar{u}_\tau),\bar{z}_\tau)\cap(-\partial R_1(\frac{\mathrm{d}z_\tau}{\mathrm{d}t})-\gamma\sqrt{\tau}\frac{\partial z_\tau}{\partial t})$; cf Remark 3.3. Besides, we have also (4.12). Note that, in view of (4.19), $\varphi'_e(e(\bar{u}_\tau),\bar{z}_\tau,\nabla\bar{z}_\tau)=\varphi'_e(e(\bar{u}_\tau),\bar{z}_\tau)$ in (4.12).

We select subsequences $\{\bar{u}_\tau\}$ and $\{\bar{z}_\tau\}$ as in the proof of Proposition 4.3 and, now, by (3.7c), we can still select $\sigma_{\rm in}\in L^2(Q;\mathbb{R}^m)$ and a subsequence so that $\bar{\sigma}_{{\rm in},\tau}\to\sigma_{\rm in}$ weakly in $L^2(Q;\mathbb{R}^m)$. Let us realize that, in view of (2.4), ζ_1 continuous is just related with S bounded so that (3.7d) yields indeed an L^2-estimate for $\bar\sigma_{\mathrm{in},\tau}$.

We want to pass to the limit in (4.12). Let us take v_τ and $\bar v_\tau$ respectively a piecewise affine approximation of u and the corresponding approximation piecewise constant in time on the partition of $[0,T]$ such that $v_\tau \to u$ strongly in $L^p(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)) \cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$ and $\bar v_\tau \to u$ strongly in $L^p(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$; such approximation is always possible since u lies in this space due to (3.7a). Using the p-strong monotonicity (4.18) and the identity (4.12) tested by $u_\tau - v_\tau$ and $\bar u_\tau - \bar v_\tau$ instead of v_τ and $\bar v_\tau$, respectively, we obtain

$$\begin{aligned}
\alpha\Big(\big\|e(\bar u_\tau)\big\|^{p-1}_{L^p(Q;\mathbb{R}^{n\times n})} - \big\|e(\bar v_\tau)\big\|^{p-1}_{L^p(Q;\mathbb{R}^{n\times n})}\Big)\Big(\big\|e(\bar u_\tau)\big\|_{L^p(Q;\mathbb{R}^{n\times n})} - \big\|e(\bar v_\tau)\big\|_{L^p(Q;\mathbb{R}^{n\times n})}\Big)&\\
\le \int_Q \alpha\big(|e(\bar u_\tau)|^{p-2}e(\bar u_\tau) - |e(\bar v_\tau)|^{p-2}e(\bar v_\tau)\big){:}e(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t&\\
\le \int_Q \big(\varphi_e'(e(\bar u_\tau),\bar z_\tau) - \varphi_e'(e(\bar v_\tau),\bar z_\tau)\big) : e(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t&\\
= \int_Q \bar f_\tau\cdot(\bar u_\tau - \bar v_\tau) - \mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar u_\tau - \bar v_\tau) - \varphi_e'(e(\bar v_\tau),\bar z_\tau) : e(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t&\\
+ \int_\tau^T\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(\cdot - \tau)\cdot\frac{\partial(u_\tau - v_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t&\\
- \int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(T)\cdot[u_\tau - v_\tau](T) - \varrho\dot u_0\cdot\big[u_\tau - v_\tau\big](\tau)\,\mathrm{d}x \to 0.&
\end{aligned} \tag{4.22}$$

We are to prove the last convergence. In fact, it suffices to prove that the limit superior is nonpositive. Obviously, $\int_Q \bar f_\tau\cdot(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t \to 0$. As for the $\mathbb{D}$-term, we have

$$\begin{aligned}
\limsup_{\tau\to0}\int_Q -\mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar u_\tau - \bar v_\tau)\,\mathrm{d}x\mathrm{d}t &\le -\liminf_{\tau\to0}\int_\Omega \frac12\mathbb{D}e(u_\tau(T)) : e(u_\tau(T))\,\mathrm{d}x\\
&\quad + \int_\Omega \frac12\mathbb{D}e(u_0) : e(u_0)\,\mathrm{d}x - \lim_{\tau\to0}\int_Q \mathbb{D}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e(\bar v_\tau)\,\mathrm{d}x\mathrm{d}t\\
\le -\int_\Omega \frac12\mathbb{D}e(u(T)) : e(u(T)) + \frac12\mathbb{D}e(u_0) : e(u_0)\,\mathrm{d}x &- \int_Q \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e(u)\,\mathrm{d}x\mathrm{d}t = 0.
\end{aligned} \tag{4.23}$$

The first inequality in (4.23) used $\mathbb{D}e(u^k - u^{k-1}) : e(u^k) \ge \frac12\mathbb{D}e(u^k) : e(u^k) - \frac12\mathbb{D}e(u^{k-1}) : e(u^{k-1})$ so that $\int_Q \mathbb{D}e(\frac{\partial u_\tau}{\partial t}) : e(\bar u_\tau)\,\mathrm{d}x\mathrm{d}t \ge \frac12\int_\Omega \mathbb{D}e(u_\tau(T)) : e(u_\tau(T)) - \mathbb{D}e(u_0) : e(u_0)\,\mathrm{d}x$.

We use the Aubin-Lions theorem (again generalized as [56, Cor.7.9]) so that $\bar z_\tau \to z$ strongly in $L^{q^*-\epsilon}(Q;\mathbb{R}^m)$ with $\epsilon > 0$. This gives $\varphi_e'(e(\bar v_\tau),\bar z_\tau)) : e(\bar u_\tau - v_\tau) \to \varphi_e'(e(u),z)) : e(u-u) = 0$ weakly in $L^1(Q)$; here the growth (4.19a) of φ_e' has been used.

Also we use $\frac{\partial u_\tau}{\partial t} \to \frac{\partial u}{\partial t}$ strongly in $L^2(Q;\mathbb{R}^n)$, which can be proved by Aubin-Lions theorem (again generalized as [56, Cor.7.9]) based on the estimate of $\frac{\partial u_\tau}{\partial t}$ in $L^2(I;W^{1,2}(\Omega;\mathbb{R}^n)) \cap \mathrm{M}(\bar I;W^{1,2}(\Omega;\mathbb{R}^n)^*)$ from (3.7a) and (3.10) where $\mathrm{M}(\bar I;X)$ denotes the space of X-valued measures on $\bar I = [0,T]$. Thus $\int_\tau^T\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(\cdot - \tau)\cdot\frac{\partial(u_\tau - v_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t \to \int_Q \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial(u-u)}{\partial t}\,\mathrm{d}x\mathrm{d}t = 0$.

Further we use also $\frac{\partial u_\tau}{\partial t}(T) \to \frac{\partial u}{\partial t}(T)$ weakly in $L^2(\Omega;\mathbb{R}^n)$ and $u_\tau(T) \to u(T)$ weakly in $W^{1,2}(\Omega;\mathbb{R}^n)$ hence strongly in $L^2(\Omega;\mathbb{R}^n)$, so that $\int_\Omega \varrho\frac{\partial u_\tau}{\partial t}(T)\cdot[u_\tau - v_\tau](T)\,\mathrm{d}x \to \int_\Omega \varrho\frac{\partial u}{\partial t}(T)\cdot [u-u](T)\,\mathrm{d}x = 0$.

Eventually, for limiting the last term in (4.22) we need $u_\tau(\tau) = u_\tau^1 \to u_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$ and $v_\tau(\tau) \to u_0$ in $L^2(\Omega;\mathbb{R}^n)$.

Altogether, from (4.22), we get $\|e(\bar{u}_\tau)\|_{L^p(Q;\mathbb{R}^{n\times n})} \to \|e(u)\|_{L^p(Q;\mathbb{R}^{n\times n})}$. As we already know that $e(\bar{u}_\tau) \to e(u)$ weakly in $L^p(Q;\mathbb{R}^{n\times n})$, by the well-known fact that $L^p(Q;\mathbb{R}^{n\times n})$ is a uniformly convex space, we obtain $e(\bar{u}_\tau) \to e(u)$ strongly in $L^p(Q;\mathbb{R}^{n\times n})$. Then the limit passage in (4.12) is easy and we thus obtain (4.8).

The limit passage in (4.20) to (4.14) is just by weak lower semicontinuity, similarly as in the proof of Proposition 4.3.

Eventually, the limit passage in (4.21) uses the already established convergence $\bar{z}_\tau \to z$ strongly in $L^1(Q;\mathbb{R}^m)$, $\bar{\sigma}_{\text{in},\tau} \to \sigma_{\text{in}}$ weakly in $L^2(Q;\mathbb{R}^m)$, and $e(\bar{u}_\tau) \to e(u)$ strongly in $L^p(Q;\mathbb{R}^{n\times n})$. □

Remark 4.6 If φ is convex, (4.14) can be obtained more straightforwardly by testing (3.1a) by $\frac{u_\tau^k - u_\tau^{k-1}}{\tau}$ and imitating the procedure (4.2)–(4.4). The convexity of φ yields the inequality "$\geq$" in (4.2b) and thus also in (4.4) which also uses the inequality "$\geq$" in (4.3) due to convexity of $\dot{u} \mapsto \frac{\varrho}{2}|\dot{u}|^2$.

Remark 4.7 (*Concepts based on maximal monotonicity.*) Alternatively, following [17], we can rely on maximal monotonicity of the multivalued mapping $\partial\zeta_1$ without benefiting from its potentiality, i.e. the inclusion in (2.1b) leads to $\int_Q (\omega + \sigma_{\text{in}}) \cdot (v - \frac{\partial z}{\partial t})\,\mathrm{d}x\mathrm{d}t \geq 0$ for all v and ω such that $\omega \in \partial\zeta_1(v)$ a.e. on Q. Then we could avoid (2.9). E.g., instead of (4.5), by using (4.4), we would obtain

$$\int_Q (\omega + \sigma_{\text{in}}) v \,\mathrm{d}x\mathrm{d}t - \Big\langle \omega, \frac{\partial z}{\partial t} \Big\rangle \geq \int_\Omega \frac{\varrho}{2} \Big| \frac{\partial u}{\partial t}(T) \Big|^2 + \varphi\big(e(u(T)), z(T), \nabla z(T)\big)\,\mathrm{d}x$$
$$- \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x - \int_Q f \cdot \frac{\partial u}{\partial t} + 2\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t \tag{4.24}$$

for all v and ω such that $\omega \in \partial\zeta_1(v)$ a.e. on Q. Unlike (4.5), $\frac{\partial z}{\partial t}$ is now directly tested by general test functions ω, which may need a suitably extended sense of $\frac{\partial z}{\partial t}$ that may naturally be a measure. In fact, the idea of a definition based on (4.24) has been, in an appropriately modified way, used already in [57] for a rate-independent evolution of microstructure in magnetization in ferromagnets. Theory of maximal monotone mappings that are homogeneous degree-0 has been developed in [22], called maximal responsive mappings.

