Nečas Center for Mathematical Modeling

Incompressible ionized non-Newtonean fluid mixtures

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INCOMPRESSIBLE IONIZED NON-NEWTONEAN FLUID MIXTURES[1]

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Abstract: The model combining Navier-Stokes' equation in a non-Newtonean p-power-law modification for barycentric velocity together with Nernst-Planck's equation for concentrations of particular mutually reacting constituents, the heat equation, and the Poisson equation for self-induced quasistatic electric field is formulated, existence of its (very) weak solutions is proved for $p > 11/5$, and its thermodynamics is discussed.

Keywords: chemically reacting fluids, Eckart-Prigogine concept, Navier-Stokes equation, Nernst-Planck equation, Poisson equation, heat equation.

AMS Subj. Class. 35Q35, 76T30, 80A20, 92C05.

1 Introduction

Chemically reacting mixtures represent a framework for modelling of various complicated processes in biology and chemistry. The research in this area, resulting to a model [32], has been initiated by J. Nečas who, during many years before he passed away, spoke about "living fluids", although he never elaborated any concept of such fluids. The model proposed in [32] uses incompressible Newtonian framework with the barycentric impulse balance. This "barycentric" approach is called the Eckart-Prigogine's [11, 25] concept, simplifying phenomenologically the description by considering only one temperature and one velocity of the whole mixture and having been awarded in the context of non-equilibrium thermodynamics of dissipative structures by Nobel prize in chemistry in 1977; in the compressible case, see also [2, 4, 10, 15]. The incompressibility refers here both to each particular constituent and, through a volume-additivity hypothesis (i.e. Amagat's law) as in e.g. [20, 27, 36], also to the overall mixture. To cover biological applications on a cellular or subcellular level where intensity of electric field on cell membranes is very high, the self-induced electrostatic field must be considered; recall that the intra-cellular electric potential ranges usually over 60-100 mV while the thickness of cell membranes is of the order of 10-100 nm, which results to intensity of electric field of the order of 1 MV/m.

In comparison with [29, 30, 32] or [31, Sect. 12.6], we consider here a more general model exploiting the non-Newtonian concept with a (possibly temperature-dependent) shear-thickening p-power-law stress tensor and admitting diffusive fluxes with different mobilities, and we prove existence of its solution in a fully coupled and fully nonlinear case. The key mathematical tool is a non-variational technique for the heat equation based on integrability of temperature gradient observed in [5, 6, 7] combined with a regularization of the Navier-Stokes equation and a sophisticated limit passage. Finally, in Sect. 4, thermodynamics of a specific model is discussed.

2 The model: a general framework

We consider a 3-dimensional incompressible flow of a mixture of L mutually reacting chemical ionic constituents; the $\ell^{\rm th}$-constitutent having a specific charge z_ℓ, $\ell = 1, ..., L$. Our model consists in a system of 4+L+1+1 differential equations combining the *non-Newtonian* modification of the *Navier-Stokes equation* (balancing the barycentric momentum ϱv) with the incompressibility constraint $\mathrm{div}(v) = 0$, the *Nernst-Planck equation* modified for moving media (balancing the mass of particular constituents), the *heat equation* (balancing the heat part $c_{\rm v}\theta$ of the internal energy u, cf. (4.17) below), and the

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quasistatic *Poisson equation* for the electrostatic field (balancing the electric induction $\varepsilon\nabla\phi$):

$$\varrho\frac{\partial v}{\partial t} - \mathrm{div}\big(\tau(\mathrm{D}v,c,\theta) - \varrho v\otimes v\big) + \nabla\pi = -q\nabla\phi, \qquad q = c\cdot z, \tag{2.1a}$$

$$\mathrm{div}(v) = 0 \tag{2.1b}$$

$$\frac{\partial c}{\partial t} - \mathrm{div}\big(\mathfrak{D}(c,\theta)\nabla c + \mathfrak{m}(c,\theta)\otimes\nabla\phi - c\otimes v\big) = r(c,\theta)\,, \tag{2.1c}$$

$$c_{\rm v}\frac{\partial\theta}{\partial t} - \mathrm{div}\big(\kappa\nabla\theta - c_{\rm v}v\,\theta\big) = \tau(\mathrm{D}v,c,\theta){:}\mathrm{D}v + \big(\mathfrak{D}(c,\theta)\nabla c + \mathfrak{m}(c,\theta)\otimes\nabla\phi\big) : (z\otimes\nabla\phi) + h(c,\theta), \tag{2.1d}$$

$$-\,\mathrm{div}(\varepsilon\nabla\phi) = q\ . \tag{2.1e}$$

The variables v, π, c, θ, and ϕ have the following meaning:
$v = (v_1, v_2, v_3)$ barycenter velocity,
π pressure,
$c = (c_1, ..., c_L)$ the vector of concentrations of particular constituents,
ϕ electrostatic potential,
θ temperature,
where the concentration vector c is to satisfy the constraint

$$\forall\ell = 1,...,L: \quad c_\ell(t,x) \ge 0 \quad \text{and} \quad \sum_{\ell-1}^{L} c_\ell(t,x) = 1 \quad \text{for a.a. } (t,x). \tag{2.2}$$

We will write briefly (2.2) as $c(t,x)\in G_1^+$ for a.a. (t,x), where we denoted

$$G_1^+ := \{c\in\mathbb{R}^L;\quad c\cdot\mathbf{1} = 1,\quad \forall\ell = 1,..,L:\quad c_\ell \ge 0\} \tag{2.3}$$

with $\mathbf{1}\in\mathbb{R}^L$ denoting the "unit" vector $(1,...,1)$; usually G_1^+ is called the *Gibbs simplex*. The meaning of the scalar or tensorial products (denoted by "$\cdot$" and "$\otimes$", respectively) is standard, while ":" means $[\tau_{ij}]{:}[e_{ij}] = \sum_{i=1}^n\sum_{j=1}^n \tau_{ij}e_{ij}$, i.e. (2.1a,b) means

$$\varrho\frac{\partial v_i}{\partial t} - \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(\tau_{ij}(\mathrm{Dv},c,\theta) - \varrho v_i v_j\big) + \frac{\partial\pi}{\partial x_i} = -q\frac{\partial\phi}{\partial x_i}, \quad q = \sum_{\ell=1}^{L} c_\ell z_\ell, \tag{2.4a}$$

$$\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j} = 0, \tag{2.4b}$$

while (2.1c) means

$$\frac{\partial c_\ell}{\partial t} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\bigg(\sum_{k=1}^{L}\mathfrak{D}_{k\ell}(c,\theta)\frac{\partial c_k}{\partial x_i} + \mathfrak{m}_\ell(c,\theta)\frac{\partial\phi}{\partial x_i} - v_i c_\ell\bigg) = r_\ell(c,\theta) \tag{2.5}$$

for any $\ell = 1,...,L$, and (2.1d) means

$$c_{\rm v}\frac{\partial\theta}{\partial t} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\bigg(\kappa\frac{\partial\theta}{\partial x_i} - c_{\rm v}v_i\,\theta\bigg) = \sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}(\mathrm{D}v,c,\theta)\bigg(\frac{1}{2}\frac{\partial v_j}{\partial x_i} + \frac{1}{2}\frac{\partial v_i}{\partial x_j}\bigg) + \sum_{\ell=1}^{L}\sum_{i=1}^{3}\bigg(\sum_{k=1}^{L}\mathfrak{D}_{k\ell}(c,\theta)\frac{\partial c_k}{\partial x_i} + \mathfrak{m}_\ell(c,\theta)\frac{\partial\phi}{\partial x_i}\bigg)z_\ell\frac{\partial\phi}{\partial x_i} + h(c,\theta). \tag{2.6}$$

The meaning of the data is:

$\tau = [\tau_{ij}]_{i,j=1}^3$ the stress tensor, depending on $(\mathrm{D}v, c, \theta)$,
$\mathrm{D}v = \frac{1}{2}(\nabla v)^\top + \frac{1}{2}\nabla v$ the symmetrized velocity gradient,
$\varrho > 0$ mass density (without loss of generality assumed equal 1 in what follows),
$z = (z_1, ..., z_L)$ the vector of specific charges of particular constituents,
$q = c \cdot z$ the total charge, depending on time t and space x,
$\varepsilon > 0$ permitivity,
$r = (r_1, ..., r_L)$ the vector of chemical production rates, depending on (c, θ),
h heat production rate due to all chemical reactions, depending on (c, θ),
$\mathfrak{D} = [\mathfrak{D}_{kl}]_{k,l=1}^L$ the matrix of diffusion coefficients, depending on (c, θ),
$\mathfrak{m} = (\mathfrak{m}_1, ..., \mathfrak{m}_L)$ the vector of effective mobilities, depending on (c, θ),
$\kappa > 0$ thermal conductivity, and
$c_\mathrm{v} > 0$ heat capacity.

The system (2.1) is to be completed by the initial conditions

$$v(0,\cdot) = v_0\,, \qquad c(0,\cdot) = c_0\,, \qquad \theta(0,\cdot) = \theta_0 \tag{2.7}$$

on the considered fixed bounded C^2-domain $\Omega \subset \mathbb{R}^3$, and by the boundary conditions corresponding, e.g., to a closed container, which, in some simplified version, leads respectively to:

$$v = 0, \tag{2.8a}$$

$$\big(\mathfrak{D}(c,\theta)\nabla c + \mathfrak{m}(c,\theta) \otimes \nabla\phi\big)\nu = 0, \tag{2.8b}$$

$$\varepsilon\frac{\partial\phi}{\partial\nu} = \alpha(\phi_\Sigma - \phi), \tag{2.8c}$$

$$\kappa\frac{\partial\theta}{\partial\nu} = 0 \tag{2.8d}$$

on $(0,T) \times \partial\Omega$, where ν is the unit outward normal to the boundary $\partial\Omega$ and ϕ_Σ is a prescribed external electric potential and $\alpha > 0$ is a "lumped capacity" of the boundary $\partial\Omega$.

Remark 2.1 (*Right-hand sides of (2.1).*) The meaning of the right-hand side of (2.1a) is the *Lorenz force* $q\nabla\phi$ due to Coulomb electrostatic interactions. The particular terms on the right-hand side of (2.1d) mean respectively the production rate of the *dissipative heat* due to friction in the fluid $\tau(\mathrm{D}v, c, \theta){:}\mathrm{D}v$, the power $(\mathfrak{D}(c,\theta)\nabla c) : (z \otimes \nabla\phi)$ of the electric current arising by the diffusion flux $z^\top(\mathfrak{D}(c,\theta)\nabla c)$ in the electric field gradient $\nabla\phi$ (so-called *Peltier effect*), the power of the *Joule heat* produced by the electric current $(\mathfrak{m} \otimes \nabla\phi) : (z \otimes \nabla\phi) = (z{\cdot}\mathfrak{m})|\nabla\phi|^2$, and, as already said, heat production rate due to all chemical reactions h; see also Remark 4.6 below.

Remark 2.2 (*Fourier, Fick, Ohm's laws.*) The model (2.1) involves various phenomenological laws. Certainly, (2.1d) relies on the conventional *Fourier law* in linear isotropic homogenous medium, i.e. the heat flux $-\kappa\nabla\theta$ is proportional to the negative temperature gradient. Further, (2.1c) involves a certain generalization of *Fick's law* saying that diffusive fluxes are proportional to negative concentration gradients; here, however, cross-effects make it more complicated and a non-constant diffusivity matrix $\mathfrak{D}$ occurs instead of a single constant, cf. also (3.8) below. In view of Remark 2.1, the effective *electric conductivity* is $\sigma := z{\cdot}\mathfrak{m}$, and we can identify *Ohm's law* that the electric current $(z{\cdot}\mathfrak{m})\nabla\phi$ is proportional to the gradient $\nabla\phi$ of the electric field just via σ. Naturally, σ now depends through $\mathfrak{m} = \mathfrak{m}(c,\theta)$ on the ion concentrations c.

Remark 2.3 (*Simplifying assumptions.*) It should be emphasized that many simplifications are adopted in the presented model. In particular, we have considered small electrical currents (i.e. magnetic field is neglected), we adopted the mentioned volume-additivity and incompressibility assumption, we have further assumed mass densities equal for all constituents and the diffusion fluxes independent of the temperature gradient (i.e. Soret's effect is neglected) and then, in agreement with Onsager's reciprocity principle, we also consider the heat flux independent of the concentration gradients (i.e. Dufour's effect is neglected). Detailed identification of simplifying assumptions related to (2.1) in comparison with the rational Truesdell concept [22, 26, 34, 35, 37, 38] was made by Samohýl [36].

3 Analysis of the model: existence of a solution

We will prove the existence of a very weak solution, defined in Section 3.1, in several steps. First, in Section 3.2 we treat an auxiliary, so-called *multipolar* regularization of the Navier-Stokes equation and prove existence of its solution by Schauder's fixed point technique similarly like it was done in [29] for spatial regular case (except that [29] had thus assumed composition/temperature-independent potential stress-tensor τ). Then, in Section 3.3 we pass this regularization to zero. This two-step approach allows us to avoid any regularity results for p-power-law non-Newtonean fluids (and thus any qualifications of data related with them) and to admit temperature-dependent stress tensor and rather low exponent $p > 11/5$; this bound even improves some particular known results, cf. Remark 3.4. The advantage of the smoothening is that it avoids difficult (or even unrealistic) requirements of uniqueness or convexity of the set of solutions of decoupled systems needed for Schauder's or Kakutani's fixed point theorems. It is also particularly important for the multi-component fluids to have $\frac{\partial}{\partial t}c$ in duality with the negative part of c to get $c \geq 0$, cf. (3.57). As a side-efect, it simplifies some other arguments, e.g., it ensures the sum-equal-one property $\sum_{\ell=1}^{L} c_\ell = 1$ in (3.47) for smooth velocity field v, and the unique response (cf. e.g. (3.50)) for the fixed-point mapping to use simply Schauder's fixed point theorem.

3.1 Definition of a very weak solution and data qualification

We consider an evolution of (2.1) on a fixed time interval $(0, T)$. We use a standard notation $C^1(\cdot;\mathbb{R}^n)$ of continuously differentiable $\mathbb{R}^n$-valued functions, $L^p(\cdot;\mathbb{R}^n)$ for Lebesgue L^p-spaces as well as $W^{k,p}(\cdot;\mathbb{R}^n)$ for the Sobolev spaces on the domain indicated. Let us abbreviate $I := (0, T)$, $Q := I \times \Omega$, $\Sigma := I \times \Gamma$, $\Gamma := \partial\Omega$, and let $W^{k,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$ denote the space of functions from the zero-trace Sobolev space $W^{k,p}_0(\Omega;\mathbb{R}^3)$ but with zero divergence (in the distributional sense), and later also we will use $L^2_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$ denoting the closure of $W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$ in $L^2(\Omega;\mathbb{R}^3)$. Usage of such divergence-free test functions makes the usual effect that the pressure π disappears from the (very) weak formulation. If $\mathbb{R}^n$ is replaced by a Banach space X, then $L^p(I;X)$ refers to the L^p-Bochner space of Banach-space-valued functions while $W^{k,p}(I;X)$ is a respective Sobolev-Bochner space. We also denote standardly $W^{-k,2}(\Omega) = W^{k,2}_0(\Omega)^*$.