5 Energetic solutions to the system (2.1)

The above concept of the weak solutions in Definitions 4.1 and 4.4 essentially works for any $\zeta_1 \geq 0$ convex and coercive, without relying essentially on its degree-1 homogeneity, although this homogeneity was occasionally exploited in the proofs especially if φ was not convex. The degree-1 homogeneity of ζ_1 offers, however, also an alternative way to treat the inclusion (2.1b) itself, i.e. the inclusion $-\sigma_{\text{in}} \in \partial\zeta_1(\frac{\partial z}{\partial t})$, which avoids usage of φ'_z and φ'_Z like in Definition 4.4 and, in addition, records more information allowing, e.g., for Proposition 5.4. By this homogeneity and by (2.4), we have

$$\operatorname{div} \varphi'_Z(e(u), z, \nabla z) - \varphi'_z(e(u), z, \nabla z) = -\sigma_{\text{in}} \in \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) \subset \partial\zeta_1(0) = S. \tag{5.1}$$

Referring to (2.7) and the definition of the subdifferential $\partial R_1(0)$ and properties of ζ_1, (5.1) means $0 = R_1(0) \leq R_1(v) + \langle V_z'(e(u),z),v\rangle$ for any $v \in W^{1,q}(\Omega;\mathbb{R}^m)$, cf. also (5.7) below. Written it for $v-z$ instead of v, we have

$$0 \leq R_1(v{-}z) + \langle V_z'(e(u),z),v{-}z\rangle. \tag{5.2}$$

Assuming $\varphi(e,\cdot,\cdot)$ convex, we have $V(e,\cdot)$ convex and thus

$$V(e,z) \leq V(e,v) - \langle V_z'(e,z),v{-}z\rangle, \tag{5.3}$$

cf. also (4.21) above. Altogether, summing (5.2) with (5.3) at a current time level t for $e=e(u)$, we have

$$\int_\Omega \varphi\big(e(u(t)),z(t),\nabla z(t)\big)\,\mathrm{d}x \leq \int_\Omega \varphi\big(e(u(t)),v,\nabla v\big) + \zeta_1\big(v{-}z(t)\big)\,\mathrm{d}x \tag{5.4}$$

for all $v \in W^{1,q}(\Omega;\mathbb{R}^m)$. If $z(t)$ satisfies (5.4), we say that z is *semi-stable* at t; the adjective “semi” distinguishes (5.4) from (6.9) below. As before, we have also the energy inequality arising from (4.5) for $v=0$ at our disposal, as well as the momentum equilibrium (2.1a) considered in the weak formulation (4.8). It leads to the following definition:

Definition 5.1 *Let (3.9) hold. We call (u,z) qualified as in (4.7) an energetic solution to the problem (2.1) with the initial/boundary conditions (4.1) and (2.6) if*

(i) *(2.1a) holds in the weak sense, i.e. for $v\in C^1(\bar{Q};\mathbb{R}^n)$ with $v|_{\Gamma_0}=0$:*

$$\int_\Omega \varrho\frac{\partial u}{\partial t}(T)\cdot v(T)\,\mathrm{d}x + \int_Q \Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \varphi_e'(e(u),z,\nabla z)\Big):\nabla v \\ - \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial t} - f\cdot v\,\mathrm{d}x\mathrm{d}t = \int_\Omega \varrho\dot{u}_0\cdot v(0)\,\mathrm{d}x, \tag{5.5}$$

(ii) *the energy inequality holds, i.e.*

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi\big(e(u(T)),z(T),\nabla z(T)\big)\,\mathrm{d}x + \mathrm{Var}_S(z;0,T) + 2\int_Q \zeta_2\big(e(\frac{\partial u}{\partial t})\big)\,\mathrm{d}x\mathrm{d}t \\ \leq \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi\big(e(u_0),z_0,\nabla z_0\big)\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t, \tag{5.6}$$

(iii) *the semi-stability (5.4) holds for all $v\in W^{1,q}(\Omega;\mathbb{R}^m)$ and for a.a. $t\in I$,*

(iv) *the initial conditions $u(0)=u_0$ and $z(0)=z_0$ are satisfied.*

Note that this definition admits $\varphi(e,\cdot,\cdot)$ nonsmooth and φ valued in $\mathbb{R}\cup\{+\infty\}$. Also note that the semi-stability is not required for each $t\in\bar{I}$ but only for a.e. t, which facilitates the existence proofs and still allows for equality in (5.6) in qualified cases, cf. Proposition 5.4 below. Also, (5.5) slightly modifies (4.8) to have more direct usage for the proof of Proposition 5.2(iii).

Proposition 5.2 *Let φ be smooth, satisfying (4.6). Then:*

(i) *Any energetic solution due to Definition 5.1 is also a weak solution according to Definition 4.1.*

Let, moreover, $\varphi(e,\cdot,\cdot)$ be convex. Then also:

(ii) *Any energetic solution due to Definition 5.1 is also a weak solution according to Definition 4.4.*
(iii) *Conversely, any weak solution (u,z) according to Definition 4.1 such that $z \in L^1(I;W^{2,1}(\Omega;\mathbb{R}^m)) \cap W^{1,1}(I;L^1(\Omega;\mathbb{R}^m))$ is also an energetic solution due to Definition 5.1.*
(iv) *If $(u,z,\sigma_{\rm in})$ is a weak solution due to Definition 4.4 such that $z \in L^1(I;W^{2,1}(\Omega;\mathbb{R}^m)) \cap W^{1,1}(I;L^1(\Omega;\mathbb{R}^m))$, then (u,z) is an energetic solution due to Definition 5.1 and $\sigma_{\rm in} = \varphi'_z\big(e(u),z,\nabla z\big) - {\rm div}\,\varphi'_Z\big(e(u),z,\nabla z\big)$.*

Proof. Using subsequently the definition of the directional derivative $\mathrm{D}_z V(e(u(t)),z(t),v)$ of $V(e(u(t)),\cdot) : W^{1,q}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ at $z(t)$ in the direction v with V from (2.7), the semi-stability (5.4) of z at time t with respect to $z(t)+\varepsilon v$ and the degree-1 homogeneity of ζ_1, we obtain

$$\begin{aligned}\int_\Omega \varphi'_z\big(e(u(t)),z(t),\nabla z(t)\big)\cdot v + \varphi'_Z\big(e(u(t)),z(t),\nabla z(t)\big) : \nabla v \,\mathrm{d}x \\ = \mathrm{D}_z V(e(u(t)),z(t),v) := \lim_{\varepsilon\downarrow 0} \frac{V(e(u(t)),z(t){+}\varepsilon v) - V(e(u(t)),z(t))}{\varepsilon} \\ = \lim_{\varepsilon\downarrow 0}\int_\Omega \frac{\varphi(e(u(t)),z(t){+}\varepsilon v,\nabla z(t){+}\varepsilon\nabla v) - \varphi(e(u(t)),z(t),\nabla z(t))}{\varepsilon}\,\mathrm{d}x \\ \ge -\lim_{\varepsilon\downarrow 0}\int_\Omega \frac{\zeta_1(z(t){+}\varepsilon v{-}z(t))}{\varepsilon}\,\mathrm{d}x = -\int_\Omega \zeta_1(v)\,\mathrm{d}x.\end{aligned} \tag{5.7}$$

By Definition 5.1(iii), (5.7) holds for a.a. $t \in I$. Integrating it over $[0,T]$ and summing it with (5.6), we get (4.9), i.e. the weak solution due to Definition 4.1. Thus (i) is proved.

As to (ii), in addition to what was proved in (i), it suffices still to define $\sigma_{\rm in} := V'_z(e(u),z)$ and, by convexity of $V(e,\cdot)$, we have $V(e(u),z) \le V(e(u),v) + \langle\sigma_{\rm in}, z-v\rangle$, which is just (4.17).

As to (iii), if z is smooth as specified, we have

$$\int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi'_z\big(e(u),z,\nabla z\big)\cdot\frac{\partial z}{\partial t} + \varphi'_Z\big(e(u),z,\nabla z\big) : \nabla\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = 0. \tag{5.8}$$

To show the inequality "$\le$" in (5.8), we use that (4.9) and (5.5) by a backward procedure from Section 4 imply (2.8), which is equivalent to ${\rm div}\varphi'_Z(e(u),z,\nabla z) - \varphi'_z(e(u),z,\nabla z) \in \partial\zeta_1(\frac{\partial z}{\partial t})$ a.e. on Q and, by the definition of the subdifferential of ζ_1 used at $\frac{\partial z}{\partial t}$, we have $\int_Q \zeta_1(\frac{\partial z}{\partial t}) - \sigma\cdot\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t \le 0$ for any $\sigma \in \partial\zeta_1(\frac{\partial z}{\partial t})$, in particular for $\sigma = \sigma_{\rm in} = {\rm div}\varphi'_Z(e(u),z,\nabla z) - \varphi'_z(e(u),z,\nabla z)$. The opposite inequality "$\ge$" in (5.8) follows from the degree-1 homogeneity of ζ_1 and definition of its subdifferential used at 0. Namely,

$$\sigma\cdot\frac{\partial z}{\partial t} = \zeta_1(0) + \sigma\cdot\Big(\frac{\partial z}{\partial t} - 0\Big) \le \zeta_1\Big(\frac{\partial z}{\partial t}\Big) \tag{5.9}$$

for any $\sigma \in \partial\zeta_1(0) = S$ but $S \supset \partial\zeta_1(\frac{\partial z}{\partial t})$ because of the degree-1 homogeneity of ζ_1, hence again we can take $\sigma = \sigma_{\rm in} = {\rm div}\varphi'_Z(e(u),z,\nabla z) - \varphi'_z(e(u),z,\nabla z)$.

The semi-stability (5.4) integrated over I, i.e.

$$\int_Q \varphi\big(e(u),z,\nabla z\big)\,\mathrm{d}x\mathrm{d}t \le \int_Q \varphi\big(e(u),v,\nabla v\big) + \zeta_1\big(v{-}z\big)\,\mathrm{d}x\mathrm{d}t \tag{5.10}$$

for all $v \in L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))$, follows again by (5.2)–(5.3) because we already proved that $\sigma_{\rm in} \in \partial\zeta_1(\frac{\partial z}{\partial t})$. Note that here we used convexity of φ.

Now we use that $\frac{\partial^2 u}{\partial t^2}$ is in duality with $\frac{\partial u}{\partial t}$. To see that indeed $\frac{\partial^2 u}{\partial t^2} \in L^1(I; W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*)+ L^2(I; W^{1,2}(\Omega;\mathbb{R}^n)^*)$, just realize that we have the information $\frac{\partial^2 u}{\partial t^2} = \mathrm{div}(\mathbb{D}e(\frac{\partial u}{\partial t})+\varphi'_e(e(u),z,\nabla z))+ f\cdot u$ in the sense of distributions. Note that such an information does not follow directly from (3.10). We can therefore use such test for (5.5) extended for $u \notin C^1(\bar{Q};\mathbb{R}^n)$ by continuity, which yields just (4.3). Subtracting it from the energy inequality (5.6), we obtain

$$\int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm el} : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x; \tag{5.11}$$

note that we assumed $\frac{\partial z}{\partial t} \in L^1(Q;\mathbb{R}^m)$. Then (5.11) implies, by (4.2),

$$\int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in}\cdot \frac{\partial z}{\partial t}\,\mathrm{d}x\mathrm{d}t \le 0. \tag{5.12}$$

The calculations (5.7) holds on Q instead of Ω if based on (5.10). Summing thus modified (5.7) with (5.12) and using Green's formula with the boundary conditions (2.6), we obtain

$$\int_Q \sigma_{\rm in}\cdot\Big(v - \frac{\partial z}{\partial t}\Big) + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \ge \int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \tag{5.13}$$

cf. (2.8). The inequality (4.9) is obtained by realizing that residuum in (5.6) is nonpositive, hence lesser than the left-hand side of (5.8) which can be further estimated by the left-hand side of (4.9) by using (5.13).

Eventually, from (5.10) we would like to see (5.4) for a.a. $t \in I$. Assuming it would not be true, we could find $\varepsilon > 0$ and $J \subset I$ with positive measure such that

$$\forall t\in J\ \ \exists v \in W^{1,q}(\Omega;\mathbb{R}^m): \qquad \int_\Omega \varphi\big(e(u(t)), z(t), \nabla z(t)\big)\,\mathrm{d}x \\ \ge \int_\Omega \varphi\big(e(u(t)), v, \nabla v\big) + \zeta_1\big(v{-}z(t)\big)\,\mathrm{d}x + \varepsilon. \tag{5.14}$$

Let $M(t)$ denote the set of all v satisfying the inequality in (5.14). Each $M(t)$ is nonempty and closed, and the set-valued mapping $t \mapsto M(t)$ is measurable and bounded; note that the boundedness of M follows from the boundedness of $t \mapsto \int_\Omega \varphi\big(e(u(t)), z(t), \nabla z(t)\big)\,\mathrm{d}x$ which is guaranteed by (4.6a) with (4.7a,c). Then it is well-known that there is a measurable selection of M, let us denote it as $\tilde{v}$. Considering $v \in L^\infty(I; W^{1,q}(\Omega;\mathbb{R}^m))$ as $v(t) = \tilde{v}(t)$ for $t \in J$ and $v(t) = z(t)$ for $t \in I \setminus J$, we obtain

$$\begin{aligned}\int_Q \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\mathrm{d}t &= \int_J\int_\Omega \varphi\big(e(u), z, \nabla z\big)\,\mathrm{d}x\mathrm{d}t + \int_{I\setminus J}\int_\Omega \varphi\big(e(u), v, \nabla v\big)\,\mathrm{d}x\mathrm{d}t \\ &\ge \int_J \bigg(\int_\Omega \varphi\big(e(u), v, \nabla v\big) + \zeta_1(v{-}z)\,\mathrm{d}x + \varepsilon\bigg)\mathrm{d}t + \int_{I\setminus J}\int_\Omega \varphi\big(e(u), v, \nabla v\big)\,\mathrm{d}x\mathrm{d}t \\ &= \int_Q \varphi\big(e(u), v, \nabla v\big) + \zeta_1(v{-}z)\,\mathrm{d}x\mathrm{d}t + \varepsilon\,\mathrm{meas}(J),\end{aligned} \tag{5.15}$$

which would contradict (5.10) since $\varepsilon\,\mathrm{meas}(J) > 0$.

As to (iv), (4.5) with $v = 0$ implies (5.6). Further, (4.17) and smoothness of φ implies $\sigma_{\rm in} = \varphi'_z(e(u), z, \nabla z) - \mathrm{div}\,\varphi'_Z(e(u), z, \nabla z)$. Like in the point (iii), $\frac{\partial^2 u}{\partial t^2}$ is in duality with $\frac{\partial u}{\partial t}$.

Then (4.8) implies (4.4), and then (4.5) gives $\int_Q \sigma_{\rm in}\cdot(v-\frac{\partial z}{\partial t})+\zeta_1(v)\mathrm{d}x\mathrm{d}t \geq \int_Q \zeta_1(\frac{\partial z}{\partial t})\mathrm{d}x\mathrm{d}t$, i.e. $\sigma_{\rm in} \in -\partial\zeta_1(\frac{\partial z}{\partial t})$. By degree-1 homogeneity of ζ_1 we have also $-\sigma_{\rm in} \in \partial\zeta_1(0)$ a.e., i.e. $\int_Q \zeta_1(0)\,\mathrm{d}x\mathrm{d}t \leq \int_Q \zeta_1(v) + \sigma_{\rm in}\cdot v\,\mathrm{d}x\mathrm{d}t$, hence $\int_Q \sigma_{\rm in}\cdot(z-v)\,\mathrm{d}x\mathrm{d}t \leq \int_Q \zeta_1(v-z)\,\mathrm{d}x\mathrm{d}t$ because $\zeta_1(0)=0$, and combining it with (4.17) eventually yields the semi-stability (5.10) integrated over I, from which one gets (5.4) as above via (5.14)–(5.15). $\Box$

Proving existence of an energetic solution by merging Propositions 4.3/4.5 and 5.2(iii)/(iv) is not fruitful due to the regularity hypothesis about z in Proposition 5.2(iii)/(iv) that hardly could be verified. Instead, we will prove the existence of energetic solutions directly. We will however need a discrete semi-stability which can be get from (3.1) only if $\gamma = 0$. Hence, unfortunately, we must exclude (e,z)-semiconvex potentials φ as used in Lemma 3.1(ii).

Proposition 5.3 *Let (3.6) and the assumptions of Lemma 3.1(i) hold, and one of the three cases hold:*

$$\begin{aligned}
&\zeta_1 \textit{ continuous, } p>1 \textit{ and } q>1 \textit{ in (2.12a),}\\
&\qquad \varphi(e,z,Z) = \phi_1(e,z)+\phi_2(z,Z), \textit{ and}\\
&\qquad \phi_1(\cdot,z) \textit{ has a } p\textit{-strongly monotone derivative in the sense (4.18),} \qquad (5.16a)\\
&\varphi \textit{ satisfies (4.18), } q=2 \textit{ in (2.12a), and } \varphi(e,\cdot,\cdot) \textit{ quadratic, and again}\\
&\qquad \varphi(e,z,Z) = \phi_1(e,z)+\phi_2(z,Z) \textit{ with}\\
&\qquad \big|\varphi'_{(z,Z)}(e,z,Z)\big| \leq C\big(1+|e|^{p(q^*-2)/(2q^*)}\big)\big(|z|+|Z|\big), \qquad (5.16b)\\
&\varphi \textit{ satisfies (2.12a) with } p=2=q \textit{ and has the form}\\
&\qquad \varphi(e,z,Z) = \phi_1(e,z,Z)+\phi_2(e,z,Z) \textit{ with}\\
&\qquad \phi_1(\cdot,z,\cdot) \textit{ affine, and } \phi_2 \textit{ quadratic,}\\
&\qquad \big|\phi_1(e,z,Z)\big| \leq C\big(1+|z|^{q^*/2-\epsilon}\big)\big(1+|e|+|Z|\big). \qquad (5.16c)
\end{aligned}$$

Then the energetic solution to (2.1) with (4.1) exist.

Note that, like in Proposition 4.5, (5.16a) admits ϕ_1 valued in $\mathbb{R}\cup\{+\infty\}$. Moreover, "quadratic" in (5.16b) can easily be augmented by an affine part, too.

Proof of Proposition 5.3. Again we use those discrete solutions (3.1) that are obtained by solving the minimization problem (3.11) as before, cf. Lemma 3.1. Its existence is by direct argument based on (3.11). For clarity, we divide the proof into steps that are (except the last one) usual in the theory of rate-independent processes, cf. [23, 42, 48]:

Step 1: A-priori estimates (3.7) are derived as before, see the proof of Lemma 3.1.

Step 2: Selection of subsequences is then made by Banach's and Helly's principles, in particular $e(\bar u_\tau(t)) \to e(u(t))$ weakly in $L^p(\Omega;\mathbb{R}^{n\times n})$ and $\bar z_\tau(t) \to z(t)$ weakly in $W^{1,q}(\Omega;\mathbb{R}^m)$ for any $t\in[0,T]$. Note that, because of viscosity, we have $e(\bar u_\tau) \in \mathrm{BV}(\bar I; L^2(\Omega;\mathbb{R}^{n\times n}_{\rm sym}))$. Moreover, in case (5.16a,b), we use the procedure (4.22) to obtain the strong convergence $e(\bar u_\tau) \to e(u)$ in $L^p(Q;\mathbb{R}^{n\times n})$; note that this procedure does not need any convexity of $\varphi(e,\cdot,\nabla z)$ but needs φ'_e independent of ∇z, as assumed (5.16a,b).

Step 3: Taking (u_τ^k, z_τ^k) a solution to (3.11) and fixing u_τ^k, we can see that z_τ^k fulfills $R_1(z_\tau^k - z_\tau^{k-1}) + V(e(u_\tau^k), z_\tau^k) \leq R_1(v - z_\tau^{k-1}) + V(e(u_\tau^k), v)$ for all $v\in W^{1,2}(\Omega;\mathbb{R}^m)$. Using also degree-1 homogeneity of R_1, we also know $R_1(v - z_\tau^{k-1}) - R_1(z_\tau^k - z_\tau^{k-1}) \le R_1(v - z_\tau^k)$ because of the triangle inequality for ζ_1. Altogether, we get

$$V(e(u_\tau^k), z_\tau^k) \le V(e(u_\tau^k), v) + R_1(v - z_\tau^k). \tag{5.17}$$

In other words, after summation for $k = 1, ..., T/\tau$, we get the "integrated" semi-stability for the discrete solution in the sense

$$\int_Q \varphi\big(e(\bar u_\tau), \bar z_\tau, \nabla \bar z_\tau\big)\, \mathrm{d}x\mathrm{d}t \le \int_Q \varphi\big(e(\bar u_\tau), v, \nabla v\big) + \zeta_1(v - \bar z_\tau)\, \mathrm{d}x\mathrm{d}t. \tag{5.18}$$

A limit passage in (5.18) to (5.10) is to be made case by case, while the further passage from (5.10) to (5.4) holding for a.a. $t \in I$ is as in the proof of Proposition 5.2.

Step 3a: Let us begin with the case (5.16a), which allows limit passage in (5.18) simply by continuity as far as ϕ_1 and ζ_1 concerns and by weak lower semicontinuity as far as $\int_Q \phi_2(\bar z_\tau, \nabla \bar z_\tau)\, \mathrm{d}x\mathrm{d}t$ concerns. Here we also use the Aubin-Lions theorem (again generalized as [56, Cor.7.9]) so that $\bar z_\tau \to z$ strongly in $L^{q^*-\epsilon}(Q; \mathbb{R}^m)$ with $\epsilon > 0$.