The adjective "very weak" wants to emphasize that, contrary to conventional weak solutions, the very weak solutions have less regularity than possible test functions, which concerns particularly the temperature.

Definition 3.1 ***(Very weak solution.)*** *We will call*

$$v \in L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)) \cap W^{1,p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*), \tag{3.1a}$$

$$c \in L^\infty(I;L^2(\Omega;\mathbb{R}^L)) \cap L^2(I;W^{1,2}(\Omega;\mathbb{R}^L)) \cap W^{1,r/(r-1)}(I;W^{1,r}(\Omega;\mathbb{R}^L)^*), \tag{3.1b}$$

$$\theta \in L^\infty(I;L^1(\Omega)) \cap L^{5/4-\xi}(I;W^{1,5/4-\xi}(\Omega)) \cap W^{1,1}(I;W^{-3,2}(\Omega)), \tag{3.1c}$$

$$\phi \in L^\infty(I;W^{1,2}(\Omega)) \tag{3.1d}$$

with any $\xi > 0$ *and* $r = \max(2, 10p/(7p-6))$ *a very weak solution to the system (2.1)–(2.2)* *with the initial and boundary conditions (2.7) and (2.8) if*

$$\int_Q \tau(\mathrm{D}v,c,\theta){:}\mathrm{D}w - (v\otimes v){:}\nabla w + (z\cdot c)(\nabla\phi\cdot w) - v\frac{\partial w}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_\Omega v_0(x)\cdot w(0,x)\,\mathrm{d}x \tag{3.2}$$

for any $w\in C^1(Q;\mathbb{R}^3)$ *with* $\operatorname{div} w=0$, $w|_\Sigma=0$ *and* $w(T,\cdot)=0$,

$$\int_Q \big(\mathfrak{D}(c,\theta)\nabla c + \mathfrak{m}(c,\theta)\otimes\nabla\phi - c\otimes v\big){:}\nabla w + r(c,\theta)w - c\cdot\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega c_0(x)\cdot w(0,x)\,\mathrm{d}x \tag{3.3}$$

with the test-function $w\in C^1(Q;\mathbb{R}^L)$ *arbitrary with* $w(T,\cdot)=0$,

$$\int_Q \varepsilon\nabla\phi\cdot\nabla w - qw\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \phi\alpha w\,\mathrm{d}S\,\mathrm{d}t = \int_\Sigma \alpha\phi_\Sigma w\,\mathrm{d}S\,\mathrm{d}t \tag{3.4}$$

for any $w\in C^1(Q)$, *and*

$$\int_Q \big(\kappa\nabla\theta - c_{\rm v}v\theta\big)\cdot\nabla w - \Big(\tau(\mathrm{D}v,c,\theta){:}\mathrm{D}v + \big(\mathfrak{D}(c,\theta)\nabla c + \mathfrak{m}(c,\theta)\otimes\nabla\phi\big) : (z\otimes\nabla\phi) + h\Big)w - c_{\rm v}\theta\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = c_{\rm v}\int_\Omega \theta_0(x)w(0,x)\,\mathrm{d}x \tag{3.5}$$

for any $w\in C^1(Q)$ *with* $w(T,\cdot)=0$ *on* Ω. *Finally, (2.2) is to be satisfied, too.*

We naturally assume the mass conservation in all chemical reactions, and the volume-additivity constraint holding for the initial conditions c_0, i.e.

$$r(c,\theta)\cdot\mathbf{1} = 0, \tag{3.6a}$$

$$c_0\cdot\mathbf{1} = 1, \qquad [c_0]_\ell \ge 0 \quad \forall \ell=1,...,L. \tag{3.6b}$$

Other important qualification concerns the diffusion matrix $\mathfrak{D}$ and the effective-mobility vector $\mathfrak{m}$:

$$\mathfrak{D}: G_1^+\times\mathbb{R}\to\mathbb{R}^{L\times L},\ \ \mathfrak{m}: G_1^+\times\mathbb{R}\to\mathbb{R}^L \ \text{ continuous and bounded,} \tag{3.7a}$$

$$\exists\eta_0>0\ \forall c{\in}G_1^+,\ \theta{\in}\mathbb{R},\ d{\in}\mathbb{R}^L:\quad d^\top\mathfrak{D}(c,\theta)d := \sum_{\ell=1}^L\sum_{k=1}^L \mathfrak{D}_{k\ell}(c,\theta)d_kd_\ell \ge \eta_0|d|^2, \tag{3.7b}$$

$$\exists\beta\ge 0\ \forall k=1,...,L:\quad \sum_{\ell=1}^L \mathfrak{D}_{k\ell}(c,\theta) = \beta, \tag{3.7c}$$

$$\sum_{\ell=1}^L \mathfrak{m}_\ell(c,\theta) = 0, \tag{3.7d}$$

where ${G_1}^+$ is from (2.3). Note that, if $\beta=0$, (3.7c,d) means that the sum $\sum_{\ell=1}^L j_\ell$ of the diffusive fluxes

$$j_\ell := \sum_{k=1}^L \mathfrak{D}_{k\ell}(c,\theta)\nabla c_k + \mathfrak{m}_\ell(c,\theta)\nabla\phi \tag{3.8}$$

is identically zero, which is to hold the equality constraint in (2.2). Essentially the same effect is made by (3.7c,d) also if $\beta>0$, cf. the arguments around (3.47).

As to the stress tensor $\tau : \mathbb{R}^{3\times 3}_{\rm sym} \times G_1^+ \times \mathbb{R} \to \mathbb{R}^{3\times 3}_{\rm sym}$, where $\mathbb{R}^{3\times 3}_{\rm sym}$ denotes the set of symmetric 3×3-matrices, we assume that, for some $\eta_1 > 0$, $C \in \mathbb{R}$, it satisfies

$$\tau : \mathbb{R}^{3\times 3}_{\rm sym} \times G_1^+ \times \mathbb{R} \to \mathbb{R}^{3\times 3}_{\rm sym} \quad \text{continuously differentiable,} \tag{3.9a}$$

$$\forall D_1, D_2 {\in} \mathbb{R}^{3\times 3}_{\rm sym},\ c{\in} G_1^+,\ \theta{\in}\mathbb{R}: \quad \big(\tau(D_1,c,\theta){-}\tau(D_2,c,\theta)\big){:}(D_1{-}D_2) \ge \eta_1\big|D_1{-}D_2\big|^p, \tag{3.9b}$$

$$\forall D \in \mathbb{R}^{3\times 3}_{\rm sym},\ c \in G_1^+,\ \ \theta \in \mathbb{R}: \quad |\tau(D,c,\theta)| \le C\big(1+|D|^{p-1}\big), \tag{3.9c}$$

$$\big|\tau'_D(D,c,\theta)\big| \le C\big(1+|D|^{p-2}\big), \tag{3.9d}$$

$$\big|\tau'_{(c,\theta)}(D,c,\theta)\big| \le C, \tag{3.9e}$$

$$\tau(0,c,\theta) = 0. \tag{3.9f}$$

Note that (3.9b,f) yields the coercivity $\tau(D,c,\theta) : D \ge \eta_1|D|^p$. Other important assumptions ensure non-negativity of concentrations during their evolution, namely by nonnegative production rate and by a natural direction of the flux j_ℓ of the ℓth constituent from (3.8) if the concentration of this particular constituent vanishes:

$$\mathfrak{D}_{k\ell}(c_1,...,c_{\ell-1},0,c_{\ell+1},...,c_L,\theta) \begin{cases} \ge 0 & \text{for } k=\ell, \\ = 0 & \text{for } k \ne \ell, \end{cases} \tag{3.10a}$$

$$\mathfrak{m}_\ell(c_1,...,c_{\ell-1},\ 0,\ c_{\ell+1},...,c_L,\theta) \ = 0\,, \tag{3.10b}$$

$$r_\ell\,(c_1,...,c_{\ell-1},\ 0,\ \ c_{\ell+1},...,c_L,\theta) \ \ge 0\,, \tag{3.10c}$$

for each $\ell = 1,...,L$. Eventually, we still assume

$$r_\ell, h : G_1^+ \times \mathbb{R} \to \mathbb{R} \ \text{ continuous and bounded.} \tag{3.11}$$

Remark 3.2 (*Extension convention.*) For the purpose of the proof of Proposition 3.9, we consider $\mathfrak{D}$, $\mathfrak{m}$, τ, r_ℓ, and h extended suitably from the Gibbs' simplex G_1^+ defined by (2.3) on the affine manifold

$$G_1 := \Big\{c{\in}\mathbb{R}^L;\ \sum_{\ell=1}^L c_\ell = 1\Big\}. \tag{3.12}$$

We assume contionuos and bounded extension so that (3.7b-d) and (3.9b-e) hold even for $c \in G_1 \setminus G_1^+$. Moreover, (3.10) allows us to consider non-negative extensions of r_ℓ and zero-extension of $\mathfrak{m}_\ell$ if $c_\ell \le 0$.

Remark 3.3 (*Data qualification versus reality.*) The assumption (3.11) represent a rather drastic mathematical simplification contrasting with the usual feature that the rate of chemical reactions r and the corresponding heat production h depends rather exponentially on temperature θ. In fact, making the estimates in Section 3.2 still in a more complicated (and less lucid) way, a certain (although only sub-linear) growth of $r(c,\cdot)$ and $h(c,\cdot)$ may be admitted, too; cf. also [32]. The mentioned exponential growth would allow for "explosive" blow-ups which we do not have in mind, especially in context of usual biological applications. Also, (3.7b) is not directly relevant and contradicts an Einstein law if $\theta \searrow 0$, cf. also the arguments in Remark 4.6. Yet, considering $\mathfrak{D}(c,\theta)$ approaching zero if $\theta \searrow 0$ would inevitably make the analysis of the problem extremely difficult, if possible at all. Anyhow, the model of fluid mixtures looses its validity much earlier than the absolute temperature θ approaches zero because of ultimate phase transition to solid state.

Remark 3.4 (*Special case: single-component fluids.*) A sub-system (2.1a,b,d) with $\mathfrak{D}$ and $\mathfrak{m}$ vanishing and with a general heat flux $j_0(\theta,\nabla\theta)$ instead of $\kappa\nabla\theta$ together with a fixed right-hand side instead of $q\nabla\phi$ was considered in [8]. Assuming monotonicity and

p_0-polynomial structure of $j_0(\theta,\cdot):\mathbb{R}^3\to\mathbb{R}^3$, existence of a weak solution was proved for $p\ge 5/2$ and $p_0\ge 10p/(5p{-}1)>2$. Physically, the heat flux j_0 may depend substantially on θ but the dependence of $\nabla\theta$ is rather linear, which corresponds to the case $p_0=2$ not covered by [8]. Our results cover, in particular, the case $11/5<p<5/2$ and enable to treat the physically more relevant case $p_0=2$ for such a sub-system (2.1a,b,d). Also, [8] assumed $\tau(\cdot,\theta)$ to have a potential, even with a special structure $\sum_{l=1}^L\mu_l(\theta)F_l(|\cdot|)$ (which we do not need at all) but, on the other hand, allowed for a temperature dependence of $c_{\rm v}$ and κ.

3.2 Auxiliary multipolar regularization

We will regularize (2.1a) by a $2k^{\rm th}$-order term $(-1)^k\epsilon\Delta^k v$ with a regularization parameter $\epsilon>0$ and with an integer $k\ge 5$ specified later (see (3.21) with (3.29)) as follows:

$$\frac{\partial v}{\partial t}-{\rm div}\big(\tau({\rm D}v,c,\theta)-v\otimes v\big)+\nabla\pi+(-1)^k\epsilon\Delta^k v=-c{\cdot}z\nabla\phi, \tag{3.13a}$$

$${\rm div}(v)=0. \tag{3.13b}$$

Such a "multipolar" regularization is even physically motivated, cf. [23]. Let us emphasize that we distinquish ε (the permitivity) from ϵ (the regularizing parameter. The boundary conditions (2.8) are now to be completed by another higher-order condition for the Δ^k-operator. In fact, its choice is not important as this term has only an auxiliary character; let us choose, say, the homogeneous Dirichlet condition

$$\frac{\partial^l v}{\partial\nu^l}=0,\qquad l=1,...,k-1. \tag{3.14}$$

We modify Definition 3.1 for a weak solution to the system (2.1c-e)-(3.13) with the initial and boundary conditions (2.7) and (2.8) and (3.14).

Definition 3.5 ***(Weak solution to (2.1c-e)-(3.13).)*** *We will call (v,c,θ,ϕ) satisfying*

$$v\in L^\infty(I;W^{k,2}_{0,{\rm DIV}}(\Omega;\mathbb{R}^3))\cap W^{1,2}(I;L^2(\Omega;\mathbb{R}^3)), \tag{3.15a}$$

$$c\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))\cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^L)^*), \tag{3.15b}$$

$$\theta\in L^\infty(I;W^{1,2}(\Omega))\cap W^{1,2}(I;L^2(\Omega)), \tag{3.15c}$$

$$\phi\in L^\infty(I;W^{1,2}(\Omega)) \tag{3.15d}$$

a weak solution to the system (2.1c-e)-(3.13) with the initial and boundary conditions (2.7) and (2.8) and (3.14) if (3.3), (3.4), (3.5) and (2.2) hold while, instead of (3.2), we require for any $w\in C^1(Q;\mathbb{R}^3)$ with ${\rm div}(w)=0$, $\frac{\partial^l}{\partial\nu^l}w|_\Sigma=0$ for $l=0,...,k{-}1$, and $w(T,\cdot)=0$ the following identity to hold

$$\int_Q\tau({\rm D}v,c,\theta){:}{\rm D}w-(v{\otimes}v):\nabla w+z{\cdot}c\nabla\phi{\cdot}w+\epsilon\nabla^k v\,\vdots\,\nabla^k w-v\frac{\partial w}{\partial t}\,{\rm d}x\,{\rm d}t =\int_\Omega v_0(x)\cdot w(0,x)\,{\rm d}x\ , \tag{3.16}$$

where "$\vdots$" denotes the scalar product of $k^{\rm th}$-order tensors; for $k=1$ or $k=2$ we already used "$\cdot$" or "$:$", respectively.

To correct the concentration satisfying the constraint $\sum_{\ell=1}^L c_\ell=1$ but possibly not being positive, we define a retract $K:G_1\to G_1^+$ by

$$K_\ell(c):=\frac{c_\ell^+}{\sum_{l=1}^L c_l^+},\qquad c_\ell^+:=\max(c_\ell,0), \tag{3.17}$$

where G_1 is from (3.12). Let us note that K is continuous and bounded on G_1, and leaves G_1^+ fixed, and even $K_\ell(c) = 0$ if $c_\ell \le 0$. Further, we consider r, h, $\mathfrak{D}$ and $\mathfrak{m}$ continuously and boundedly extended on G_1. Considering $\gamma = (\gamma_1, ..., \gamma_L) =$ "old" concentrations and $\vartheta =$ an "old" temperature field, we define the quadruple (v, c, θ, ϕ) as the weak solution to the *de-coupled regularized system*:

(3.18a) $$-\operatorname{div}(\varepsilon\nabla\phi) = q, \qquad\qquad q = z{\cdot}K(\gamma),$$

(3.18b) $$\frac{\partial v}{\partial t} - \operatorname{div}\big(\tau(\mathrm{D}v, \gamma, \vartheta) - v{\otimes}v\big) + \nabla\pi + (-1)^k\epsilon\Delta^k v = -q\nabla\phi,$$

(3.18c) $$\operatorname{div}(v) = 0\,,$$

(3.18d) $$\frac{\partial c}{\partial t} - \operatorname{div}\big(\mathfrak{D}(\gamma,\vartheta)\nabla c + \mathfrak{m}(\gamma,\vartheta)\otimes\nabla\phi - c\otimes v\big) = r(\gamma,\vartheta)\,,$$

(3.18e) $$c_{\mathrm{v}}\frac{\partial\theta}{\partial t} - \operatorname{div}\big(\kappa\nabla\theta - c_{\mathrm{v}} v\,\theta\big) = \tau(\mathrm{D}v,\gamma,\vartheta) : \mathrm{D}v + \big(\mathfrak{D}(\gamma,\vartheta)\nabla c + \mathfrak{m}(\gamma,\vartheta)\otimes\nabla\phi\big) : (z\otimes\nabla\phi) + h(\gamma,\vartheta),$$

(3.18f) $$c\cdot\mathbf{1} := \sum_{\ell=1}^{L} c_\ell = 1$$

with the boundary conditions (2.8) and (3.14) and with the initial conditions

(3.19) $$v(0,\cdot) = v_{0\epsilon}\,, \qquad c(0,\cdot) = c_0\,, \qquad \theta(0,\cdot) = \theta_{0\epsilon}.$$

Obviously, given (γ, ϑ), we are to solve first (3.18a), and after knowing also ϕ we can solve (3.18b,c) to get v, and then we can solve (3.18d) to obtain c, and finally (3.18e) to obtain also θ. In (3.19), we have made a regularization of the original initial conditions $v_0 \in L^2_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$ and $\theta_0 \in L^1(\Omega)$ respectively by $v_{0\epsilon} \in W^{k,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$ and $\theta_{0\epsilon} \in W^{1,2}(\Omega)$ in such a way that

(3.20a) $$\big\|v_{0\epsilon}\big\|_{W^{k,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)} \le \frac{C}{\epsilon}, \qquad \big\|\theta_{0\epsilon}\big\|_{W^{1,2}(\Omega)} \le \frac{C}{\epsilon}, \quad \text{and}$$

(3.20b) $$\big\|v_{0\epsilon}\big\|_{L^2(\Omega;\mathbb{R}^3)} \le C, \qquad \big\|\theta_{0\epsilon}\big\|_{L^1(\Omega)} \le C.$$

Proposition 3.6 ***(A-priori estimates for (3.18).)*** *Let the assumptions (3.6), (3.7), (3.9), (3.10), (3.11) hold, let $v_{0\epsilon} \in W^{k,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$, $c_0 \in L^\infty(\Omega;\mathbb{R}^L)$, $\theta_{0\epsilon} \in W^{1,2}(\Omega)$ satisfy (3.20), and let Ω be a bounded C^2-domain, and let $p \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfy*

(3.21) $$p > \frac{11}{5}, \quad \textit{and} \quad k \ge \frac{5p-3}{2}.$$

Let further $(\gamma,\vartheta) \in L^2(I;W^{1,2}(\Omega))^{L+1}$ be given such that $\sum_{\ell=1}^L \gamma_\ell = 1$ a.e. on Q. Then, (3.18) with the boundary condition (2.8) and the initial condition (3.19) has a weak solution (which need not satisfy $c_\ell \ge 0$, however, but) which satisfies, for any $\xi > 0$ and some

$C_0,...,C_{11}<+\infty$ independent of ϵ, the following a-priori estimates:

(3.22a) $$\big\|\phi\big\|_{L^\infty(I;W^{2,r}(\Omega))}\le C_1\quad\text{with}\ \ r<+\infty,$$

(3.22b) $$\big\|v\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))\,\cap\,L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le C_2,$$

(3.22c) $$\big\|v\big\|_{L^\infty(I;W^{k,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))\,\cap\,W^{1,2}(I;L^2(\Omega;;\mathbb{R}^3))}\le\frac{C_3\mathrm{e}^{C_0/\epsilon^2}}{\epsilon},$$

(3.22d) $$\Big\|\frac{\partial v}{\partial t}+(-1)^k\epsilon\Delta^k v\Big\|_{L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)}\le C_4,$$

(3.22e) $$\big\|c\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^L))\,\cap\,L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))}\le C_5,$$

(3.22f) $$\Big\|\frac{\partial c}{\partial t}\Big\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^L)^*)}\le\frac{C_6}{\sqrt{\epsilon}}\,,$$

(3.22g) $$\Big\|\frac{\partial c}{\partial t}\Big\|_{L^{r/(r-1)}(I;W^{1,r}(\Omega;\mathbb{R}^L)^*)}\le C_7\qquad\text{with}\ \ r=\max\Big(2,\frac{10p}{7p{-}6}\Big),$$

(3.22h) $$\big\|\theta\big\|_{L^2(I;W^{1,2}(\Omega))}\le\frac{C_8}{\sqrt{\epsilon}}\,,$$

(3.22i) $$\big\|\theta\big\|_{L^\infty(I;W^{1,2}(\Omega))\,\cap\,W^{1,2}(I;L^2(Q))}\le\frac{C_9\mathrm{e}^{C_0/\epsilon^2}}{\epsilon}\,,$$

(3.22j) $$\big\|\theta\big\|_{L^\infty(I;L^1(\Omega))\,\cap\,L^{5/4-\xi}(I;W^{1,5/4-\xi}(\Omega))}\le C_{10},$$

(3.22k) $$\Big\|\frac{\partial\theta}{\partial t}\Big\|_{L^1(I;W^{-3,2}(\Omega))}\le C_{11}.$$

Moreover, except C_0, C_3 and C_9, the constants C's are independent of (γ,ϑ), while C_0, C_3 and C_9 depend on $\|(\nabla\gamma,\nabla\vartheta)\|_{L^2(Q;\mathbb{R}^3))^{L+1}}$ due to (3.32) below. The meaning $\frac{\partial v}{\partial t}+(-1)^k\epsilon\Delta^k v$ in (3.22d) as a linear continuous functional on $L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$ is a continuous extension of the weak form of $\mathrm{div}(\tau(\mathrm{D}v,\gamma,\vartheta)-v\otimes v)-q\nabla\phi$ which has a good sense for smooth functions, cf. (3.26).

Proof. First, we realize that the total charge $z{\cdot}K(\gamma)$ in (3.18a) is always bounded, namely $\|z{\cdot}K(\gamma)\|_{L^\infty(Q)}\le\max_{\ell=1,..,L}|z_\ell|$, and then (3.22a) follows by usual $W^{2,r}$-regularity of the Δ-operator with (2.8) for any $r<+\infty$; cf. [1]. Then also the driving force $q\nabla\phi=(zK(\gamma))\nabla\phi$ in (3.18b) is bounded in $L^\infty(Q;\mathbb{R}^3)$, hence certainly in $L^1(I;L^2(\Omega;\mathbb{R}^3))$. Then, by a test of (3.13) by v itself and by using the Korn inequality

(3.23) $$\exists\eta_2>0\quad\forall v\in W^{1,p}_0(\Omega;\mathbb{R}^3):\qquad\eta_2\big\|v\big\|_{W^{1,p}_0(\Omega;\mathbb{R}^3)}\le\big\|\mathrm{D}v\big\|_{L^p(\Omega;\mathbb{R}^{3\times3})}$$

with $\eta_2>0$ depending on the Lipschitz domain Ω, and by using the usual trick that $\int_\Omega\nabla\pi\cdot v\,\mathrm{d}x=-\int_\Omega\pi\,\mathrm{div}(v)\,\mathrm{d}x=0$ as well as $\int_\Omega(v{\otimes}v){:}\nabla v\,\mathrm{d}x=0$, and by using still (3.9b), we obtain the estimate

(3.24) $$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big\|v\big\|^2_{L^2(\Omega;\mathbb{R}^3)}&+\eta_1\eta_2^p\big\|v\big\|^p_{W^{1,p}_0(\Omega;\mathbb{R}^3)}+\epsilon\big\|\nabla^m v\big\|^2_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}\\&\le\int_\Omega\frac{\partial v}{\partial t}\cdot v+\tau(\mathrm{D}v,\gamma,\theta){:}\mathrm{D}v+\epsilon\nabla^m v\,\vdots\,\nabla^m v\;\mathrm{d}x\\&=-\int_\Omega q\nabla\phi{\cdot}v\;\mathrm{d}x\le\max_{\ell=1,..,L}|z_\ell|\,\big\|\nabla\phi\big\|_{L^2(\Omega;\mathbb{R}^3)}\big\|v\big\|_{L^2(\Omega;\mathbb{R}^3)};\end{aligned}$$

let us recall the convention pronounced in Remark 3.2 so $\tau(\mathrm{D}v,\gamma,\theta)$ behaves well even if some γ's are negative. By Young's and Gronwall's inequalities, we obtain (3.22b) and

(3.25) $$\big\|v\big\|_{L^2(I;W^{k,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le\frac{C_3}{\sqrt{\epsilon}},$$

by a test of (3.13) by v itself and using the usual trick that $\int_\Omega \nabla\pi\cdot v\,\mathrm{d}x = -\int_\Omega \pi\,\mathrm{div}(v)\,\mathrm{d}x = 0$ as well as $\int_\Omega (v\otimes v){:}\nabla v\,\mathrm{d}x = 0$. Note that, because of the retract K used in (3.18a), the bounds in (3.22b) and (3.25) are completely independent of γ.

The estimate (3.22d) can be obtained by testing (3.18b) by $w \in L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$ as follows:

$$\begin{aligned}(3.26)\qquad &\Big\|\frac{\partial v}{\partial t} + (-1)^k\epsilon\Delta^k v\Big\|_{L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)}\\ &:= \sup_{\|w\|_{L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le 1} \Big\langle \frac{\partial v}{\partial t} + (-1)^k\epsilon\Delta^k v, w\Big\rangle\\ &= \sup_{\|w\|_{L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le 1} \int_Q \tau(\mathrm{D}v,\gamma,\vartheta){:}\mathrm{D}w - (v\otimes v){:}\nabla w + q\,\nabla\phi\cdot w\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The boundedness of $\int_Q (v\otimes v){:}\nabla w\ \mathrm{d}x\,\mathrm{d}t$ just requires $p \ge 11/5$ because, by interpolation, (3.22b) guarantees v bounded in $L^{11/3}(Q;\mathbb{R}^3)$ so that $(v\otimes v){:}\nabla w \in L^1(Q)$ if $\nabla w \in L^{11/5}(Q;\mathbb{R}^3)$; cf. also e.g. [19, Chap.5, Lemma 2.44(iii)].

To get (3.22e), we test (3.18d) by c. We realize that

$$(3.27)\qquad \int_\Omega c_\ell v\cdot\nabla c_\ell\ \mathrm{d}x = \frac12\int_\Omega v\cdot\nabla c_\ell^2\ \mathrm{d}x = -\frac12\int_\Omega (\mathrm{div}\,v)c_\ell^2\ \mathrm{d}x = 0$$

for each $\ell = 1,...,L$, i.e. $\int_\Omega (c\otimes v){:}\nabla c\ \mathrm{d}x = 0$. By (3.27), we obtain

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |c|^2\,\mathrm{d}x \ + \ \eta_0\int_\Omega |\nabla c|^2\,\mathrm{d}x &\le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |c|^2\,\mathrm{d}x + \int_\Omega \nabla c^\top \mathfrak{D}(\gamma,\vartheta)\nabla c\ \mathrm{d}x\\ &= \int_\Omega (c\otimes v){:}\nabla c - \big(\mathfrak{m}(\gamma,\vartheta)\otimes\nabla\phi\big){:}\nabla c + r(\gamma,\vartheta)\ \mathrm{d}x\\ &\le \frac{\eta_0}{2}\|\nabla c\|^2_{L^2(\Omega;\mathbb{R}^L)} + \frac{1}{2\eta_0}\|\mathfrak{m}\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^L)}\|\nabla\phi\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|r\|_{L^1(\mathbb{R}^2;\mathbb{R}^L)}.\end{aligned}$$

Then (3.22e) follows by Young and Gronwall inequalities when using also (3.11) and (3.22a). Again, Remark 3.2 applies, of course.

As for (3.22f), let us realize that, by (3.22e), c is bounded in $L^\infty(I;L^2(\Omega;\mathbb{R}^L))$ and, by (3.25), $\sqrt{\epsilon}v$ is bounded $L^2(I;L^\infty(\Omega;\mathbb{R}^3))$ so that $\sqrt{\epsilon}\,c\otimes v$ is certainly bounded in $L^2(Q;\mathbb{R}^{3\times L})$. Then we obtain (3.22f) by testing (3.18d) by an arbitrary w from $L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))$ when using (3.11) and (3.22a).

The estimate (3.22g) can be obtained similarly as (3.22d) by testing (3.18d) by arbitrary $w \in L^r(I;W^{1,r}(\Omega;\mathbb{R}^L))$ with a suitable r. The resulting term $\int_Q (c\otimes v){:}\nabla w\ \mathrm{d}x\,\mathrm{d}t$ is now to be estimated as

$$(3.28)\qquad \int_Q (c\otimes v) : \nabla w\ \mathrm{d}x\,\mathrm{d}t \le C\|v\|_{L^{5p/3}(Q;\mathbb{R}^3)}\|c\|_{L^{10/3}(Q;\mathbb{R}^L)}\|\nabla w\|_{L^r(Q;\mathbb{R}^3)}$$

provided $r \ge 10p/(7p-6)$. The other resulting term $\int_Q \nabla w^\top\mathfrak{D}(\gamma,\vartheta)\nabla c\ \mathrm{d}x\,\mathrm{d}t$ requires $r \ge 2$, which eventually gives the restriction (3.22g) on r.