Step 3b: In case (5.16b), the limit passage in (5.18) can rely on the binomial formula for the quadratic functional $V(e, \cdot)$ from (2.7):

$$V(e, z) - V(e, \tilde z) = \frac{1}{2}\big\langle V_z'(e, z{-}\tilde z), z{+}\tilde z\big\rangle. \tag{5.19}$$

Thus, for any test function $\tilde v \in L^2(I; W^{1,2}(\Omega; \mathbb{R}^m))$, we can use (5.18) with $v := v_\tau = \tilde v - z + \bar z_\tau$ in place of v, and use the binomial formula (5.19) with $\bar z_\tau$ instead of z, i.e.

$$\begin{aligned}
&\int_Q \varphi\big(e(\bar u_\tau), \bar z_\tau, \nabla \bar z_\tau\big)\, \mathrm{d}x\mathrm{d}t - \int_Q \varphi\big(e(\bar u_\tau), v_\tau, \nabla v_\tau\big)\, \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_\Omega \big[\varphi'_{(z,Z)}\big(e(\bar u_\tau), \bar z_\tau{-}v_\tau, \nabla(\bar z_\tau{-}v_\tau)\big)\big]\big(\bar z_\tau{+}v_\tau, \nabla(\bar z_\tau{+}v_\tau)\big)\, \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_Q \big[\varphi'_{(z,Z)}\big(e(\bar u_\tau), z{-}\tilde v, \nabla(z{-}\tilde v)\big)\big]\big(\bar z_\tau{+}v_\tau, \nabla(\bar z_\tau{+}v_\tau)\big)\, \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{5.20}$$

which then converges to $\frac{1}{2}\int_Q [\varphi'_{(z,Z)}(e(u), z{-}\tilde v, \nabla(z{-}\tilde v))](z{+}v, \nabla(z{+}v))\, \mathrm{d}x\mathrm{d}t$ which equals to $\int_Q \varphi(e(u), z, \nabla z)\, \mathrm{d}x\mathrm{d}t - \int_Q \varphi(e(u), v, \nabla v)\, \mathrm{d}x\mathrm{d}t$ by (5.19). Here we used the growth conditions for $\varphi'_{(u,Z)}$ in (5.16b) to guarantee continuity of the Nemytskiĭ mapping induced by this integrand. Moreover, we have simply $\zeta_1(v_\tau - \bar z_\tau) = \zeta_1(\tilde v - z)$, so that the limit passage from (5.18) to (5.4) is proved in case (5.16b).

Step 3c: In case (5.16c), the limit passage in (5.18) can rely on the binomial formula for the quadratic part ϕ_2:

$$\phi_2(e, z, \nabla z) - \phi_2(\tilde e, \tilde z, \nabla \tilde z) = \frac{1}{2}\big[\phi_2(e{-}\tilde e, z{-}\tilde z, \nabla(z{-}\tilde z))\big]'(e{+}\tilde e, z{+}\tilde z, \nabla(z{+}\tilde z)). \tag{5.21}$$

Thus, for any test function $\tilde v \in L^2(I; W^{1,2}(\Omega; \mathbb{R}^m))$, we can use (5.18) with $v := v_\tau = \tilde v - z + \bar z_\tau$ in place of v, and use the binomial formula

$$\begin{aligned}
&\phi_2\big(e(\bar u_\tau), \bar z_\tau, \nabla \bar z_\tau\big) - \phi_2\big(e(\bar u_\tau), v_\tau, \nabla v_\tau\big) \\
&= \frac{1}{2}\big[\phi_2\big(0, \bar z_\tau{-}v_\tau, \nabla(\bar z_\tau{-}v_\tau)\big)\big]'\big(2e(\bar u_\tau), \bar z_\tau{+}v_\tau, \nabla(\bar z_\tau{+}v_\tau)\big) \\
&= \frac{1}{2}\big[\phi_2\big(0, z - \tilde v, \nabla(z - \tilde v)\big)\big]'\big(2e(\bar u_\tau), \bar z_\tau + v_\tau, \nabla(\bar z_\tau + v_\tau)\big)
\end{aligned} \tag{5.22}$$

which then converges weakly in $L^1(Q)$ to $\frac{1}{2}[\phi_2(0, z-\tilde{v}, \nabla(z-\tilde{v}))]'(2e(u), z+v, \nabla(z+v)) = \phi_2(e(u), z, \nabla z) - \phi_2(e(u), v, \nabla v)$.

The limit passage both $\phi_1(e(\bar{u}_\tau), \bar{z}_\tau, \nabla\bar{z}_\tau) \to \phi_1(e(u), z, \nabla z)$ and $\phi_1(e(\bar{u}_\tau), v_\tau, \nabla v_\tau) \to \phi_1(e(u), v, \nabla v)$ is simply by (weak×norm×weak)-continuity due to the compact embedding in z (hence $\bar{z}_\tau \to z$ in $L^{q^*-\epsilon}(Q;\mathbb{R}^m)$) and affinity of ϕ_1 in e and ∇z.

Again, we have simply $\zeta_1(v_\tau - \bar{z}_\tau) = \zeta_1(\tilde{v} - z)$ as in case (5.16b), which accomplishes the limit passage from (5.18) to (5.4) in case (5.16.

Step 4: The limit passage in this discrete energy inequality (3.8), which we proved in Lemma 3.1(i), is then possible by weak lower-semicontinuity with possible compactness of $\{\bar{z}_\tau\}$ by Aubin-Lions' theorem as before.

Step 5: Eventually, in cases (5.16a,b), the limit passage in the discrete momentum equation (4.12) is by strong convergence of both $e(\bar{u}_\tau)$ and $\bar{z}_\tau$ as before, cf. the proof of Proposition 4.5, while in case (5.16c) it is just by weak continuity. □

It is an important application of the method developed in the theory of rate-independent processes [18, 23, 42, 43, 47] that one can prove even that the equality holds in (5.6).

Proposition 5.4 ***(Energy equality.)*** *Let (3.9) be strengthened to* $|\varphi_e'(e,z,Z)\big| \le C(1+|e|^{p/2}+|z|^{q^*/2}+|Z|^{q/2}\big)$ *and* $\varphi_e'(\cdot,z,Z)$ *be Lipschitz continuous uniformly with respect to* (z,Z)*, and let* (u,z) *be an energetic solution in accord with Definition 5.1 and the initial conditions* z_0 *be semi-stable with respect to* $u(0) = u_0$ *in the sense*

$$\int_\Omega \varphi\big(e(u_0), z_0, \nabla z_0\big)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(u_0), v, \nabla v\big) + \zeta_1\big(v - z_0\big)\,\mathrm{d}x \tag{5.23}$$

for all $v \in W^{1,q}(\Omega;\mathbb{R}^n)$*. Then the equality in (5.6) holds.*

Proof. Since $\frac{\partial^2 u}{\partial t^2}$ is, in fact, in duality with $\frac{\partial u}{\partial t}$ (cf. the proof of Proposition 5.2), the energy balance holds as an equality for the visco-elastic part, see (4.3). To show the equality in (5.6), we thus need only to show the converse inequality in (5.11). For this, we consider now $\varepsilon > 0$, and a partition $0 = t_0^\varepsilon < t_1^\varepsilon < ... < t_{k_\varepsilon}^\varepsilon = T$ with $\max_{i=1,...,k_\varepsilon}(t_i^\varepsilon - t_{i-1}^\varepsilon) \le \varepsilon$. Moreover, as (5.4) holds a.e. $t \in I$ and by (5.23) also at $t = 0$, we can consider the above partition so that the semi-stability holds at t_i^ε for each $i = 0, ..., k_\varepsilon - 1$. Using this semi-stability of z at time t_{i-1}^ε gives, when tested by $v := z(t_i^\varepsilon)$, the estimate

$$\begin{aligned}\int_\Omega \varphi\big(e(u(t_{i-1}^\varepsilon)), z(t_{i-1}^\varepsilon), \nabla z(t_{i-1}^\varepsilon)\big)\,\mathrm{d}x &\le \int_\Omega \varphi\big(e(u(t_{i-1}^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) + \zeta_1\big(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon)\big)\,\mathrm{d}x \\ &= \int_\Omega \bigg(\varphi\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) + \zeta_1\big(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon)\big) \\ &\quad - \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \varphi_e'\big(e(u(t)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}t\bigg)\mathrm{d}x.\end{aligned} \tag{5.24}$$

Summing (5.24) for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon-1}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$\begin{aligned}
&\int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big) - \int_\Omega \varphi\big(e(u_0), z_0, \nabla z_0)\big)\,\mathrm{d}x + \mathrm{Var}_S(z;0,T)\\
&\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\geq \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t - \delta_\varepsilon\\
&= \sum_{i=1}^{k_\varepsilon-1} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\\
&\quad + \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'\big(e(u(t)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) - \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big)\Big) : e\Big(\frac{\partial u}{\partial t}\Big)\mathrm{d}x\mathrm{d}t\\
&\quad + \sum_{i=1}^{k_\varepsilon-1} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(\frac{\partial u}{\partial t} - \Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}t - \delta_\varepsilon\\
&=: S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon - \delta_\varepsilon
\end{aligned} \tag{5.25}$$

where

$$\delta_\varepsilon := \Big| \int_{t_{k_\varepsilon-1}^\varepsilon}^T \int_\Omega \varphi_e'\big(e(u(t)), z(T), \nabla z(T)\big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t\Big|. \tag{5.26}$$

As to S_2^ε, using the Lipschitz continuity of $\varphi_e'(\cdot, z, \nabla z) : \mathbb{R}^{n\times n}_{\rm sym} \to \mathbb{R}^{n\times n}_{\rm sym}$ (with ℓ denoting the Lipschitz constant), we can estimate

$$\begin{aligned}
|S_2^\varepsilon| &\leq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \ell \big\|e\big(u(t) - u(t_i^\varepsilon)\big)\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \Big\|e\Big(\frac{\partial u}{\partial t}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\mathrm{d}t\\
&\leq \ell \max_{i=1,\dots,k_\varepsilon} \max_{t\in[t_{i-1}^\varepsilon, t_i^\varepsilon]} \big\|e\big(u(t) - u(t_i^\varepsilon)\big)\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \big\|e(u)\big\|_{W^{1,1}(I;L^2(\Omega;\mathbb{R}^{n\times n}))}.
\end{aligned} \tag{5.27}$$

Since certainly $e(u) \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^{n\times n}))$, the "max max"-term tends to zero with $\varepsilon\downarrow 0$, hence $\lim_{\varepsilon\downarrow 0} S_2^\varepsilon = 0$. As to S_3^ε, by Fubini's theorem, we can estimate

$$\begin{aligned}
|S_3^\varepsilon| &= \Big| \sum_{i=1}^{k_\varepsilon} \int_\Omega \varphi_e'\big(e(u(t_i^\varepsilon)), z(t_i^\varepsilon), \nabla z(t_i^\varepsilon)\big) : e\Big(u(t_i^\varepsilon) - u(t_{i-1}^\varepsilon) - (t_i^\varepsilon - t_{i-1}^\varepsilon)\Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\,\mathrm{d}x\Big|\\
&\leq \big\|\varphi_e'\big(e(u), z, \nabla z\big)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))} \sum_{i=1}^{k_\varepsilon} \Big\|e\Big(u(t_i^\varepsilon) - u(t_{i-1}^\varepsilon) - (t_i^\varepsilon - t_{i-1}^\varepsilon)\Big[\frac{\partial u}{\partial t}\Big](t_i^\varepsilon)\Big)\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}.
\end{aligned}$$