We want now to show boundedness of ∇v in $L^{2p}(Q;\mathbb{R}^{3\times3})$, which will guarantee the dissipative heat $\tau(\mathrm{D}v,\gamma,\vartheta){:}\mathrm{D}v$ bounded in $L^2(Q)$ to allow for a test of (3.18e) by $\frac{\partial\theta}{\partial t}$. We get it by the Gagliardo-Nirenberg inequality $\|w\|_{W^{1,2p}(\Omega)} \le C\|w\|^\lambda_{W^{k,2}(\Omega)}\|w\|^{1-\lambda}_{L^2(\Omega)}$ which holds for $1/(2p) + \lambda k/3 \ge 5/6$. To have also the interpolation $\|w\|_{L^{2p}(I)} \le C\|w\|^\lambda_{L^2(I)}\|w\|^{1-\lambda}_{L^\infty(I)}$, which holds for $0 \le \lambda \le 1/p$, we put $\lambda = 1/p$. Choosing k large enough, namely as specified in (3.21), we obtain the desired interpolation

$$(3.29)\ \|v\|_{L^{2p}(I;W^{1,2p}(\Omega;\mathbb{R}^3))} \le \|v\|^\lambda_{L^2(I;W^{k,2}(\Omega;\mathbb{R}^3))}\|v\|^{1-\lambda}_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))} = \mathcal{O}\Big(\frac{1}{\epsilon^{\frac{\lambda}{2}}}\Big) = \mathcal{O}\Big(\frac{1}{\epsilon^{\frac{1}{2p}}}\Big)$$

where the order with respect to the parameter ϵ comes from (3.22b) and (3.25). Thus

$$\big\|\tau(\mathrm{D}v,\gamma,\vartheta){:}\mathrm{D}v\big\|_{L^2(Q)}=\mathcal{O}\Big(\frac{1}{\sqrt{\epsilon}}\Big). \tag{3.30}$$

The other terms on the right-hand side of (3.18e) are bounded in $L^2(Q)$, too; note that $\nabla c_\ell\cdot\nabla\phi\in L^2(Q)$ because $\nabla c_\ell\in L^2(Q;\mathbb{R}^3)$ due to (3.22e) while $\nabla\phi\in L^\infty(Q;\mathbb{R}^3)$ due to (3.22a). Hence the total right-hand side of (3.18e), let us denote it by $h_{\rm tot}$, is bounded in $L^2(Q)$. Then the test of (3.18e) by θ gives (3.22h) with the order $\mathcal{O}(1/\sqrt{\epsilon})$ coming from (3.30), the constant C_8 being still independent of γ and ϑ.

Now we test simultaneously (3.18b) by $\frac{\partial v}{\partial t}$ and (3.18e) by $\frac{\partial\theta}{\partial t}$. (Rigorously, this step is not legal unless we have an L^2-information about $\frac{\partial v}{\partial t}$ and $\frac{\partial\theta}{\partial t}$ we want just to derive but one can, for a moment, imagine e.g. a Galerkin approximation of (3.18b) and (3.18e) to make these tests and a subsequent limit passage.) We sum them to obtain, for a.a. $t\in I$,

$$\begin{aligned}&\Big\|\frac{\partial v}{\partial t}\Big\|_{L^2(\Omega;\mathbb{R}^3)}^2+c_{\rm v}\Big\|\frac{\partial\theta}{\partial t}\Big\|_{L^2(\Omega)}^2+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\epsilon}{2}\|\nabla^k v\|_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}^2+\frac{\kappa}{2}\|\nabla\theta\|_{L^2(\Omega;\mathbb{R}^3)}^2\Big)\\&=\int_\Omega\mathrm{div}\big(\tau(\mathrm{D}v,\gamma,\vartheta)\big)\cdot\frac{\partial v}{\partial t}-(v\cdot\nabla)v\cdot\frac{\partial v}{\partial t}+q\nabla\phi\cdot\frac{\partial v}{\partial t}-c_{\rm v}(v\cdot\nabla)\theta\frac{\partial\theta}{\partial t}+h_{\rm tot}\frac{\partial\theta}{\partial t}\\&\le 2\big\|\mathrm{div}\big(\tau(\mathrm{D}v,\gamma,\vartheta)\big)\big\|_{L^2(\Omega;\mathbb{R}^3)}^2+2\big\|(v\cdot\nabla)v\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\\&\quad+2\big\|q\nabla\phi\big\|_{L^2(\Omega;\mathbb{R}^3)}^2+\frac12\Big\|\frac{\partial v}{\partial t}\Big\|_{L^2(\Omega;\mathbb{R}^3)}^2\\&\quad+\frac{1}{c_{\rm v}}\big\|(v\cdot\nabla)\theta\big\|_{L^2(\Omega)}^2+\frac{1}{c_{\rm v}}\big\|h_{\rm tot}\big\|_{L^2(\Omega)}^2+\frac{c_{\rm v}}{2}\Big\|\frac{\partial\theta}{\partial t}\Big\|_{L^2(\Omega)}^2.\end{aligned} \tag{3.31}$$

The particular right-hand-side terms can be estimated as follows: the first one allows for the estimate:

$$\begin{aligned}&\big\|\mathrm{div}\big(\tau(\mathrm{D}v,\gamma,\vartheta)\big)\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\le\big\|\tau_D'(\mathrm{D}v,\gamma,\vartheta)\big\|_{L^2(\Omega;\mathbb{R}^{3\times3\times3})}^2\big\|\nabla^2 v\big\|_{L^\infty(\Omega;\mathbb{R}^{3^{k+1}})}^2\\&+\big\|\tau_\gamma'(\mathrm{D}v,\gamma,\vartheta)\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}^2\big\|\nabla\gamma\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\\&+\big\|\tau_\theta'(\mathrm{D}v,\gamma,\vartheta)\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}^2\big\|\nabla\vartheta\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\\&\le\tilde C\Big(1+\|\nabla v\|_{L^{2p}(\Omega;\mathbb{R}^{3\times3})}^{2p}\Big)\big\|\nabla^k v\big\|_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}^2+\tilde C\big\|\nabla\gamma\big\|_{L^2(\Omega;\mathbb{R}^3)}^2+\tilde C\big\|\nabla\vartheta\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\end{aligned} \tag{3.32}$$

where (3.9e) and the embedding $W^{k,2}(\Omega)\subset W^{2,\infty}(\Omega)$ have been used as well as the growth condition (3.9d) to estimate

$$\begin{aligned}\big\|\tau_D'(\mathrm{D}v,\gamma,\vartheta)\big\|_{L^2(\Omega;\mathbb{R}^{3\times3\times3})}^2&\le\int_\Omega C^2\big(1+|\nabla v|^{p-2}\big)^2\,\mathrm{d}x\\&\le 2C^2\Big(\mathrm{meas}(\Omega)+\int_\Omega|\nabla v|^{2p-4}\,\mathrm{d}x\Big)\le\tilde C\Big(1+\int_\Omega|\nabla v|^{2p}\,\mathrm{d}x\Big)\end{aligned}$$

with C from (3.9d). In view of (3.22b), this term can then be handled by Gronwall's inequality because, due to (3.29), $t\mapsto\|\nabla v(t,\cdot)\|_{L^{2p}(\Omega;\mathbb{R}^{3\times3})}^{2p}$ is integrable; note that (3.29) implies that the $L^1(0,T)$-norm of this function is of the order $\mathcal{O}(1/\epsilon)$, which gives the factors $\mathrm{e}^{C_0/\epsilon}$ in (3.22c,i). The further term can be estimated as

$$\big\|(v\cdot\nabla)v\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\le\big\|v\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\big\|\nabla v\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}^2\le C_2^2N^2\big\|\nabla^k v\big\|_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}^2$$

where C_2 is from (3.22b) and N is the norm of the embedding $W^{k,2}(\Omega)\subset W^{1,\infty}(\Omega)$, so that also this term can be handled by Gronwall's inequality. The term $q\nabla\phi$ is already estimated in (3.22a). The term $(v\cdot\nabla)\theta$ is to be estimated as

$$\big\|(v\cdot\nabla)\theta\big\|_{L^2(\Omega)}^2\le\big\|v\big\|_{L^\infty(\Omega;\mathbb{R}^3)}^2\big\|\nabla\theta\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\le N^2\big\|\nabla^k v\big\|_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}^2\big\|\nabla\theta\big\|_{L^2(\Omega;\mathbb{R}^3)}^2$$

where N is the norm of the embedding $W^{k,2}(\Omega)\subset L^\infty(\Omega)$, and again we can treat it by Gronwall's inequality if (3.25) is taken into account. The boundedness of $h_{\rm tot}$ in $L^2(Q)$ has already been mentioned. Therefore, (3.31) yields both (3.22c) and (3.22i); note that (3.25) gives $\|\nabla^k v\|^2_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}=\mathcal{O}(1/\epsilon)$ and (3.30) gives $\|h_{\rm tot}\|^2_{L^2(\Omega)}=\mathcal{O}(1/\epsilon)$, which (together with the already mentioned factor $\mathrm{e}^{C_0/\epsilon}$) eventually determines the order both in (3.22c) and in (3.22i).

Having ∇v bounded in $L^p(Q;\mathbb{R}^{3\times3}))$, see (3.22b), $\tau(\mathrm{D}v,\gamma,\vartheta){:}\mathrm{D}v$ is then certainly bounded in $L^1(Q)$ (independently of ϵ) while the other right-hand-side terms of (3.18e) are bounded in this space, too, because of (3.22a,e) and (3.11). This allows us to use fine results about integrability of temperature gradient [5, 6, 7] modified for the initial-boundary-value problem

$$c_{\rm v}\frac{\partial\theta}{\partial t}-\mathrm{div}\big(\kappa\nabla\theta-c_{\rm v}v\,\theta\big)=h_{\rm tot}\ \text{on}\ Q,\qquad \kappa\frac{\partial\theta}{\partial\nu}=0\ \text{on}\ \Sigma,\qquad \theta(0,\cdot)=\theta_0\ \text{on}\ \Omega; \tag{3.33}$$

recall that $h_{\rm tot}\in L^1(Q)$ denotes the total right-hand side of (3.18e). First, let us test (3.33) by $\mathrm{sign}(\theta)$ or, more rigorously, by a regularization of it, say $\max(-1,\min(1,n\theta))$, and then make a limit passage with $n\to\infty$, which gives the first part of the estimate (3.22j), i.e. a bound for θ in $L^\infty(I;L^1(\Omega))$; for more details about this rather standard technique see e.g. [31, Sect.9.4 with 3.2.3]. The second part of (3.22j) is more involved. Following [6, 7], we test (3.33) by $\psi_n(\theta)$ with $\psi_n:\mathbb{R}\to[-1,1]$ a bounded Lipschitz function defined, for $n\in\mathbb{N}$, by

$$\psi_n(\theta):=\begin{cases}0 & \text{if } |\theta|\le n,\\ \mathrm{sign}(\theta)(|\theta|-n) & \text{if } n\le|\theta|\le n+1,\\ \mathrm{sign}(\theta) & \text{if } |\theta|\ge n+1.\end{cases} \tag{3.34}$$

We use

$$\begin{aligned}\int_\Omega\theta\,v\cdot\nabla\psi_n(\theta)\,\mathrm{d}x &= \int_\Omega\theta\,\psi_n'(\theta)\,v\cdot\nabla\theta\,\mathrm{d}x\\ &= \int_\Omega v\cdot\nabla\hat\phi_n(\theta)\,\mathrm{d}x=-\int_\Omega\mathrm{div}(v)\hat\phi_n(\theta)\,\mathrm{d}x=0\end{aligned} \tag{3.35}$$

where $\hat\phi_n:\mathbb{R}\to\mathbb{R}$ denotes a primitive function to $\phi_n:\theta\mapsto\theta\,\psi_n'(\theta)$. Further, we denote by $\hat\psi_n$ the primitive function of ψ_n such that $\hat\psi_n(0)=0$; note that $0\le\hat\psi_n(\theta)\le|\theta|$. Testing (3.33) by $\psi_n(\theta)$ and denoting $B_n:=\{(t,x)\in Q:\ n\le|\theta(t,x)|\le n{+}1\}$ then gives

$$\begin{aligned}\kappa\int_{B_n}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t &= \kappa\int_Q\psi_n'(\theta)|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t=\int_Q\kappa\nabla\theta{\cdot}\nabla\psi_n(\theta)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_Q\kappa\nabla\theta\cdot\nabla\psi_n(\theta)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega c_{\rm v}\hat\psi_n(\theta(T,\cdot))\,\mathrm{d}x\\ &= \int_\Omega c_{\rm v}\hat\psi_n(\theta_{0\epsilon})\,\mathrm{d}x+\int_Q h_{\rm tot}\psi_n(\theta)\,\mathrm{d}x\,\mathrm{d}t\\ &\le c_{\rm v}\big\|\theta_{0\epsilon}\big\|_{L^1(\Omega)}+\big\|h_{\rm tot}\big\|_{L^1(Q)}.\end{aligned} \tag{3.36}$$

For $\mu>0$ fixed, we get

$$\begin{aligned}\int_Q\frac{|\nabla\theta|^2}{(1+\theta)^{1+\mu}}\,\mathrm{d}x\,\mathrm{d}t &= \sum_{n=0}^\infty\int_{B_n}\frac{|\nabla\theta|^2}{(1+\theta)^{1+\mu}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \sum_{n=0}^\infty\frac{1}{(1+n)^{1+\mu}}\int_{B_n}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{c_{\rm v}\big\|\theta_{0\epsilon}\big\|_{L^1(\Omega)}+\big\|h_{\rm tot}\big\|_{L^1(Q)}}{\kappa}\sum_{n=0}^\infty\frac{1}{(1+n)^{1+\mu}}\le C_\mu\end{aligned} \tag{3.37}$$

with some C_μ. Further, we simplify [6, 7] which estimate $\nabla\theta$ in an anisotropic space. For our purposes, an estimate of $\nabla\theta$ in an "isotropic" space $L^\zeta(Q;\mathbb{R}^3)$ will suffice. For this, let us take $1 \le \zeta < 2$. By Hölder's inequality,

$$
\begin{aligned}
(3.38)\qquad \int_Q |\nabla\theta|^\zeta \,\mathrm{d}x\,\mathrm{d}t &= \int_Q \frac{|\nabla\theta|^\zeta}{(1+\theta)^{(1+\mu)\zeta/2}}(1+\theta)^{(1+\mu)\zeta/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \Big(\int_Q \frac{|\nabla\theta|^2}{(1{+}\theta)^{1+\mu}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\zeta/2}\Big(\int_Q (1{+}\theta)^{(1+\mu)\zeta/(2-\zeta)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(2-\zeta)/2} \\
&\le C_\mu^{\zeta/2}\Big(\int_0^T \|1+\theta(t,\cdot)\|_{L^{(1+\mu)\zeta/(2-\zeta)}(\Omega)}^{(1+\mu)\zeta/(2-\zeta)}\,\mathrm{d}t\Big)^{(2-\zeta)/2}
\end{aligned}
$$