Note that $u(t_i^\varepsilon) - u(t_{i-1}^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ although particular terms are in $W^{1,p}(\Omega;\mathbb{R}^n)$ need not belong to $W^{1,2}(\Omega;\mathbb{R}^n)$ if $p < 2$ and that the assumed growth of φ_e' together with (4.7a,c) indeed guarantees $\varphi_e'(e(u), z, \nabla z) \in L^\infty(I; L^2(\Omega;\mathbb{R}^{n\times n}))$. We have still a freedom to choose the partition $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ in such a way that both $\lim_{\varepsilon\downarrow 0} S_3^\varepsilon = 0$ and that the Riemann sum S_1^ε approaches the corresponding Lebesgue integral, namely $\lim_{\varepsilon\downarrow 0} S_1^\varepsilon = \int_0^T \int_\Omega \varphi_e'(e(u(t)), z(t), \nabla z(t)) : e(\frac{\partial u}{\partial t})\,\mathrm{d}x\mathrm{d}t$; cf. [18, Lemma 4.12] or [23, Lemma 4.5], following the idea of Hahn [30]. Eventually, $\lim_{\varepsilon\downarrow 0} \delta_\varepsilon = 0$ the integrand in (5.26) is absolutely continuous and $t^\varepsilon_{k_\varepsilon-1}\uparrow T$ for $\varepsilon\downarrow 0$. This allows for a limit passage in (5.25) for $\varepsilon\downarrow 0$, which gives the desired opposite inequality in (5.11) with $\mathrm{Var}_S(z;0,T)$ instead of $\int_Q \zeta_1(\frac{\partial z}{\partial t})\,\mathrm{d}x\mathrm{d}t$. $\square$

6 Fully rate-independent processes in (2.1)

An interesting question is how solutions to (2.1) will behave for very slow loading regimes f. By proper scaling of time like εt on the fixed time interval $[0,T]$, we can replace ϱ by $\varepsilon^2\varrho$ and ζ_2 by $\varepsilon\zeta_2$ while ζ_1, φ, and f remain unchanged. Let $(u_\varepsilon,z_\varepsilon)$ denote the corresponding energetic solution. By method from Lemma 3.1, one can easily get a-priori estimates such solution $(u_\varepsilon,z_\varepsilon)$:

$$\Big\|\varrho\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))}\le\frac{C}{\varepsilon}, \tag{6.1a}$$

$$\Big\|e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\Big\|_{L^2(Q;\mathbb{R}^{n\times n})}\le\frac{C}{\sqrt{\varepsilon}}, \tag{6.1b}$$

$$\big\|e(u_\varepsilon)\big\|_{L^\infty(I;L^p(\Omega;\mathbb{R}^{n\times n}))}\le C, \tag{6.1c}$$

$$\big\|z_\varepsilon\big\|_{L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))}\le C. \tag{6.1d}$$

Let us investigate convergence of $(u_\varepsilon,z_\varepsilon)$ for $\varepsilon\to 0$ (in terms of subsequences) to some (u,z). One can expect that, in such "slow-loading" limit, the inertial as well as the viscous effects actually disappear, i.e. $\varrho=0$ and $\zeta_2=0$, and the whole system becomes fully *rate independent.* This means that (2.1) reduces to

$$-\mathrm{div}\,\varphi'_e(e(u),z,\nabla z)=f,\quad \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\varphi'_z(e(u),z,\nabla z)-\mathrm{div}\,\varphi'_Z(e(u),z,\nabla z)\ni 0. \tag{6.2}$$

Definition 6.1 ***("Semi"-energetic solution to (6.2).)*** *We call the couple (u,z) with $u\in\mathrm{B}(\bar I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ and $z\in\mathrm{B}(\bar I;W^{1,q}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))$ an energetic solution to (6.2) if*

(i) *the (weakly-formulated) momentum-equilibrium*

$$\int_Q\varphi'_e\big(e(u),z,\nabla z\big):e(v)-f\cdot v\,\mathrm{d}x\mathrm{d}t=0 \tag{6.3}$$

holds for all $v\in L^p(I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, cf. (6.6),

(ii) *the semi-stability (5.4) holds for all $v\in W^{1,q}(\Omega;\mathbb{R}^m)$ and a.a. $t\in[0,T]$,*

(iii) *the Gibbs-type energy inequality holds, i.e.*

$$\int_\Omega\varphi\big(e(u(T)),z(T),\nabla z(T)\big)-f(T)\cdot u(T)\,\mathrm{d}x+\mathrm{Var}_S(z;0,T)$$
$$\le\int_\Omega\varphi\big(e(u_0),z_0,\nabla z_0\big)-f(0)\cdot u_0\,\mathrm{d}x-\int_Q\frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t, \tag{6.4}$$

(iv) *the initial condition $z(0)=z_0$ holds.*

Contrary to the viscous/inertial evolution, the initial condition on u is not involved now. Also, as we now loose any estimate on $\frac{\partial u}{\partial t}$, we used the by-part integration in time

$$\int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t=\int_\Omega f(T)\cdot u(T)-f(0)\cdot u_0\,\mathrm{d}x-\int_Q\frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t \tag{6.5}$$

so that the Helmholtz-type energy inequality (5.6) turns into the Gibbs-type energy inequality (6.4). Of course, we must now qualify f better to make $\frac{\partial f}{\partial t}$ in (6.4) sense.

Proposition 6.2 *Let* $\mathrm{meas}_{n-1}(\Gamma_0) > 0$*,* $f \in W^{1,1}(I; L^{p^{*\prime}}(\Omega;\mathbb{R}^n))$*, the assumptions of Proposition 5.3 hold with* $p = 2$ *and, in addition to (5.16a,b), let* $\varphi(e,z,\nabla z) = \phi_1(e) + \phi_2(e,z)+\phi_3(z,\nabla z)$ *with* ϕ_1 *quadratic convex and* $\phi_2(\cdot,z)$ *affine. Then the sequence of energetic solutions* $\{(u_\varepsilon,z_\varepsilon)\}_{\varepsilon>0}$ *contains a subsequence converging weakly* in* $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^n))\times (L^\infty(I;W^{1,q}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m)))$ *to some* (u,z) *and every* (u,z) *obtained by this way is "semi"-energetic solution to (6.2).*

Proof. Existence of $(u_\varepsilon,z_\varepsilon)$ is by Proposition 5.3, together with the a-priori estimates (6.1) inherited from the discrete solutions. We use $\mathrm{meas}_{n-1}(\Gamma_0) > 0$ to have $\{u_\varepsilon\}_{\varepsilon>0}$ bounded in $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^n))$ by Korn's inequality and (6.1c).

By (6.1a,b), we have $|\int_Q \varepsilon\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t}) : e(v)\mathrm{d}x\mathrm{d}t| = \mathcal{O}(\sqrt{\varepsilon})$ and $|\int_Q \varepsilon^2\varrho\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial v}{\partial t}\mathrm{d}x\mathrm{d}t| = \mathcal{O}(\varepsilon)$, and we can pass to the limit in the (weakly formulated) momentum equilibrium:

$$\begin{aligned}0 &= \int_Q \varepsilon\mathbb{D}e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big) : e(v) - \varepsilon^2\varrho\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial v}{\partial t} + \varphi_e'\big(e(u_\varepsilon),z_\varepsilon,\nabla z_\varepsilon\big) : e(v) - f\cdot v\,\mathrm{d}x\mathrm{d}t\\ &\to \int_Q \varphi_e'(e(u),z,\nabla z) : e(v) - f\cdot v\,\mathrm{d}x\mathrm{d}t, \end{aligned}\tag{6.6}$$

note that $\varphi_e'(\cdot,z,\cdot)$ is now assumed linear. Convergence for $\varepsilon\to 0$ in the semi-stability

$$\int_\Omega \varphi\big(e(u_\varepsilon(t)),z_\varepsilon(t),\nabla z_\varepsilon(t)\big)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(u_\varepsilon(t)),v,\nabla v\big) + \zeta_1\big(v - z_\varepsilon(t)\big)\,\mathrm{d}x \tag{6.7}$$

is as before, cf. Step 3 in the proof of Proposition 5.3 modified by using only weak* convergence of u_ε in Step 3a,b. Using (6.5) for u_ε, we obtain the Gibbs-type energy inequality

$$\begin{aligned}&\int_\Omega \varepsilon^2\frac{\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi\big(e(u_\varepsilon(T)),z_\varepsilon(T),\nabla z_\varepsilon(T)\big) - f(T)\cdot u_\varepsilon(T)\,\mathrm{d}x\\ &\qquad + \mathrm{Var}_S(z_\varepsilon;0,T) + 2\int_Q \varepsilon\zeta_2\Big(e(\frac{\partial u_\varepsilon}{\partial t})\Big)\mathrm{d}x\mathrm{d}t\\ &\qquad \le \int_\Omega \varepsilon^2\frac{\varrho}{2}|\dot u_0|^2 + \varphi\big(e(u_0),z_0,\nabla z_0\big) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q\frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{6.8}$$

The limit passage in (6.8) to obtain (6.4) is simple just by omitting the nonnegative terms $\varepsilon^2\frac{\varrho}{2}|\frac{\partial u_\varepsilon}{\partial t}(T)|^2$ and $\varepsilon\zeta_2(e(\frac{\partial u_\varepsilon}{\partial t}))$, passing $\varepsilon^2\frac{\varrho}{2}|\dot u_0|^2\to 0$, and using weak lower-semicontinuity. □

Theory for such rate-independent systems was developed by Mielke et al. [42, 50, 51, 52]. The energetic solution introduced by Mielke and Theil [50, 51] is based on the full stability:

Definition 6.3 ***("Fully" energetic solution to (6.2).)*** *We call the couple* (u,z) *with* $u\in \mathrm{B}(\bar I;W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ *and* $z\in \mathrm{B}(\bar I;W^{1,q}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))$ *an energetic solution to (6.2) if*

(i) *the (full) stability*

$$\int_\Omega \varphi\big(e(u(t)),z(t),\nabla z(t)\big) - f(t)\cdot u(t)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(w(t)),v,\nabla v\big) - f(t)\cdot w + \zeta_1\big(v-z(t)\big)\,\mathrm{d}x \tag{6.9}$$

holds for all $(w,v)\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m)$ *and all* $t\in[0,T]$,

(ii) *the energy inequality (6.4) holds, and*

(iii) *the initial condition* $z(0)=z_0$ *holds.*

Proposition 6.4 *Let* $f\in W^{1,1}(I;L^{p^{*\prime}}(\Omega;\mathbb{R}^n))$, *let* (u_0,z_0) *be stable in the sense* $V(u_0,z_0)\le V(w,z)+R_1(v-z_0)$ *for all* $(w,v)\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m)$, $\mathrm{meas}_{n-1}(\Gamma_0)>0$ *and* $\varphi(\cdot,z,\nabla z)$ *be strictly convex. Then Definitions 6.1 and 6.3 are equivalent to each other.*

Proof. Putting $w:=u(t)$ into (6.9) gives the semi-stability in Definition 6.1(ii), while putting $v:=z(t)$ shows that $u(t)$ minimizes the functional $u\mapsto\int_\Omega\varphi(e(u),z(t))-f(t)\cdot u$, and writing the corresponding Euler-Lagrange equation weakly and integrating it over $[0,T]$ gives (6.3).

Conversely, strict convexity of $\varphi(\cdot,z,\nabla z)$ and nontrivial Dirichlet boundary conditions make sure that the boundary-value problem for $-\mathrm{div}\big(\varphi'_e(e(u),z,\nabla z)\big)=f(t)$ with the boundary conditions $u|_{\Gamma_0}=0$ and $\varphi'_e(e(u),z,\nabla z)|_{\Gamma_1}\cdot\nu=0$, cf. (2.6), has the only solution $u=\mathfrak{u}(t,z)\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Let us define the reduced stored energy $G(t,z):=V(e(\mathfrak{u}(t,z)),z)-\int_\Omega f(t)\cdot\mathfrak{u}(t,z)\,\mathrm{d}x$ with V from (2.7), i.e. $G(t,z)=\int_\Omega\varphi(e(\mathfrak{u}(t,z))),z,\nabla z)-f(t)\cdot\mathfrak{u}(t,z)\,\mathrm{d}x$. It holds

$$G(t,z)=\min_{w\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)}\int_\Omega\varphi(w,z,\nabla z)-f(t)\cdot w\,\mathrm{d}x. \tag{6.10}$$

The derivative $G'_z(t,z)$ can be calculated by the adjoint equation method. Denoting $\Pi: W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)\times W^{1,q}(\Omega;\mathbb{R}^m)\to W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)^*$ defined by $\langle\Pi(u,z),v\rangle:=\int_\Omega\varphi'_e(e(u),z,\nabla z):e(v)-f(t)\cdot v\,\mathrm{d}x$, and $\lambda\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)^{**}\cong W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ the solution of the so-called adjoint equation $\Pi'_u(u,z)^*\lambda=V'_u(u,z)-f(t)$, we have $G'_z(t,z)=V'_z(u,z)-\Pi'_z(u,z)^*\lambda$. Formally, this means that $\lambda\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ is thus to solve in the weak sense the adjoint equation $\mathrm{div}(\varphi''_{ee}(e(u),z,\nabla z)e(\lambda))=\mathrm{div}(\varphi'_e(e(u),z,\nabla z)-f$, and then $G'_z(t,z)=\varphi'_z(e(u),z,\nabla z)-\mathrm{div}\varphi'_Z(e(u),z,\nabla z)-\varphi''_{ez}(e(u),z,\nabla z)\lambda+\mathrm{div}(\varphi''_{eZ}(e(u),z,\nabla z)\lambda)$ in the sense of distributions. Yet, the right-hand side $V'_u(u,z)-f(t)$ of the adjoint equation identically vanishes if $u=\mathfrak{u}(t,z)$ and thus $\lambda\equiv0$. (This is well-known effect in optimization of systems with so-called compliance cost functionals.) Therefore, $G'_z(t,z)=V'_z(\mathfrak{u}(t,z),z)$ even if φ is not twice-differentiable as formally needed for derivation of the above adjoint-equation argument.

Thus $G'_z(t,z)=\sigma_{\rm in}:=\varphi'_z(e(\mathfrak{u}(t,z)),z,\nabla z)-\mathrm{div}\varphi'_Z(e(\mathfrak{u}(t,z)),z,\nabla z)$ in the sense of distributions. Comparing it with (2.1b) and the definition of R_1 in (2.7), we can see that the evolution of z is then governed by the "reduced" inclusion $\partial R_1(\frac{\mathrm{d}u}{\mathrm{d}t})+G'_z(t,z)\ni0$. Stability for this reduced evolution leads to stability condition

$$G(t,z(t))\le G(t,v)+R_1(v-z(t)) \tag{6.11}$$

for any $v\in W^{1,q}(\Omega;\mathbb{R}^m)$. The assumed stability of the initial condition ensures, by arguments of the theory of rate independent processes [23, 42, 48, 51], that (6.11) holds even for all $t \in \bar{I}$. Thus

$$\begin{aligned}\int_\Omega \varphi\big(e(u(t), z, \nabla z\big) - f(t)\cdot u(t)\,\mathrm{d}x &= \min_{w\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)} \int_\Omega \varphi(w, z, \nabla z) - f(t)\cdot w\,\mathrm{d}x \\ &\le \min_{w\in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)} \int_\Omega \varphi(w, v, \nabla v) - f(t)\cdot w\,\mathrm{d}x + \int_\Omega \zeta_1\big(v - z(t)\big)\,\mathrm{d}x \\ &\le \int_\Omega \varphi(w, v, \nabla v) - f(t)\cdot w + \zeta_1\big(v - z(t)\big)\,\mathrm{d}x \qquad (6.12)\end{aligned}$$

for any $w \in W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n)$, i.e. full stability (6.9) of $(u(t), z(t))$ with $u(t) = \mathfrak{u}(t, z(t))$. □

By the above "slow-loading" limit procedure and Proposition 6.2, we obtain existence of the (fully) energetic solution to (6.2). However, the full stability (6.9) allows for a simpler proof because the stored-energy in the right-hand side in (6.9) is fixed. For an extensive theory applicable to it see [42]. For rate-independent generalized standard solids even at large strains see [43].

We can also have a look how Definitions 4.1 transforms in the rate-independent case. Note that (4.9) now also uses (6.5):

Definition 6.5 (*Weak solution to* (6.2).) *We call (u, z) with $u \in \mathrm{B}(\bar{I}; W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ and $z \in \mathrm{B}(\bar{I}; W^{1,q}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}(\bar{I}; L^1(\Omega;\mathbb{R}^m))$ a weak solution to (6.2) if*

(i) *the (weakly-formulated) momentum-equilibrium (6.3) holds,*

(ii) *the variational inequality*

$$\begin{aligned}\int_Q \varphi'_z\big(e(u), z, \nabla z\big)\cdot v + \varphi'_Z\big(e(u), z, \nabla z\big) : \nabla v + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \ge \mathrm{Var}_S(z; 0, T) \\ + \int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big) - f(T)\cdot u(T) \\ - \varphi\big(e(u_0), z_0, \nabla z_0\big) + f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t, \qquad (6.13)\end{aligned}$$

for all $v \in C^1(\bar{Q};\mathbb{R}^m)$,

(iii) *the initial condition $z(0) = z_0$ holds.*

Proposition 6.6 *Any energetic solution is also the weak one. Conversely, if the assumptions of Proposition 6.4 hold and $\varphi(e,\cdot,\cdot)$ is convex, then any weak solution (u, z) with $\frac{\partial u}{\partial t} \in L^p(Q;\mathbb{R}^n)$ and $\frac{\partial z}{\partial t}$ integrable is also an energetic one.*

Proof. The first assertion follows from (6.4) with (5.7) integrated over I, which gives (6.13), while the momentum equilibrium (6.3) follows from (6.9) for $v := z(t)$.

Conversely, $\frac{\partial u}{\partial t} \in L^p(Q;\mathbb{R}^n)$ allows for testing (6.3) to get $\int_Q \varphi'_e(e(u), z, \nabla z) : e(\frac{\partial u}{\partial t})\,\mathrm{d}x\mathrm{d}t = \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t$. Hence the energy balance $V(e(u(T)), z(T)) - V(e(u_0), z_0)) = \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t + \langle V'_z(e(u), z), \frac{\partial z}{\partial t}\rangle$ follows. Using also the by-part integration (6.5) and substituting it into (6.13) yields (2.8). By the procedure (5.2)–(5.4), one gets the semi-stability integrated over I, i.e. (5.10); here convexity of $\varphi(e,\cdot,\cdot)$ has been used. The semi-stability (5.4) for a.a. t then follows by the procedure (5.14)–(5.15). Thus we come to "semi"-energetic solution in accord with Definition 6.1. Then usage of Proposition 6.4 gives the "fully" energetic solution. □

Just for completeness, let us present also a rate-independent modification of Definitions 4.4. Again, (6.5) is used for (4.5):

Definition 6.7 (***Weak solution to* (6.2) *alternatively.***) *We call* $(u, z, \sigma_{\rm in})$ *with* $u \in {\rm B}(\bar{I}; W^{1,p}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, $z \in {\rm B}(\bar{I}; W^{1,q}(\Omega;\mathbb{R}^m)) \cap {\rm BV}(\bar{I}; L^1(\Omega;\mathbb{R}^m))$, *and* $\sigma_{\rm in} \in L^2(Q;\mathbb{R}^m)$ *a weak solution to (6.2) if*

(i) *the (weakly formulated) momentum equilibrium (6.3) holds,*

(ii) *the variational inequality*

$$\int_Q \sigma_{\rm in} \cdot v + \zeta_1(v)\,{\rm d}x{\rm d}t \geq {\rm Var}_S(z;0,T) + \int_\Omega \varphi\big(e(u(T)), z(T), \nabla z(T)\big) - f(T)\cdot u(T) - \varphi\big(e(u_0), z_0, \nabla z_0\big) + f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u\,{\rm d}x{\rm d}t \qquad (6.14)$$

for all $v \in C^1(\bar{Q};\mathbb{R}^m)$,

(iii) *(4.17) holds for all* $v \in C^1(\bar{Q};\mathbb{R}^m)$, *and*

(iv) *the initial condition* $z(0) = z_0$ *holds.*

Remark 6.8 (*"Truly" viscous regularization converges to weak solutions.*) We can now consider, in addition, also a "viscous" regularization of ζ_1 itself in the form

$$\zeta_1(\dot{z}) = \delta^*_K(\dot{z}) + \frac{1}{2}|\dot{z}|^2. \qquad (6.15)$$

The evolution of z would then look as

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in} = \partial\delta^*_K\Big(\frac{\partial z}{\partial t}\Big) + \frac{\partial z}{\partial t} + \sigma_{\rm in} \ni 0. \qquad (6.16)$$

Scaling for slow-loading time εt would lead to

$$\partial\delta^*_K\Big(\frac{\partial z}{\partial t}\Big) + \varepsilon\frac{\partial z}{\partial t} + \sigma_{\rm in} \ni 0. \qquad (6.17)$$

Denoting by $(u_\varepsilon, z_\varepsilon)$ a weak solution to (6.17) and (2.1a) with ϱ replaced by $\varepsilon^2\varrho$ and ζ_2 by $\varepsilon\zeta_2$, we have, in addition to (6.1), also the estimate

$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^m)} \leq \frac{C}{\sqrt{\varepsilon}}. \qquad (6.18)$$

Convergence to the weak solution according to Definition 6.5 is simple. For this, we need to pass to the limit in

$$\int_Q \varphi'_z\big(e(u_\varepsilon), z_\varepsilon, \nabla z_\varepsilon\big)\cdot v + \varphi'_Z\big(e(u_\varepsilon), z_\varepsilon, \nabla z_\varepsilon\big) : \nabla v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2\,{\rm d}x{\rm d}t \geq \int_Q \delta^*_K\Big(\frac{\partial z_\varepsilon}{\partial t}\Big) + \frac{\varepsilon}{2}\Big|\frac{\partial z_\varepsilon}{\partial t}\Big|^2 + 2\varepsilon\zeta_2\Big(e(\frac{\partial u_\varepsilon}{\partial t})\Big){\rm d}x{\rm d}t + \int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi\big(e(u_\varepsilon(T)), z_\varepsilon(T), \nabla z_\varepsilon(T)\big) - f(T)\cdot u_\varepsilon(T)\,{\rm d}x - \int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot{u}_0|^2 + \varphi\big(e(u_0), z_0, \nabla z_0\big) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t; \qquad (6.19)$$

this inequality follows again by (4.2)–(4.5) when using (2.8) regularized. To pass to the limit, we can just drop $\frac{1}{2}\varepsilon|\frac{\partial z_\varepsilon}{\partial t}|^2$, $\frac{1}{2}\varepsilon^2\varrho|\frac{\partial u_\varepsilon}{\partial t}(T)|^2$, and $2\varepsilon\zeta_2(e(\frac{\partial u_\varepsilon}{\partial t}))$ out, limit $\frac{\varepsilon}{2}|v|^2 \to 0$, and then use weak (lower semi)continuity to arrive to (6.13) under the additional assumption that $\varphi'_{(z,Z)}(\cdot, z, \cdot)$ is affine. The limit passage (6.6) is as before.