The proved first part of (3.22j), i.e. $\|\theta\|_{L^\infty(I;L^1(\Omega))} \le C_{10}$, allows us further to estimate, by using Gagliardo-Nirenberg's inequality,

$$
\begin{aligned}
(3.39)\qquad \|1{+}\theta(t,\cdot)\|_{L^{(1+\mu)\zeta/(2-\zeta)}(\Omega)} &\le C_{\rm GN}\|\nabla\theta(t,\cdot)\|_{L^\zeta(\Omega)}^\lambda \|1{+}\theta(t,\cdot)\|_{L^1(\Omega)}^{1-\lambda} \\
&\le C_{\rm GN}\big(|\Omega|{+}C_{10}\big)^{1-\lambda}\|\nabla\theta(t,\cdot)\|_{L^\zeta(\Omega)}^\lambda
\end{aligned}
$$

for some $C_{\rm GN} \in \mathbb{R}$ provided

$$
(3.40)\qquad \frac{2-\zeta}{(1+\mu)\zeta} \ge \lambda\Big(\frac{1}{\zeta} - \frac{1}{3}\Big) + 1 - \lambda.
$$

We rice (3.39) to the power $(1{+}\mu)\zeta/(2-\zeta)$, exploit it for (3.38), and choose $\lambda := (2{-}\zeta)/(1{+}\mu)$, which yields:

$$
\begin{aligned}
(3.41)\qquad &\Big(\int_0^T \|1+\theta(t,\cdot)\|_{L^{(1+\mu)\zeta/(2-\zeta)}(\Omega)}^{(1+\mu)\zeta/(2-\zeta)}\,\mathrm{d}t\Big)^{(2-\zeta)/2} \le \\
&\le \Big(\int_0^T C_{\rm GN}^{\frac{(1+\mu)\zeta}{2-\zeta}}\big(|\Omega|{+}C_{10}\big)^{\frac{(1-\lambda)(1+\mu)\zeta}{2-\zeta}}\|\nabla\theta(t,\cdot)\|_{L^\zeta(\Omega)}^{\frac{\lambda(1+\mu)\zeta}{2-\zeta}}\,\mathrm{d}t\Big)^{\frac{2-\zeta}{2}} \\
&\le \Big(\int_0^T C_{\rm GN}^{\frac{(1+\mu)\zeta}{2-\zeta}}\big(|\Omega|{+}C_{10}\big)^{\frac{(1-\lambda)(1+\mu)\zeta}{2-\zeta}}\|\nabla\theta(t,\cdot)\|_{L^\zeta(\Omega)}^{\zeta}\,\mathrm{d}t\Big)^{\frac{2-\zeta}{2}} \\
&= C_{\rm GN}^{(1+\mu)\zeta/2}\big(|\Omega|{+}C_{10}\big)^{\zeta(\zeta-1+\mu)/2}\Big(\int_Q |\nabla\theta|^\zeta\,\mathrm{d}x\,\mathrm{d}t\Big)^{(2-\zeta)/2}
\end{aligned}
$$

Merging (3.38) with (3.41) gives the estimate

$$
(3.42)\qquad \|\nabla u\|_{L^\zeta(Q;\mathbb{R}^3)}^\zeta \le C_\mu C_{\rm GN}^{1+\mu}\big(|\Omega|{+}C_{10}\big)^{\zeta-1+\mu}.
$$

Putting our choice of $\lambda := (2{-}\zeta)/(1{+}\mu)$ into (3.40), one obtains, after some algebra, the conditions $\zeta \le (5{-}3\mu)/4$ so that (3.42) just gives the second part of the estimate (3.22j) with $\xi := \frac{3}{4}\mu$.

To prove (3.22k), we must, in particular, estimate the term $\mathrm{div}(v\theta)$ by the following way

$$
\begin{aligned}
(3.43)\qquad \big\|\mathrm{div}(v\theta)\big\|_{L^1(I;W^{-3,2}(\Omega))} &= \sup_{\|w\|_{L^\infty(I;W_0^{3,2}(\Omega))}\le 1} \int_Q v\theta\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t \\
&\le \sup_{\|w\|_{L^\infty(I;W_0^{3,2}(\Omega))}\le 1} C\big\|v\big\|_{L^{5p/3}(Q;\mathbb{R}^3)}\big\|\theta\big\|_{L^{5/3-\delta}(Q)}\big\|\nabla w\big\|_{L^\infty(Q;\mathbb{R}^3)} \\
&= C\big\|v\big\|_{L^{5p/3}(Q;\mathbb{R}^3)}\big\|\theta\big\|_{L^{5/3-\delta}(Q)}
\end{aligned}
$$

with a sufficiently small $\delta > 0$ and with a suitable constant C, where we used the embedding $W^{3,2}(\Omega) \subset W^{1,\infty}(\Omega)$ and, by the Gagliardo-Nirenberg inequality, also the embedding

$$L^p(I;W^{1,p}(\Omega)) \cap L^\infty(I;L^2(\Omega)) \subset L^{5p/3}(Q), \tag{3.44}$$

cf. [9, Sect.I.3], and finally we used also the embedding

$$L^\infty(I;L^1(\Omega)) \,\cap\, L^{5/4-\xi}(I;W^{1,5/4-\xi}(\Omega)) \subset L^{5/3-\delta}(Q) \tag{3.45}$$

again by Gagliardo-Nirenberg inequality; alternatively, we could use here Sobolev embeddings and usual interpolation of Lebesgue spaces; note that (3.43) works even for $p > 3/2$.

Eventually, we prove (3.18f). Let us abbreviate $\sigma := \sum_{\ell=1}^L c_\ell = 1$. By summing (3.18e) for $\ell = 1, ..., L$ and by (3.6a) and (3.7c,d), one gets

$$\begin{aligned} \frac{\partial\sigma}{\partial t} &= \sum_{\ell=1}^L r_\ell\big(\gamma,\vartheta\big) + \mathrm{div}\Big(\sum_{\ell=1}^L\sum_{k=1}^L \mathfrak{D}_{k\ell}(\gamma,\vartheta)\nabla c_k + \mathfrak{m}_\ell(\gamma,\vartheta)\nabla\phi - vc_\ell\Big) \\ &= 0 + \mathrm{div}\Big(\beta\sum_{k=1}^L \nabla c_k + \Big(\sum_{\ell=1}^L \mathfrak{m}_\ell(\gamma,\vartheta)\Big)\nabla\phi - v\Big(\sum_{\ell=1}^L c_\ell\Big)\Big) \\ &= \mathrm{div}(\beta\nabla\sigma) - v\cdot\nabla\sigma. \end{aligned} \tag{3.46}$$

Due to (3.6b) and (2.8b), a solution to thus obtained initial-boundary-value problem for a parabolic (if $\beta > 0$) or hyperbolic (if $\beta = 0$) equation, i.e.

$$\frac{\partial\sigma}{\partial t} - \mathrm{div}(\beta\nabla\sigma) + v\cdot\nabla\sigma = 0 \text{ on } Q, \qquad \frac{\partial\sigma}{\partial\nu} = 0 \text{ on } \Sigma, \qquad \sigma(0,\cdot) = 1 \text{ on } \Omega, \tag{3.47}$$

is $\sigma \equiv 1$. This solution is unique, which can be proved by testing the difference of (3.47) for two solutions σ_1 and σ_2 by $\sigma_1 - \sigma_2$. The important fact is that the resulting term $\int_\Omega (v\cdot\nabla(\sigma_1-\sigma_2))(\sigma_1-\sigma_2)\,\mathrm{d}x$ vanishes as in (3.27); note that our estimates (3.25) and (3.22e) ensures integrability of all integrands occurring in (3.27) with $\sigma_1-\sigma_2$ in place of c_ℓ. □

Remark 3.7 The parabolic/hyperbolic equation (3.47) can be found in the literature in this context, cf. [15, Sect.7.3.5].

Proposition 3.8 ***(Continuity.)*** *Let the assumptions of Proposition 3.6 hold. Then the weak solution to (3.18)–(3.19) with the boundary conditions (2.8) and (3.14) is determined uniquely and the mapping*

$$(\gamma,\vartheta) \mapsto \big\{(v,c,\theta,\phi) \text{ is a weak solution to } (3.18),(3.19),(2.8),(3.14)\big\} \tag{3.48}$$

with $\sum_{\ell=1}^L \gamma_\ell = 1$ *is continuous from the weak topology on* $\mathcal{W}^L \times \mathcal{W}$ *with*

$$\mathcal{W} := L^2(I;W^{1,2}(\Omega)) \,\cap\, W^{1,2}(I;W^{1,2}(\Omega)^*) \tag{3.49}$$

to the weak topology related to the spaces from the estimates (3.22a,c,f,i).*

Proof. The uniqueness of the solution to (3.18a) follows standardly because of linearity and because $\varepsilon > 0$ and $\alpha > 0$ is assumed. As to (3.18b,c), the uniqueness is due to the monotonicity (3.9b) of $\tau(\cdot,\gamma,\vartheta)$ and because the term $\mathrm{div}(v\otimes v)$ can be estimated on the right-hand side: indeed, considering two solutions v_1 and v_2, by the test by v_1-v_2 and by using Green's formula several times and $\mathrm{div}\, v_1 = 0 = \mathrm{div}\, v_2$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|v_1-v_2\big\|^2_{L^2(\Omega;\mathbb{R}^3)} + \epsilon\big\|\nabla^k v_1-\nabla^k v_2\big\|^2_{L^2(\Omega;\mathbb{R}^{3^{k+1}})} \\ &\quad\le \int_\Omega \big(v_1\otimes v_1 - v_2\otimes v_2\big) : \nabla(v_1-v_2)\,\mathrm{d}x \\ &\quad= \int_\Omega \big((v_1\cdot\nabla)v_1 - (v_2\cdot\nabla)v_2\big)\cdot(v_1-v_2)\,\mathrm{d}x \\ &\quad= \int_\Omega \big(((v_1-v_2)\cdot\nabla)v_1\big)\cdot(v_1-v_2)\,\mathrm{d}x \le \big\|\nabla v_1\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}\big\|v_1-v_2\big\|^2_{L^2(\Omega;\mathbb{R}^3)} \end{aligned} \tag{3.50}$$

from which $v_1 = v_2$ follows by Gronwall's inequality when counting still the estimates $\|\nabla v_1\|_{L^\infty(\Omega;\mathbb{R}^{3\times 3})} \le N\|\nabla^k v_1\|_{L^2(\Omega;\mathbb{R}^{3^{k+1}})}$ and (3.22c). The uniqueness of solutions to (3.18d,e) then follows standardly because these equations are de-coupled and linear and all time-derivatives are in duality with the corresponding solutions.

Take a sequence $\{(\gamma_n,\vartheta_n)\}_{n\in\mathbb{N}}$ converging weakly to some (γ,ϑ) in $\mathcal{W}^L\times\mathcal{W}$. Take the corresponding $(v_n,c_n,\theta_n,\phi_n)$ and choose a subsequence converging weakly* in the spaces specified in the estimates (3.22). By Aubin-Lions' compact-embedding theorem [3, 17], cf. also e.g. [31, Lemma 7.7], the estimates (3.22e) and (3.22f) imply that

$$\gamma_n \to \gamma \ \text{ in } \ L^2(I;L^{6-\xi}(\Omega;\mathbb{R}^L)) \tag{3.51}$$

in the norm topology with any $\xi>0$. This allows us to pass to the limit $K(\gamma_n)\to K(\gamma)$ and also ensures $\nabla\phi_n\to\nabla\phi$ strongly in $L^r(Q;\mathbb{R}^3)$ for any $r<+\infty$ to be exploited for (3.18e). Using again Aubin-Lions' theorem, we obtain $\vartheta_n\to\vartheta$ strongly in $L^2(I;L^{6-\xi}(\Omega))$, which allows us to pass to the limit $h(\gamma_n,\vartheta_n)\to h(\gamma,\vartheta)$ and $r_\ell(\gamma_n,\vartheta_n)\to r_\ell(\gamma,\vartheta)$. Moreover, again by Aubin-Lions' theorem and by interpolation like in (3.29) in the proof of Proposition 3.6,

$$\nabla v_n \to \nabla v \ \text{ in } \ L^{2p}(Q;\mathbb{R}^3) \tag{3.52}$$

in the norm topology, hence

$$\tau(\mathrm{D}v_n,\gamma_n,\vartheta_n){:}\mathrm{D}v_n \to \tau(\mathrm{D}v,\gamma,\vartheta){:}\mathrm{D}v \ \text{ in } \ L^2(Q), \tag{3.53}$$

which is essential for the limit passage in (3.18e) to obtain a weak solution. For the convective term in (3.18e), let us realize that $v_n\to v$ weakly* in $L^\infty(Q;\mathbb{R}^3)$ and, due to (3.22i), $\theta_n\to\theta$ weakly in $W^{1,2}(Q)$, hence strongly in $L^2(Q)$ just by Rellich's theorem, which easily implies $v_n\theta_n\to v\theta$ weakly in $L^2(Q;\mathbb{R}^3)\subset L^1(Q;\mathbb{R}^3)$.

The limit passage in (3.18) is then rutine. The uniqueness already proved above ensures eventually the convergence of the whole sequence. □

Proposition 3.9 ***(Existence of a weak solution to (2.1c-e) and (3.13).)*** *Let again the assumptions of Proposition 3.6 hold, then the mapping $\mathcal{F}:(\gamma,\vartheta)\mapsto(c,\theta)$, where (c,θ) is uniquely determined by (3.18), maps the set*

$$\mathcal{S} := \Big\{(c,\theta)\in\mathcal{W}^L\times\mathcal{W}: \quad \|c\|_{\mathcal{W}^L}\le\max\Big(C_5,\frac{C_6}{\sqrt{\epsilon}}\Big), \quad \|\theta\|_{\mathcal{W}}\le\max\Big(\frac{C_8}{\sqrt{\epsilon}},\frac{C_9\mathrm{e}^{C_0/\epsilon^2}}{\epsilon}\Big),\ \ \sum_{\ell=1}^L c_\ell=1\Big\} \tag{3.54}$$

where C_0,C_5,C_6,C_8,C_9 are from (3.22e,f,h,i) with C_0 and C_9 depending on C_5 and C_8, into itself and has a fixed point $(c,\theta)\in\mathcal{S}$. Moreover, every such a fixed point satisfies also $c_\ell\ge0$ for each $\ell=1,...,L$ and, considering the corresponding ϕ and v, the quadruple (v,c,θ,ϕ) is a weak solution to (2.1c-e) and (3.13) with (2.8), (3.14) and (3.19).

Proof. The fact that $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ follows from Proposition 3.6 because C_5, C_6 and C_8 from (3.22e,f,h) do not depend on (γ,ϑ) at all while C_0 and C_9 from (3.22i) are fixed when C_5 and C_8 are fixed, hence $\mathcal{F}$ indeed maps $\mathcal{S}$ into itself. We use $\mathcal{S}$ equipped with the weak topology $\mathcal{W}^{L+1}$. The continuity of $\mathcal{F}$ in this topology was proved in Proposition 3.8. The fixed point then exists by Schauder's theorem (in Tikhonov's modification).