Considering Definition 6.7, we are to limit (6.19) with $\sigma_{\mathrm{in},\varepsilon}$ in place of $\int_Q \varphi'_z\big(e(u_\varepsilon), z_\varepsilon, \nabla z_\varepsilon\big)\cdot v + \varphi'_Z\big(e(u_\varepsilon), z_\varepsilon, \nabla z_\varepsilon\big) : \nabla v \,\mathrm{d}x\mathrm{d}t$. If assuming ζ_1 continuous, this limit passage is even simpler as before because, in view of (6.17) with (6.18), we have $\sigma_{\mathrm{in},\varepsilon}$ uniformly bounded $L^2(Q;\mathbb{R}^m)$, hence (up to a subsequence) converging weakly in $L^2(Q;\mathbb{R}^m)$. Assuming $q^* > 2$, the limit passage in (4.17) can be done by lower semicontinuity and by using the compact embedding of $L^\infty(I;W^{1,q}(\Omega)) \cap \mathrm{BV}(\bar{I};L^1(\Omega)) \subset L^2(Q)$.

Remark 6.9 Note that, contrary to (4.5), putting $v = 0$ into (6.19) yields only suboptimal energy estimate due to the factor $\frac{1}{2}$ in front of $\varepsilon|\frac{\partial z_\varepsilon}{\partial t}|^2$. Contrary to convergence claimed in Remark 6.8, any convergence to an energetic solution is not clear for viscous regularization (6.17), however. For special cases, see [20, 36].

7 Examples

We end with some nontrivial examples to illustrate applicability of the presented results. In particular, we will see that, for nonconvex φ, one must often resign on usage of energetic-solution concept from Section 5 since (e)-semiconvexity (3.4) is not easy to check in general.

Example 7.1 (*Viscoplasticity with hardening.*) The internal variable $z = (\pi, \eta) \in \mathbb{R}^{n\times n}_{\mathrm{sym},0} \times \mathbb{R}$ have now the meaning of the plastic deformation π and the hardening parameter η, where $\mathbb{R}^{n\times n}_{\mathrm{sym},0} := \big\{A \in \mathbb{R}^{n\times n}_{\mathrm{sym}};\ \mathrm{tr}(A) = 0\big\}$. For some nonlinearities $a : \mathbb{R}^{n\times n}_{\mathrm{sym},0} \to \mathbb{R}^{n\times n}_{\mathrm{sym},0}$ and $b : \mathbb{R} \to \mathbb{R}$ and a regularization parameter $\kappa > 0$, one can consider

$$\varphi(e, z, \nabla z) \equiv \varphi(e, \pi, \eta, \nabla\pi, \nabla\eta) = \frac{1}{2}\mathbb{C}\big(e - a(\pi)\big):\big(e - a(\pi)\big) + b(\eta) + \frac{\kappa}{2}|\nabla\pi|^2 + \frac{\kappa}{2}|\nabla\eta|^2 \tag{7.1}$$

where $\mathbb{C}$ is the positive-definite elasticity tensor exhibiting the usual symmetries $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$. The parameter $\kappa > 0$ determines an internal length scale of a possible microstructure in plastic strain and in (7.1) the κ-terms have just a simplified form; cf. also [55, 61, 62]. Let $P \subset \mathbb{R}^{n\times n}_{\mathrm{sym},0}$ be a convex closed neighbourhood of the origin, δ_P is its indicator function, and δ^*_P the conjugate functional to δ_P with respect to the duality pairing $\sigma : e = \sum_{i,j=1}^n \sigma_{ij} e_{ij}$. Then we consider the cone $S := \{z = (\pi, \eta);\ \eta \ge \delta^*_P(\pi)\ \text{ a.e. on } \Omega\}$. The degree-1 homogeneous dissipation potential is

$$\zeta_1(\dot{\pi}, \dot{\eta}) = \delta^*_P(\dot{\pi}) + \delta_S(\dot{\pi}, \dot{\eta}). \tag{7.2}$$

Choosing the initial conditions $\eta_0 = 1$ makes P the initial elasticity domain that may be "inflated" within evolution of the hardening. Then the initial condition π_0 such that $z_0 := (\pi_0, \eta_0) \in S$ a.e. on Ω ensures that $z \in S$ holds also during the evolution a.e. on Q. Then we can consider φ restricted on S, which makes it coercive as (2.12a) if $b(\zeta) \ge c_0|\zeta|^q$. Note that ζ_1 is not continuous and even does not satisfy (2.12b) but (3.7c) still holds with the help of coercivity of φ. Altogether, (7.1)–(7.2) fits with the ansatz (5.16c) with $\phi_1(e, z) := -\mathbb{C}e : a(\pi) + \frac{1}{2}\mathbb{C}a(\pi) : a(\pi) + b(\zeta)$ and $\phi_2(e, z, \nabla z) := \frac{1}{2}\mathbb{C}e : e + \frac{\kappa}{2}|\nabla\pi|^2 + \frac{\kappa}{2}|\nabla\eta|^2$.

Usage of Proposition 5.3 urged ϕ_1 to be (e)-semiconvex, which needs here strong convexity of $(\pi, \zeta) \mapsto \mathbb{C}a(\pi) : a(\pi) + b(\zeta)$ and a'' bounded. Without such qualification, one can usage Proposition 4.3 based on only (e, z)-semiconvexity since (7.1) fits also with (4.10) for $\phi_0(z) := (a(\pi), \frac{1}{2}\mathbb{C}a(\pi) : a(\pi) + b(\zeta))$ and the affine $\phi_1(e, \nabla z) := (-\mathbb{C}e, 1)$. However, note that (4.19) needs ζ_1 continuous, which is not satisfied in (7.2), hence Proposition 4.5 cannot be employed.

The standard linearized model uses a as an identity and $b(\eta) = \frac{1}{2}b_0\eta^2$ with $b_0 > 0$ a hardening coefficient, cf. e.g. [1, 9, 32, 42, 54]. Then φ itself is convex and even quadratic. For this fully quadratic φ, one can even consider $\kappa = 0$ when adopting the above results by omitting ∇z; thus (7.1) turns into

$$\varphi(e, z) \equiv \varphi(e, \pi, \eta) = \frac{1}{2}\mathbb{C}(e - \pi) : (e - \pi) + \frac{1}{2}b_0\eta^2. \tag{7.3}$$

Even an inviscid case with $\zeta_2 = 0$ can be handled by a monotonicity method, see [1, Chap.4] and a series of papers [10, 11, 12, 13], in some of them even only semi-coercivity is admitted.

Example 7.2 (*Magnetostriction.*) The role of the internal parameter z in magnetostrictive materials is played by the magnetization vector $\vec{m} \in \mathbb{R}^n$. We consider a so-called anisotropic energy $\phi : \mathbb{R}^m \to \mathbb{R}$ and a function $e_\mathrm{p} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ (usually quadratic) describing dependence of the preferred strain e_p on magnetization. In contrast to most of mathematical literature, we do not count exactly the so-called Heisenberg constraint $|\vec{m}| = m_\mathrm{s}$ with $m_\mathrm{s} > 0$ being the saturation magnetization, which is actually not exactly satisfied even in reality and deviation from it might be in tens of percents, cf. [6, Fig.5.4]. Rather, we consider such a constraint involved by some "penalization" in ϕ itself. The stored energy in magnetostriction (with demagnetizing field neglected) is then considered as

$$\varphi(e, \vec{m}, \nabla\vec{m}) = \frac{1}{2}\mathbb{C}(e - e_\mathrm{p}(\vec{m})):(e - e_\mathrm{p}(\vec{m})) + \phi(\vec{m}) + \frac{\kappa}{2}|\nabla\vec{m}|^2 \tag{7.4}$$

where the last term is the so-called exchange energy, cf. e.g. [34, 60] or also [21, 42]. In evolution, we neglect any rate-dependent phenomena as far as $\vec{m}$ concerns. Then ζ_1 may describe activation energy for re-magnetization, which is related to a so-called coercive force. It determines the width of the hysteresis loop of an infinitesimally slow quasi-static cyclic magnetization process. This energy is finite, i.e. S is bounded, and hence ζ_1 continuous. Although it fits with (5.16a) with $p = 2 = q$, Proposition 5.3 needs (e)-semiconvexity of φ which is not compatible with expected nonconvexity of ϕ. On the other hand, it complies with (4.10), Propositions 4.3 and 4.5 based on only (e, z)-semiconvexity of φ can be employed. Proposition 4.3 itself needs φ'_z in (4.6b) which can be at disposal for (7.4) because we did not treated the Heisenberg constraint exactly, while Proposition 4.5 could be used if Definition 4.4 were generalized for nonconvex $\varphi(e, \cdot, \cdot)$.

Augmenting of the stored energy by the demagnetizing-field energy, which is a non-local but quadratic term of the form $\int_{\mathbb{R}^n} |\nabla\Delta^{-1}\mathrm{div}(\chi_\Omega\vec{m})|^2\,\mathrm{d}x$ with χ_Ω the characteristic function of Ω, does not brings any essential problems into the above presented theory. Beside mechanical loading, one can consider also an external magnetic field h varied in time, which then contributes by h occurring on the right-hand side of the inclusion (2.1b).