Although we cannot prove $c_\ell\ge0$ if $c\ne\gamma$, in the fixed point we have $c=\gamma$ and we can prove $c_\ell\ge0$ for each $\ell=1,...,L$ by testing (3.18d) by the negative part c^- with $c_\ell^-:=\min(c_\ell,0)$. It is important that c is a conventional weak solution so that $\frac{\partial}{\partial t}c_\ell$ is in duality with c_ℓ and also with $c_\ell^-\in L^2(I;W^{1,2}(\Omega))$. For any $\ell=1,...,L$, by (3.10a), we use

$$\sum_{k=1}^3\mathfrak{D}_{k\ell}(c,\theta)\nabla c_k{\cdot}\nabla c_\ell^- = \begin{cases}\mathfrak{D}_{\ell\ell}(c,\theta)\nabla c_\ell{\cdot}\nabla c_\ell^-\ge0 & \text{if } c_\ell(t,x)<0,\\ 0 \quad \text{as just } \ \nabla c_\ell^-=0 & \text{if } c_\ell(t,x)\ge0,\end{cases} \tag{3.55}$$

which holds for a.a. $(t,x) \in Q$; recall that $\mathfrak{D}$ is considered as extended continuously, cf. Remark 3.2, so that (3.10a) holds for c_ℓ negative, too. For the convective term, we use

$$\int_\Omega c_\ell v{\cdot}\nabla c_\ell^- \,\mathrm{d}x = \int_\Omega c_\ell^- v{\cdot}\nabla c_\ell^- \,\mathrm{d}x = \int_\Omega v{\cdot}\nabla\frac{(c_\ell^-)^2}{2}\,\mathrm{d}x = \int_\Omega(-\mathrm{div}\, v)\frac{(c_\ell^-)^2}{2}\,\mathrm{d}x = 0. \tag{3.56}$$

Recall still (3.10c) which allows us to consider $r_\ell(\cdot,\theta)$ extended continuously and non-negatively for $c_\ell \le 0$, cf. Remark 3.2, so that $r_\ell(\cdot,\theta)c_\ell^- \le 0$. By (3.10b), similar extension can be assumed for $\mathfrak{m}_\ell(\cdot,\theta)$ so that $\mathfrak{m}_\ell(\cdot,\theta)\nabla\phi\cdot\nabla c_\ell^- = 0$ a.e. on Q.

Hence the suggested test of the Nernst-Planck equation (3.18d) in the weak formulation by $c^-(t,\cdot)$ yields

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |c_\ell^-|^2\,\mathrm{d}x \le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |c_\ell^-|^2\,\mathrm{d}x + \int_\Omega\sum_{k=1}^L \mathfrak{D}_{k\ell}(c,\theta)\nabla c_k\cdot\nabla c_\ell^-\,\mathrm{d}x\\ + \int_\Omega \mathfrak{m}_\ell(c,\theta)\nabla\phi{\cdot}\nabla c_\ell^-\,\mathrm{d}x - \int_\Omega c_\ell v{\cdot}\nabla c_\ell^-\,\mathrm{d}x = \int_\Omega r_\ell(c,\theta)c_\ell^-\,\mathrm{d}x \le 0\end{aligned} \tag{3.57}$$

for a.a. $t \in (0,T)$, so that $c_\ell^- = 0$ a.e. on Q provided $c_\ell|_{t=0} \ge 0$ for any $\ell = 1,...,L$, as indeed assumed in (3.6a). Therefore $c = K(c)$ and the retract K occurring in (3.18b) can eventually be "forgotten" in the fixed point. □

3.3 Limit passage for $\epsilon \to 0$

In this section we will make a limit passage for $\epsilon \to 0$ in the weak solution to (2.1c-e) and (3.13), denoted in this Section by $(v_\epsilon, c_\epsilon, \theta_\epsilon, \phi_\epsilon)$, whose existence was proved in Proposition 3.9. This means that $(v_\epsilon, c_\epsilon, \theta_\epsilon, \phi_\epsilon)$ together with some π_ϵ solves (in the weak sense) the system

$$\frac{\partial v_\epsilon}{\partial t} - \mathrm{div}\big(\tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon) - v_\epsilon\otimes v_\epsilon\big) + \nabla\pi_\epsilon + (-1)^k\epsilon\Delta^k v_\epsilon = -c_\epsilon{\cdot}z\nabla\phi_\epsilon, \tag{3.58a}$$

$$\mathrm{div}(v_\epsilon) = 0, \tag{3.58b}$$

$$\frac{\partial c_\epsilon}{\partial t} - \mathrm{div}\big(\mathfrak{D}(c_\epsilon,\theta_\epsilon)\nabla c_\epsilon + \mathfrak{m}(c_\epsilon,\theta_\epsilon)\otimes\nabla\phi_\epsilon - c_\epsilon\otimes v_\epsilon\big) = r(c_\epsilon,\theta_\epsilon)\,, \tag{3.58c}$$

$$\begin{aligned}c_\mathrm{v}\frac{\partial\theta_\epsilon}{\partial t} - \mathrm{div}\big(\kappa\nabla\theta_\epsilon - c_\mathrm{v}v_\epsilon\,\theta_\epsilon\big) = \tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon){:}\mathrm{D}v_\epsilon\\ + \big(\mathfrak{D}(c_\epsilon,\theta_\epsilon)\nabla c_\epsilon + \mathfrak{m}(c_\epsilon,\theta_\epsilon)\otimes\nabla\phi_\epsilon\big){:}(z\otimes\nabla\phi_\epsilon) + h(c_\epsilon,\theta_\epsilon),\end{aligned} \tag{3.58d}$$

$$-\,\mathrm{div}(\varepsilon\nabla\phi_\epsilon) = c_\epsilon{\cdot}z\,, \tag{3.58e}$$

$$c_\epsilon\cdot\mathbb{1} := \sum_{\ell=1}^L [c_\epsilon]_\ell = 1 \quad\text{ and }\quad [c_\epsilon]_\ell \ge 0 \quad\text{for } \ell = 1,...,L, \tag{3.58f}$$

together with the initial conditions

$$v_\epsilon(0,\cdot) = v_{0\epsilon}\,, \qquad c_\epsilon(0,\cdot) = c_0\,, \qquad \theta_\epsilon(0,\cdot) = \theta_{0\epsilon} \tag{3.59}$$

and the boundary conditions on Σ:

$$\frac{\partial^l v_\epsilon}{\partial\nu^l} = 0, \qquad l = 0,...,k{-}1, \tag{3.60a}$$

$$\big(\mathfrak{D}(c_\epsilon,\theta_\epsilon)\nabla c_\epsilon + \mathfrak{m}(c_\epsilon,\theta_\epsilon)\otimes\nabla\phi_\epsilon\big)\nu = 0, \tag{3.60b}$$

$$\varepsilon\frac{\partial\phi_\epsilon}{\partial\nu} = \alpha(\phi_\Sigma{-}\phi_\epsilon), \tag{3.60c}$$

$$\kappa\frac{\partial\theta_\epsilon}{\partial\nu} = 0. \tag{3.60d}$$

Proposition 3.10 ***(Existence of a very weak solution to (2.1)–(2.3).)*** *Let the assumptions of Proposition 3.6 be satisfied, let $v_0 \in L^2_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)$, $c_0 \in L^\infty(\Omega;\mathbb{R}^L)$, $\theta_0 \in L^1(\Omega)$, and let $v_{0\epsilon} \to v_0$ in $L^2(\Omega;\mathbb{R}^3)$ and $\theta_{0\epsilon} \to \theta_0$ in $L^1(\Omega)$. Then any sequence $\{(v_\epsilon, c_\epsilon, \theta_\epsilon, \phi_\epsilon)\}_{\epsilon>0}$ of weak solutions obtained in Proposition 3.9 contains a subsequence converging weakly* in spaces involved in (3.22a,b,d,e,g,j,k), let us denote by (v, c, θ, ϕ) its limit, and every such (v, c, θ, ϕ) is a very weak solution due to Definition 3.1.*

Proof. We choose a subsequence that converges weakly* as claimed. Without confusion, let us denote it briefly again by $\{(v_\epsilon, c_\epsilon, \theta_\epsilon, \phi_\epsilon)\}_{\epsilon>0}$.

First, by Aubin-Lions' theorem [3, 17] and by (3.22e,g) together with the L^∞-information from Proposition 3.9, $c_\epsilon \to c$ in $L^r(Q;\mathbb{R}^L)$ for any $r < +\infty$, and from (3.58e) together with the already used $W^{2,2}$-regularity of Δ-operator, $\phi_\epsilon \to \phi$ strongly in $L^s(I;W^{2,2}(\Omega))$.

Let us prove that the weak* limit v is the sough weak solution to (2.1a,b). We use Minty's trick for the term $\operatorname{div}\tau(\mathrm{D}v, c, \theta)$ and compactness for the convective term. The important fact is that we have chosen the subsequence so that, by (3.22d), for some $\dot v \in L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)$, we have at our disposal

$$\frac{\partial v_\epsilon}{\partial t} + (-1)^k \epsilon \Delta^k v_\epsilon \to \dot v \quad \text{weakly in } L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*). \tag{3.61}$$

For any w smooth with a compact support in Q, it holds

$$\begin{aligned}\big\langle \dot v, w\big\rangle &= \lim_{\epsilon\to 0}\Big\langle \frac{\partial v_\epsilon}{\partial t} + (-1)^k\epsilon\Delta^k v_\epsilon, w\Big\rangle = \lim_{\epsilon\to0}\int_Q \frac{\partial v_\epsilon}{\partial t} w + \epsilon\nabla^k v_\epsilon \vdots \nabla^k w \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\epsilon\to0}\int_Q -v_\epsilon\frac{\partial w}{\partial t} + \epsilon\nabla^k v_\epsilon \vdots \nabla^k w\,\mathrm{d}x\,\mathrm{d}t = \int_Q -v\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.62}$$

because $v_\epsilon \to v$ weakly in $L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$ thanks to (3.22b) and also because $\|\epsilon\nabla^k v_\epsilon\|_{L^2(Q;\mathbb{R}^{3^{k+1}})} = \epsilon\mathcal{O}(1/\sqrt{\epsilon}) = \mathcal{O}(\sqrt{\epsilon}) \to 0$ due to (3.25) so that

$$\lim_{\epsilon\to0}\int_Q \epsilon\nabla^k v_\epsilon \vdots \nabla^k w\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.63}$$

This shows that $\dot v$ is the distributional derivative of v, let us denote it naturally as $\frac{\partial v}{\partial t}$. In particular, we have shown that

$$\frac{\partial v}{\partial t} = \dot v \in L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*). \tag{3.64}$$

Furthermore, for $w \in L^2(I;W^{k,2}_0(\Omega;\mathbb{R}^3))\cap L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$, by monotonicity of $\tau(\cdot, c_\epsilon(t,x), \theta_\epsilon(t,x))$, it holds

$$\begin{aligned} 0 &\le \int_Q (\tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon) - \tau(\mathrm{D}w, c_\epsilon, \theta_\epsilon)){:}\mathrm{D}(v_\epsilon - w)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q z\cdot c_\epsilon\nabla\phi_\epsilon\cdot(v_\epsilon - w) - \frac{\partial v_\epsilon}{\partial t}\cdot(v_\epsilon - w) + (v_\epsilon\otimes v_\epsilon) : \nabla(v_\epsilon - w) \\ &\qquad - \epsilon\nabla^k v_\epsilon \vdots \nabla^k(v_\epsilon - w) - \tau(\mathrm{D}w, c_\epsilon, \theta_\epsilon){:}\mathrm{D}(v_\epsilon - w)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_Q z\cdot c_\epsilon\nabla\phi_\epsilon\cdot(v_\epsilon - w) - \frac{\partial v_\epsilon}{\partial t}\cdot(v_\epsilon - w) + (v_\epsilon\otimes v_\epsilon) : \nabla(v_\epsilon - w) \\ &\qquad + \epsilon\nabla^k v_\epsilon \vdots \nabla^k w - \tau(\mathrm{D}w, c_\epsilon, \theta_\epsilon){:}\mathrm{D}(v_\epsilon - w)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.65}$$

Then we can bound from above the limit superior. The important fact is that $\frac{\partial v}{\partial t} \in L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)$ is in duality to v due to (3.64) and the estimate (3.22b). First, let us realize that (3.61) implies

$$\begin{aligned}(3.66)\qquad v_\epsilon(T) &= v_{0\epsilon}+\int_0^T\frac{\partial v_\epsilon}{\partial t}\,\mathrm{d}t = v_{0\epsilon}+\int_0^T\Big(\frac{\partial v_\epsilon}{\partial t}+(-1)^k\epsilon\Delta^k v_\epsilon\Big)\,\mathrm{d}t\\ &- (-1)^k\int_0^T\epsilon\Delta^k v_\epsilon\,\mathrm{d}t\to v_0+\int_0^T\frac{\partial v}{\partial t}\,\mathrm{d}t=v(T)\end{aligned}$$

weakly in $W^{k,2}(\Omega;\mathbb{R}^3)^*$. Due to the estimate (3.22b), $v_\epsilon(T)$ converges also weakly in $L^2(\Omega;\mathbb{R}^3)$, hence we can conclude that even

$$(3.67)\qquad v_\epsilon(T)\to v(T)\quad\text{weakly in}\ \ L^2(\Omega;\mathbb{R}^3).$$

Thus we can use the usual bound

$$\begin{aligned}(3.68)\qquad \liminf_{\epsilon\to0}\int_Q\frac{\partial v_\epsilon}{\partial t}v_\epsilon\,\mathrm{d}x\,\mathrm{d}t &= \liminf_{\epsilon\to0}\frac12\big\|v_\epsilon(T)\big\|^2_{L^2(\Omega;\mathbb{R}^3)}-\frac12\big\|v_0\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &\ge\frac12\big\|v(T)\big\|^2_{L^2(\Omega;\mathbb{R}^3)}-\frac12\big\|v_0\big\|^2_{L^2(\Omega;\mathbb{R}^3)}=\int_0^T\Big\langle\frac{\partial v}{\partial t},v\Big\rangle\,\mathrm{d}t.\end{aligned}$$

We also exploit (3.63) and the strong convergence $\tau(\mathrm{D}w,c_\epsilon,\theta_\epsilon)\to\tau(\mathrm{D}w,c,\theta)$ in $L^p(Q;\mathbb{R}^{3\times3})$. Thus, from (3.65), we eventually get

$$\begin{aligned}(3.69)\qquad 0 &\ge \int_0^T\Big(\Big\langle\frac{\partial v}{\partial t},v-w\Big\rangle-\int_\Omega z{\cdot}c\nabla\phi{\cdot}(v-w)\\ &- (v\otimes v){:}\nabla(v-w)+\tau(\mathrm{D}w,c,\theta){:}\mathrm{D}(v-w)\,\mathrm{d}x\Big)\,\mathrm{d}t.\end{aligned}$$

Now we can extent this inequality for all $w\in L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$ by continuity. Then, substituting $w=v+\delta\tilde w$, canceling $\delta>0$, passing $\delta\to0$, and choosing $\tilde w$ arbitrary, we prove that v satisfies

$$(3.70)\qquad \int_0^T\Big(\Big\langle\frac{\partial v}{\partial t},\tilde w\Big\rangle+\int_\Omega(v\otimes v){:}\nabla\tilde w+\tau(\mathrm{D}v,c,\theta){:}\mathrm{D}\tilde w-z{\cdot}c\nabla\phi{\cdot}\tilde w\,\mathrm{d}x\Big)\,\mathrm{d}t=0$$

for any $\tilde w\in L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$. Hence v is a weak solution to (2.1a).