Example 7.3 (*Piezostriction.*) The role of the internal parameter z in ferroelectric materials exhibiting piezostriction is played by the electric polarization vector $\vec{p} \in \mathbb{R}^n$. The stored-energy potential of Landau-Devonshire's type is considered as

$$\varphi(e, \vec{p}, \nabla\vec{p}) = \phi_0(e) + \phi_1(e, \vec{p}) + \phi_2(\vec{p}) + \phi_3(\nabla\vec{p}) \tag{7.5}$$

where ϕ_0, ϕ_3 is convex quadratic, $\phi_1(\cdot, \vec{p})$ affine and $\phi_1(e, \cdot)$ quadratic, ϕ_2 a 6th-order coercive polynom; for specific form of $\phi_{0,\ldots,2}$ see [41] where ϕ_2 has a convex part being a 6th-order polynom and a nonconvex part being a 4th-order polynom. This is important to make the functional $\vec{p} \mapsto \int_\Omega \phi_2(\vec{p})\mathrm{d}x$ weakly lower semicontinuous on $W^{1,2}(\Omega; \mathbb{R}^n)$ if $n \le 3$ due to the Sobolev embedding $W^{1,2}(\Omega) \subset L^6(\Omega)$.

The dissipation through a continuous ζ_1 describes an activated process of a re-polarization. For a fully rate-independent model also with ϕ_3 see [53]. Again, (7.5) fits with (5.16a) with $p = 2 = q$ but, since $\varphi(e, \cdot, Z)$ is presumably nonconvex, Proposition 5.3 cannot be employed. Yet, now (7.5) also complies with (4.10) if $\phi_1(e, \vec{p}) := \phi_{11}(e) \cdot \phi_{12}(\vec{p})$ with some ϕ_{11} affine, as indeed taken in [41], hence Proposition 4.3 based on (e, z)-semiconvexity can be used. Also, (4.19a) would be applicable together Proposition 4.5 itself if Definition 4.4 were generalized for nonconvex $\varphi(e, \cdot, \cdot)$.

Like in Example 7.2, augmenting of the stored energy by the depolarising-field energy, which is a nonlocal but quadratic term of the form $\int_{\mathbb{R}^n} |\nabla\Delta^{-1}\mathrm{div}(\chi_\Omega\vec{p})|^2\,\mathrm{d}x$, does not brings any essential problems into the above presented theory. Beside mechanical loading, one can consider also an external electric field $\vec{e}$ varied in time, which then contributes by $\vec{e}$ occurring on the right-hand side of the inclusion (2.1b).

Example 7.4 (*Shape-memory alloys.*) A popular simple model of so-called shape-memory alloys takes a "mixture" of quadratic energies in the form

$$\varphi(e, z, \nabla z) := \frac{1}{2}\sum_{\ell=1}^{m} z_\ell \mathbb{C}_\ell(e{-}e_\ell){:}(e{-}e_\ell) + \phi_0(z) + \delta_K(z) + \frac{\kappa}{2}|\nabla z|^2, \quad \text{where } \ e_\ell := \frac{U_\ell^\top {+} U_\ell}{2}, \tag{7.6}$$

with the prescribed distortion matrices U_ℓ of particular pure phases (or phase variants) and $K := \{z \in \mathbb{R}^m;\ z_\ell \ge 0\ \&\ \sum_{\ell=1}^m z_\ell = 1\}$ the so-called Gibbs' simplex. The setting here is related with the situation of martensitic transformation in a single-crystal and z's are volume fractions of the so-called austenite and of particular variants of martensite, e.g. $m = 4$ or 7 for tetragonal or orthorhombic martensite, respectively. Each (phase)variant has its own linear elastic response prescribed by the positive-definite tensor $\mathbb{C}_\ell$. On the other hand, the (simplified) viscous response prescribed by $\mathbb{D}$ is the same for every phase. The function ϕ_0 reflects the difference between chemical energies of austenite and martensite and also between pure phases and "mixtures". The philosophy of mixtures of austenite/martensite phases in shape-memory alloys has been proposed by Frémond [24]; in a rate-independent variant also presented in [25]. For its analysis and numerical implementation see [15, 16, 27, 33]. Gradients of mesoscopical volume fractions, i.e., $\kappa > 0$ in (7.6), has already been used in Frémond's model [25, p.364] or [27, Formula (7.20)]. Like in Example 7.2, ζ_1 may describe activation energy for so-called martensitic transformation or re-orientation of martensitic variants. This energies are finite, i.e. S is bounded, and hence ζ_1 continuous. Thought it fits with (5.16a) with $p = 2 = q$, (e)-semiconvexity of φ

from (7.6) is not guaranteed in general. For it, the ansatz (7.6) is to be regularized, e.g., by adding $\epsilon|e|^6$ which leads to strict (e)-semiconvexity of φ provided ϕ_0 is uniformly convex, which would then facilitate usage of Proposition 5.3. In fact, we need the (e)-semiconvexity only to derive (3.8). Once we have a correct energy inequality for the continuous problem, we can easily pass to the limit with $\varepsilon \to 0$ at the end. For (7.6) itself, usage of (4.19) and Proposition 4.5 based on (e, z)-semiconvexity is possible, yet Proposition 4.3 cannot be employed since φ'_z is not at disposal due to the indicator function δ_K in (7.6). In a special case if the elastic-moduli tensors $\mathbb{C}_\ell = \mathbb{C}$ are equal for all phases, the specific energy (7.6) reduces to

$$\varphi(e, z, \nabla z) = \frac{1}{2}\mathbb{C}\big(e - e_{\rm tr}(z)\big):\big(e - e_{\rm tr}(z)\big) + \phi_1(z) + \delta_K(z) + \frac{\kappa}{2}|\nabla z|^2 \quad \text{with} \quad e_{\rm tr}(z) = \sum_{\ell=1}^{m} z_\ell e_\ell, \tag{7.7}$$

where $e_{\rm tr}(z)$ is the so-called *transformation strain*. For such a model we refer e.g. to [2, 3, 8, 28, 29, 35, 42, 64]. It simplifies the situation substantially because now φ may be convex, depending whether ϕ_1 is so. Note that (4.10) and (5.16b,c) do not comply with the term $\delta_K(z)$ in (7.6) and (7.7), however. Phase-transformation can also be combined with plasticity, see e.g. [4, 58].

Example 7.5 (*Partial damage.*) For a scalar model of damage with a possible healing, we consider $m = 1$ and $S := [-a, b]$ where $a > 0$ is an activation threshold for damage development and $b > 0$ an activation threshold for "healing" of damage. Usually, $b \gg a$ and, if $b = +\infty$, damage is a unidirectional process without any healing possible and then $S := [-a, +\infty)$ and $\zeta_1(\dot z) = \delta^*_S(\dot z) = -a\dot z + \delta_{(-\infty,0]}(\dot z)$. The stored energy is considered as

$$\varphi(e, z, \nabla z) := \frac{1}{2}\phi(z)\mathbb{C}e : e + \frac{\kappa}{2}|\nabla z|^2 + \delta_{[0,1]}(z), \tag{7.8}$$

with $\phi : [0, 1] \to [\alpha, 1]$ increasing and $\mathbb{C}$ the 4th-order tensor of elastic moduli as in Example 7.1. Natural initial condition is $z_0 = 1$, i.e. undamaged material. We consider $\alpha > 0$, which is related to a so-called uncomplete, partial damage so that the material cannot completely disintegrate even for $z = 0$. The so-called factor of influence $\kappa > 0$ is related with a certain "hardening" effects: activation threshold a is effectively increased/decreased at a given point if its surrounding is less/more damaged, respectively, cf. also [7, 19, 25, 26, 44, 47, 49]. In general, φ is not convex but $\varphi(\cdot, z, \cdot)$ is convex quadratic and complies with (5.16a) for $p = 2 = q$ provided the healing threshold is $b < +\infty$. Certain healing may indeed occur in various biomaterials or polymer adhesives, cf. [5, 40, 65]. Mathematically, healing was used e.g. in [44, 59]. Note that $\varphi(e, \cdot, \cdot)$ is (strictly) convex if ϕ in (7.8) is so. If regularizing (7.8) by adding, say, $\epsilon|e|^6$ with $\epsilon > 0$, we get (strict) (e)-semiconvexity of φ. In this case, it also complies with (4.19) and Proposition 4.5 can be used. Again, ε can be passed to 0 at the end.

In some case, even joint convexity of φ is possible. Let us demonstrate it for $n = 1$ and ϕ itself convex and $\phi''\phi \geq 2(\phi')^2$. Then the Jacobian φ'' is just positive semi-definite; note that then $\varphi''_{ee}(z) = \phi(z) > 0$, $\varphi''_{zz}(e, z) = \frac{1}{2}\phi''(z)\mathbb{C}e^2 \geq 0$, and $\det(\varphi''_{(e,z),(e,z)}(e, z)) = (\frac{1}{2}\phi''(z)\phi(z) - \phi'(z)^2)\mathbb{C}^2e^2 \geq 0$. Then (7.8) complies also with (4.19) if $b < +\infty$. The convexity of φ is satisfied e.g. for

$$\phi(z) := \frac{\alpha}{1 - (1-\alpha)z} \tag{7.9}$$

with $0 < \alpha < 1$. Note that $\phi(0) = \alpha$, $\phi(1) = 1$, ϕ increasing, and, indeed, φ is convex since $\phi''\phi = 2(\phi')^2$.

The unidirectional damage with $b = +\infty$ is, unfortunately, not covered by any of the previous results because both ζ_1 and φ are simultaneously discontinuous. For some special techniques we refer to [7, 47, 49].

Note also that the viscosity ζ_2 is independent of z, which is a certain simplification. An alternative and more realistic model would consider $\zeta_2(\dot e, z) := \phi(z)\mathbb{D}\dot e : \dot e$. The limit passage in the term $\phi(z)\mathbb{D}e(\frac{\partial u}{\partial t}) : e(v)$ in the weak formulation of the mechanical equilibrium is simple because z is compact in any $L^{q^*-\epsilon}(Q)$ while $e(\frac{\partial u}{\partial t})$ is bounded in $L^2(Q; \mathbb{R}^{n\times n}_{\rm sym})$. However, (4.22) fails. Special tricks then must be employed, cf. [49] for ϕ linear.

Remark 7.6 (*Nonlocal regularization of z.*) Instead of the term $\frac{\kappa}{2}\int_\Omega |\nabla z|^2 {\rm d}x$ in (7.6), (7.7), or (7.8), one can use

$$|z|^2_\alpha = \frac{\kappa}{4}\int_\Omega\int_\Omega \frac{|z(x) - z(\xi)|^2}{|x-\xi|^{n+2\alpha}}\,{\rm d}x{\rm d}\xi \tag{7.10}$$

with $0 < \alpha < 1$. The boundary conditions simplifies by omitting the last condition in (2.6). In principle, more physically justified kernels with a support localized around the diagonal $\{x = \xi\}$ with the same singular behaviour as $|x - \xi|^{-n-2\alpha}$ for $|x - \xi| \to 0$ could equally be used in (7.10). Then, instead of $W^{1,q}(\Omega; \mathbb{R}^m)$, one should use the Sobolev-Slobodetskiĭ space $W^{\alpha,2}(\Omega; \mathbb{R}^m)$ with a fractional derivative, having similar compactifying effects as 1st-order derivatives.

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