Let us prove the most essential and most difficult fact, namely the strong convergence of $\mathrm{D}v_\epsilon$ to $\mathrm{D}v$ in $L^p(Q;\mathbb{R}^{3\times3})$. We will use $\int_0^T\langle\frac{\partial(v_\epsilon-v)}{\partial t},v_\epsilon-v\rangle\,dt\ge-\frac12\|v_{0\epsilon}-v_0\|^2_{L^2(\Omega;\mathbb{R}^3)}$; here again it is important that $\frac{\partial v}{\partial t}$ belongs to $L^{p/(p-1)}(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)$ and is thus in duality to v due to (3.64) and the estimate (3.22b), and also that $\frac{\partial v_\epsilon}{\partial t}$ lives in $L^2(Q;\mathbb{R}^3)$ due to (3.22c) so it is certainly in duality with $v_\epsilon{-}v$. By uniform monotonicity (3.9b) of $\tau(\cdot,c_\epsilon(t,x),\theta_\epsilon(t,x))$, we get:

$$\begin{aligned}(3.71)\qquad \eta_2\big\|\mathrm{D}v_\epsilon-\mathrm{D}v\big\|^p_{L^p(Q;\mathbb{R}^{3\times3}))} &\le\int_0^T\!\!\int_\Omega\big(\tau(\mathrm{D}v_\epsilon,c_\epsilon,\theta_\epsilon){-}\tau(\mathrm{D}v,c_\epsilon,\theta_\epsilon)\big){:}\mathrm{D}(v_\epsilon{-}v)\,\mathrm{d}x\,\mathrm{d}t\\ \le\ &\int_0^T\Big(\Big\langle\frac{\partial(v_\epsilon-v)}{\partial t},v_\epsilon-v\Big\rangle\\ &+\int_\Omega\big(\tau(\mathrm{D}v_\epsilon,c_\epsilon,\theta_\epsilon){-}\tau(\mathrm{D}v,c_\epsilon,\theta_\epsilon)\big){:}\mathrm{D}(v_\epsilon{-}v)\,\mathrm{d}x\Big)\,\mathrm{d}t+\frac12\big\|v_{0\epsilon}{-}v_0\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ =\ &\int_0^T\Big(\Big\langle\frac{\partial(v_\epsilon-v)}{\partial t},v_\epsilon{-}v\Big\rangle+\int_\Omega\big(\tau(\mathrm{D}v_\epsilon,c_\epsilon,\theta_\epsilon)-\tau(\mathrm{D}v,c,\theta)\big){:}\mathrm{D}(v_\epsilon{-}v)\,\mathrm{d}x\\ &+\int_\Omega\big(\tau(\mathrm{D}v,c,\theta)-\tau(\mathrm{D}v,c_\epsilon,\theta_\epsilon)\big){:}\mathrm{D}(v_\epsilon-v)\,\mathrm{d}x\Big)\,\mathrm{d}t+\frac12\big\|v_{0\epsilon}{-}v_0\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ =:\ &I^{(1)}_\epsilon+I^{(2)}_\epsilon+I^{(3)}_\epsilon+I^{(4)}_\epsilon.\end{aligned}$$

Using (3.70) with $\tilde{w} := v_\epsilon - v$, The integrals $I_\epsilon^{(1)}$ and $I_\epsilon^{(2)}$ can be estimated in its sum as follows:

$$
\begin{aligned}
(3.72)\qquad I_\epsilon^{(1)} + I_\epsilon^{(2)} \;=\;& \int_0^T\!\!\int_\Omega \Big(\frac{\partial v_\epsilon}{\partial t}\cdot v_\epsilon + \tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon)\!:\!\mathrm{D}v_\epsilon\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T\!\!\int_\Omega \Big(\frac{\partial v_\epsilon}{\partial t}\cdot v + \tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon)\!:\!\mathrm{D}v\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \Big(\Big\langle\frac{\partial v}{\partial t}, v_\epsilon\Big\rangle + \int_\Omega \tau(\mathrm{D}v, c, \theta)\!:\!\mathrm{D}v_\epsilon\;\mathrm{d}x\Big)\,\mathrm{d}t \\
&+ \int_0^T \Big(\Big\langle\frac{\partial v}{\partial t}, v\Big\rangle + \int_\Omega \tau(\mathrm{D}v, c, \theta)\!:\!\mathrm{D}v\;\mathrm{d}x\Big)\,\mathrm{d}t \\
=\;& \int_Q \Big(-\epsilon|\nabla^k v_\epsilon|^2 - z\cdot c_\epsilon\nabla\phi_\epsilon\cdot v_\epsilon + (v_\epsilon\otimes v_\epsilon):\mathrm{D}v_\epsilon \\
&+ \epsilon\nabla^k v_\epsilon \,\vdots\, \nabla^k\tilde{v} + z\cdot c_\epsilon\nabla\phi_\epsilon\cdot\tilde{v} - (v_\epsilon\otimes v_\epsilon):\mathrm{D}\tilde{v} \\
&+ z\cdot c\nabla\phi\cdot v_\epsilon - (v\otimes v):\mathrm{D}v_\epsilon \\
&- z\cdot c\nabla\phi\cdot v + (v\otimes v):\mathrm{D}v \\
&- \frac{\partial v_\epsilon}{\partial t}\cdot(v-\tilde{v}) - \tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon)\!:\!\mathrm{D}(v-\tilde{v})\Big)\,\mathrm{d}x\,\mathrm{d}t \\
\le\;& \int_Q \Big(-z\cdot c_\epsilon\nabla\phi_\epsilon\cdot(v_\epsilon-\tilde{v}) + (v_\epsilon\otimes v_\epsilon):\mathrm{D}(v_\epsilon-\tilde{v}) \\
&+ \epsilon\nabla^k v_\epsilon \,\vdots\, \nabla^k\tilde{v} + z\cdot c\nabla\phi\cdot(v_\epsilon - v) - (v\otimes v):\mathrm{D}(v_\epsilon - v) \\
&- \frac{\partial v_\epsilon}{\partial t}\cdot(v-\tilde{v}) - \tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon)\!:\!\mathrm{D}(v-\tilde{v})\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for any $\tilde{v} \in L^2(I; W_0^{k,2}(\Omega;\mathbb{R}^3))$. Now we can pass to the limit with $\epsilon \to 0$. The important trick is based on (3.61) with (3.64) and on by-part integration in time and on (3.67), which allows for

$$
\begin{aligned}
(3.73)\qquad & \lim_{\epsilon\to 0}\int_Q \Big(\frac{\partial v_\epsilon}{\partial t}\cdot(v-\tilde{v}) + \epsilon\nabla^k v_\epsilon \,\vdots\, \nabla^k\tilde{v}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\epsilon\to 0}\int_0^T \Big(\Big\langle(-1)^{k+1}\epsilon\Delta^k v_\epsilon - \frac{\partial v_\epsilon}{\partial t}, \tilde{v}\Big\rangle - \Big\langle\frac{\partial v}{\partial t}, v_\epsilon\Big\rangle\Big)\,\mathrm{d}t \\
&\qquad\qquad + \int_\Omega \big(v_\epsilon(T)\cdot v(T) - v_{0\epsilon}\cdot v_0\big)\;\mathrm{d}x \\
&= -\int_0^T \Big(\Big\langle\frac{\partial v}{\partial t}, \tilde{v}\Big\rangle + \Big\langle\frac{\partial v}{\partial t}, v\Big\rangle\Big)\,\mathrm{d}t + \big\|v(T)\big\|^2_{L^2(\Omega;\mathbb{R}^3)} - \big\|v_0\big\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&= \int_0^T \Big\langle\frac{\partial v}{\partial t}, v-\tilde{v}\Big\rangle\,\mathrm{d}t.
\end{aligned}
$$

The limit passage in the convective term $(v_\epsilon\otimes v_\epsilon):\mathrm{D}(v_\epsilon-\tilde{v}) \to (v\otimes v):\mathrm{D}(v-\tilde{v})$ in $L^1(Q)$ is standard, using the strong convergence $v_\epsilon \to v$ in $L^{5p/3-\xi}(Q;\mathbb{R}^3)$ which can be proved by interpolating $L^\infty(I;L^2(\Omega))$ and $L^p(I;W^{1-\delta,p}(\Omega))$ by Gagliardo-Nirenberg inequality, cf. [9, Sect.I.3], and by using Aubin-Lions' theorem to have strong convergence in $L^p(I;W^{1-\delta,p}(\Omega))$ for any $\delta > 0$; here the restriction $p > 11/5$ is originated. As for the last term in (3.71), we have $\lim_{\epsilon\to 0} I_\epsilon^{(3)} = 0$ because $\tau(\mathrm{D}, c_\epsilon, \theta_\epsilon) \to \tau(\mathrm{D}, c, \theta)$ strongly in $L^p(Q;\mathbb{R}^{3\times 3})$. By our assumption $v_{0\epsilon} \to v_0$ in $L^2(\Omega;\mathbb{R}^3)$, we have also $\lim_{\epsilon\to 0} I_\epsilon^{(4)} = 0$.

Now we can pass to the limit superior in (3.71) with $\epsilon \to 0$ to obtain

$$\begin{aligned}(3.74)\qquad &\limsup_{\epsilon\to 0} \eta_2 \big\|\mathrm{D}v_\epsilon - \mathrm{D}v\big\|^p_{L^p(Q;\mathbb{R}^{3\times 3})}\\ &\le \int_0^T \Big(\Big\langle \frac{\partial v}{\partial t}, v - \tilde v\Big\rangle + \int_\Omega (v\otimes v) : \mathrm{D}(v{-}\tilde v) - z{\cdot}c\nabla\phi{\cdot}(v{-}\tilde v)\,\mathrm{d}x\Big)\,\mathrm{d}t\\ &\quad + C \limsup_{\epsilon\to 0}\Big(1 + \big\|v_\epsilon\big\|^{p-1}_{L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\Big)\big\|v-\tilde v\big\|_{L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\end{aligned}$$

with C being the constant from (3.9c). Using the estimates (3.22b) and (3.22d), and passing with $\tilde v$ to v in the norm topology of $L^p(I;W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))$, we can see that $\limsup_{\epsilon\to 0}\|\mathrm{D}v_\epsilon - \mathrm{D}v\|_{L^p(Q;\mathbb{R}^{3\times 3})} \le 0$, i.e. $\mathrm{D}v_\epsilon \to \mathrm{D}v$ strongly.

Having this strong convergence, we can pass to the limit in the term $\tau(\mathrm{D}v_\epsilon, c_\epsilon, \theta_\epsilon) : \mathrm{D}v_\epsilon \to \tau(\mathrm{D}v, c, \theta) : \mathrm{D}v$ in $L^1(Q)$ in the right-hand side of the heat equation (3.58d). For the limit passage in the convective term in (3.58d) it suffices to prove, in the weak formulation, that $v_\epsilon\theta_\epsilon \to v\theta$ weakly in $L^1(Q)$, which is simple due to the weak convergence $v_\epsilon \to v$ in $L^{5p/3}(Q;\mathbb{R}^3)$ based on (3.22b) with (3.44) and, by Aubin-Lions' theorem with the interpolation based on (3.22j) and (3.22k), the strong convergence $\theta_\epsilon \to \theta$ in $L^{5/3-\delta}(Q)$; thus we get $v_\epsilon\theta_\epsilon \to v\theta$ weakly even in $L^{55/48-\delta}(Q;\mathbb{R}^3)$, cf. also (3.43).

Limit passage in the other terms is rutine. E.g., $\mathfrak{D}(c_\epsilon,\theta_\epsilon)\nabla c_\epsilon \cdot \nabla\phi_\epsilon \to \mathfrak{D}(c,\theta)\nabla c \cdot \nabla\phi$ because $(c_\epsilon,\theta_\epsilon) \to (c,\theta)$ strongly in $L^2(Q;\mathbb{R}^L) \times L^{5/3-\delta}(Q)$, $\nabla c_\epsilon \to \nabla c$ weakly in $L^2(Q;\mathbb{R}^{L\times 3})$ and $\nabla\phi_\epsilon \to \nabla\phi$ strongly in $L^\infty(I;L^r(\Omega;\mathbb{R}^3))$ for any $r < +\infty$, and similarly also $(\mathfrak{m}(c_\epsilon,\theta_\epsilon)\otimes\nabla\phi_\epsilon){:}(z\otimes\nabla\phi_\epsilon) \to (\mathfrak{m}(c,\theta)\otimes\nabla\phi){:}(z\otimes\nabla\phi)$ in $L^1(Q)$. The limit passage in the term $c_{\mathrm{v}}\frac{\partial\theta_\epsilon}{\partial t}$ is made easy after integration by-part based on the estimates (3.22j) and (3.22k), i.e. $\int_Q c_{\mathrm{v}}\theta_\epsilon\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega c_{\mathrm{v}}\theta_{0\epsilon}w(0,\cdot)\,\mathrm{d}x$ indeed converges to $\int_Q c_{\mathrm{v}}\theta\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega c_{\mathrm{v}}\theta_0 w(0,\cdot)\,\mathrm{d}x$ for any test function $w \in C^1(Q)$ as used in (3.5).

Eventually, the constraints $\sum_{\ell=1}^L c_{\epsilon,\ell} = 1$ and $c_{\epsilon,\ell} \ge 0$ for $\ell = 1,...,L$, which has been proved valid for the approximate solution, see (3.58f), are inherited by the limit, too. □

Remark 3.11 (*Weak solutions.*) For $p \ge 3$, the estimate (3.22g) involves $r = 2$ and then c is the conventional weak solution to the Nernst-Planck equation (2.1c). The weak solution to the whole system (2.1) needs regularity of v. This was proved in [29] for a very narrow interval of p's (of the length only about 0.0528), namely $\frac{9}{4} \le p < \frac{1+\sqrt{13}}{2}$, by using deep regularity results from [18] holding, however, only for a stress tensor τ independent of compositon and temperature and having a potential. Let us remind that regularity for Navier-Stokes equation is generally recognized as an extremely difficult problem which is, at time of creation this paper, open and, in particular for $p = 2$ and τ linear, assigned to a \$1 million Clay-institute award.

4 Discussion of the model and particular cases

The general idea of determining the phenomenological fluxes j_ℓ is a drift/diffusion model like in Roosbroeck's [28] model of semiconductors; for comparison of semiconductors and electrolytes see e.g. [33, p.20]. In the simplest linear case, the phenomenological fluxes in Roosbroeck's model look as:

$$(4.1)\qquad j_\ell := m z_\ell c_\ell \nabla\phi + d\nabla c_\ell = m c_\ell \nabla\mu_\ell \quad \text{where} \quad \mu_\ell := \rho\ln c_\ell + z_\ell\phi$$

where $m > 0$ is a mobility and $d > 0$ a diffusivity coefficient, and μ_ℓ is the *electrochemical potential* of the ℓ-th constituent involving the ratio $\rho = d/m$; any influence of the temperature and its gradient on j_ℓ (in particular Soret's cross effect) is neglected. In bi-polar semiconductors, we have $L = 2$ and $z_1 = -z_2$, but here for multi-component electrolytes we admit $L > 2$ in general. The form (4.1), however, does not satisfy (3.7d) unless the very trivial case $z_\ell m = 0$. More generally, mobilities in concrete mixtures may vary considerably for various components especially if size of molecules of particular constituents varies considerably from constituent to constituent [12], and then it is standardly considered that

$$j_\ell := \sum_{k=1}^{L} \mathbb{M}_{k\ell}(c)\nabla\mu_k, \tag{4.2}$$

where $\mu_\ell := \rho \ln c_\ell + z_\ell\phi$ is again from (4.1) now with $\rho = R\theta_{\rm R}$ with $\theta_{\rm R}$ a reference temperature and R universal gas constant; let us mention that we consider z_ℓ to involve Faraday's constant.

To satisfy zero-sum condition for the fluxes, i.e. (3.7c,d) with $\beta = 0$, the matrix $[\mathbb{M}_{k\ell}(c)]$ should satisfy

$$\forall k = 1, ..., L,\ \forall c \in G_1^+ : \qquad \sum_{\ell=1}^{L} \mathbb{M}_{k\ell}(c) = 0 \tag{4.3}$$

because then obviously

$$\sum_{\ell=1}^{L} j_\ell = \sum_{\ell=1}^{L}\sum_{k=1}^{L} \mathbb{M}_{k\ell}(c)\nabla\mu_k = \sum_{k=1}^{L} \underbrace{\sum_{\ell=1}^{L} \mathbb{M}_{k\ell}(c)}_{=0} \nabla\mu_k = 0. \tag{4.4}$$

Moreover, by the celebrated (Nobel-prize awarded) Onsager's principle [24], the matrix $[\mathbb{M}_{k\ell}(c)]$ should be symmetric.

Example 4.1 (*Symmetric models.*) The zero-sum condition (4.3) for $[\mathbb{M}_{k\ell}(c)]$ has actually been adopted e.g. in [15] or, in a bit different context of a multicomponent alloys, in [13, 14, 21], where essentially the following matrix has been considered:

$$\mathbb{M}_{k\ell}(c) := m_\ell c_\ell \Big(\delta_{k\ell} - \frac{m_k c_k}{\sum_{l=1}^{L} m_l c_l}\Big) \tag{4.5}$$

with m_ℓ being "actual" mobilities of particular constituents (assumed to be) known from experiments. Such $[\mathbb{M}_{k\ell}]$ is symmetric and satisfies (4.3) because obviously $\sum_{\ell=1}^{L} \mathbb{M}_{k\ell}(c) = m_k c_k - (\sum_{\ell=1}^{L} m_\ell c_\ell) m_k c_k / (\sum_{l=1}^{L} m_l c_l) = 0$. Moreover, (4.5) makes also j_ℓ proportional to c_ℓ, which is a natural property.

Substituting (4.5) into (4.2) gives

$$j_\ell = m_\ell\big(\rho\nabla c_\ell + c_\ell z_\ell \nabla\phi\big) - \frac{m_\ell c_\ell}{\sum_{l=1}^{L} m_l c_l}\Big(\sum_{k=1}^{L} m_k\big(\rho\nabla c_k + c_k z_k\nabla\phi\big)\Big). \tag{4.6}$$

Comparing it with (3.8), we can see that our diffusion matrix $\mathfrak{D} = [\mathfrak{D}_{k\ell}]_{\ell,k=1}^{L}$ is now

$$\mathfrak{D}_{k\ell} = \mathfrak{D}_{k\ell}(c) = \rho m_\ell \Big(\delta_{k\ell} - \frac{m_k c_\ell}{\sum_{l=1}^{L} m_l c_l}\Big) \tag{4.7}$$

and our condition (3.10a) is indeed satisfied, and also (3.7c) is satisfied with $\beta = 0$ because $\sum_{\ell=1}^{L} \mathfrak{D}_{k\ell} = \rho m_k - \rho m_k (\sum_{\ell=1}^{L} c_\ell m_\ell)/(\sum_{l=1}^{L} c_l m_l) = 0$. Therefore we can see that the coercivity and the monotonicity assumptions (3.7b) would be satisfied if and only if the matrix $[\mathfrak{D}_{k\ell}]$ given by (4.7) is positive definite uniformly with respect to $c \in G_1^+$. In fact, the positive definiteness of $[\mathfrak{D}_{k\ell}]$ in (3.7b) suffices to verify for the symmetric part of $[\mathfrak{D}_{k\ell}]$ only, and it suffices to hold on the manifold $\sum \nabla c_\ell = 0$, and in particular for

$[\mathfrak{D}_{k\ell}]+\beta(\mathbf{1}\otimes\mathbf{1})$ with some $\beta\geq 0$ so that (3.7c) then holds with this β; cf. also [15, Chap.7]. As for the effective mobilities $\mathfrak{m}_\ell$, comparing (4.6) with (3.8) yields

$$\mathfrak{m}_\ell=\mathfrak{m}_\ell(c)=m_\ell z_\ell c_\ell-m_\ell c_\ell\frac{\sum_{k=1}^L m_k c_k z_k}{\sum_{l=1}^L m_l c_l} \tag{4.8}$$

and we can see that our conditions (3.7d) and (3.10b) are indeed satisfied.

Remark 4.2 The form (4.6) was suggested in [32, Remark 4.4]; namely if $m_\ell\rho\delta_{k\ell}$ and $m_\ell\delta_{k\ell}$ are taken respectively for quantities $d_{k\ell}$ and $m_{k\ell}$ in [32].

Remark 4.3 (*Special case: equal mobilities.*) Like in (4.1), the very special situation with equal mobilities $m:=m_1=...=m_L$, the formula (4.6) gives

$$j_\ell=mz_\ell c_\ell\nabla\phi+m\rho\nabla c_\ell-mc_\ell\Big(\rho\sum_{k=1}^L\nabla c_k+\sum_{k=1}^L z_k c_k\nabla\phi\Big); \tag{4.9}$$

again (3.7) with $\beta=0$ holds. In fact, one can even omit $\sum_{k=1}^L\nabla c_k$ in (4.9) because it expectedly vanishes if $\sum_{k=1}^L c_k=1$; then (3.7c) will be satisfied with $\beta=1$ and such j_ℓ is exactly what has been considered in [29, 30, 31, 32], i.e.

$$j_\ell=mc_\ell(z_\ell-q)\nabla\phi+m\rho\nabla c_\ell\quad\text{ where again } q=\sum_{\ell=1}c_\ell z_\ell. \tag{4.10}$$

Example 4.4 (*Nonsymmetric models.*) Some other models neglect cross-effects and treat one selected constitutent, say L, in a nonsymmetric way by the formula

$$j_\ell=m_\ell c_\ell\nabla(\mu_\ell-\mu_L)\text{ for }\ell=1,...,L{-}1\text{ and }j_L=-\sum_{\ell=1}^{L-1}j_\ell, \tag{4.11}$$

see e.g. [16, Formula (2.26)]. In this case, the symmetric matrix $[\mathbb{M}_{k\ell}(c)]$ satisfying (4.3) is given by

$$\mathbb{M}_{k\ell}(c):=\begin{cases}m_\ell c_\ell\delta_{k\ell} & \text{for } k<L,\ \ell<L,\\ -m_\ell c_\ell & \text{for } k=L,\ \ell<L,\\ -m_k c_k & \text{for } k<L,\ \ell=L,\\ \sum_{l=1}^{L-1}m_l c_l & \text{for } k=L,\ \ell=L.\end{cases} \tag{4.12}$$

Substituting μ_ℓ from (4.1) into (4.11) yields

$$j_\ell=\begin{cases}m_\ell\big(\rho\nabla c_\ell-\rho\frac{c_\ell}{c_L}\nabla c_L+(z_\ell-z_L)c_\ell\nabla\phi\big) & \text{for } \ell<L,\\ \sum_{k=1}^{L-1}m_k\big(\rho\frac{c_k}{c_L}\nabla c_L-\rho\nabla c_k+(z_L-z_k)c_\ell\nabla\phi\big) & \text{for } \ell=L.\end{cases} \tag{4.13}$$

This is sometimes used in electrochemistry either for hydrogen ions as the L-component or, in case of very diluted water solutions, for water as the L-component.

Remark 4.5 (*Thermodynamics of the model.*) It is a certain internal consistency and beauty that the thermodynamics of the model based on μ_ℓ from (4.1) can be derived from a single thermodynamical potential, namely the specific *free energy* in the form

$$\psi=\psi(c,\theta,\phi,E)=\rho\sum_{\ell=1}^L c_\ell\big(\ln c_\ell-1\big)-c_{\rm v}\,\theta\ln\frac{\theta}{\theta_{\rm R}}+\sum_{\ell=1}^L c_\ell z_\ell\phi-\frac{\varepsilon}{2}E^2 \tag{4.14}$$

where E plays the role of the *intensity of electric field.* The partial derivatives of ψ define respectively the *electrochemical potential* μ_ℓ, the *entropy* s, the *total electric charge* q, and the *electric induction* D, namely

$$\mu_i = \frac{\partial\psi}{\partial c_i}, \qquad s = -\frac{\partial\psi}{\partial\theta}, \qquad q = \frac{\partial\psi}{\partial\phi}, \qquad D = -\frac{\partial\psi}{\partial E}. \tag{4.15}$$

This indeed yields expected relations, namely

$$\mu_i = \rho \ln c_i + z_i\phi, \tag{4.16a}$$

$$s = c_{\rm v}\Big(\ln\frac{\theta}{\theta_{\rm R}} + 1\Big), \tag{4.16b}$$

$$q = \sum_{\ell=1}^{L} c_\ell z_\ell, \tag{4.16c}$$

$$D = \varepsilon E. \tag{4.16d}$$

Furthermore, the *internal energy* u is then defined through Gibbs' relation as

$$u = \psi + \theta s = c_{\rm v}\theta + \rho\sum_{\ell=1}^{L} c_\ell\big(\ln c_\ell - 1\big) + \sum_{\ell=1}^{L} c_\ell z_\ell\phi - \frac{\varepsilon}{2}E^2. \tag{4.17}$$

It is interesting that $\frac{\varepsilon}{2}E^2$ has a negative sign in (4.17); let us remark that this term $-\frac{\varepsilon}{2}E^2$ can indeed be found in literature e.g. in [10, p.342].

The energetics of the model (2.1) considered, for simplicity, with $\alpha = 0$ in (2.8) has been derived in [32], resulting to:

$$\frac{\rm d}{{\rm d}t}\int_\Omega \Big(\underbrace{\frac{\varrho}{2}|v|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\frac{\varepsilon}{2}|\nabla\phi|^2}_{\substack{\text{electrostatic}\\ \text{energy}}} + \underbrace{c_{\rm v}\theta}_{\substack{\text{heat part}\\ \text{of internal}\\ \text{energy } u}} \Big)\,{\rm d}x = \int_\Omega \underbrace{h(c,\theta)}_{\substack{\text{heat production by}\\ \text{chemical reactions}}}\,{\rm d}x \tag{4.18}$$

Under the constitutive relation $E = -\nabla\phi$, we have

$$\begin{aligned}\int_\Omega q\phi - \frac{\varepsilon}{2}E^2\,{\rm d}x &= -\int_\Omega(\varepsilon\Delta\phi)\phi + \frac{\varepsilon}{2}|\nabla\phi|^2\,{\rm d}x \\ &= \int_\Omega \varepsilon\nabla\phi\cdot\nabla\phi - \frac{\varepsilon}{2}|\nabla\phi|^2\,{\rm d}x = \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2\,{\rm d}x\end{aligned} \tag{4.19}$$

and we can rewrite the above energy balance (4.18) in terms of u and μ's as

$$\frac{\rm d}{{\rm d}t}\int_\Omega \Big(\frac{\varrho}{2}|v|^2 + u - \sum_{\ell=1}^{L} c_\ell\mu_\ell + q\phi\Big)\,{\rm d}x = \int_\Omega h(c,\theta)\,{\rm d}x. \tag{4.20}$$

Alternatively, (4.20) can equally be written in terms of u and c_ℓ's as

$$\frac{\rm d}{{\rm d}t}\int_\Omega \Big(\frac{\varrho}{2}|v|^2 + u - \rho\sum_{\ell=1}^{L} c_\ell \ln c_\ell\Big)\,{\rm d}x = \int_\Omega h(c,\theta)\,{\rm d}x. \tag{4.21}$$

Besides, ϕ solving (2.1e) is just the critical point of the overall free energy $\Psi : \phi \mapsto \int_\Omega \psi(c,\theta,\phi,-\nabla\phi)\,{\rm d}x$; interestingly, as $\Psi(c,\theta,\cdot,\cdot)$ given by (4.14) is concave, this critical point is the global maximum.

The *Clausius-Duhem inequality* (under our zero-flux boundary conditions (2.8), i.e. in the isolated system) reads as

$$
\begin{aligned}
0 \ \leq\ & \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\,\mathrm{d}x = \int_\Omega \frac{c_\mathrm{v}}{\theta}\frac{\partial\theta}{\partial t}\,\mathrm{d}x \\
=\ & \int_\Omega \frac{1}{\theta}\Big(\mathrm{div}(\kappa\nabla\theta - c_\mathrm{v}v\theta) + \tau(\mathrm{D}v,c,\theta):\mathrm{D}v + \sum_{\ell=1}^L f_\ell\cdot j_\ell + h\Big)\,\mathrm{d}x \\
=\ & \int_\Omega \Big(\mathrm{div}\big(\frac{\kappa\nabla\theta}{\theta}\big) - c_\mathrm{v}v\nabla\ln\theta + \kappa\frac{|\nabla\theta|^2}{\theta^2} + \frac{\tau(\mathrm{D}v,c,\theta):\mathrm{D}v}{\theta} + \sum_{\ell=1}^L \frac{f_\ell\cdot j_\ell}{\theta} + \frac{h}{\theta}\Big)\,\mathrm{d}x
\end{aligned}
$$

where $f_\ell = -z_\ell c_\ell\nabla\phi$ is the Lorenz force acting on the ℓ-constituent. The first and the second terms in the last integral vanish in a thermally isolated system, the third and the fourth terms are always non-negative (if $\theta > 0$), while the non-negativity of $\int_\Omega(\sum_{\ell=1}^L f_\ell\cdot j_\ell + h)/\theta\,\mathrm{d}x$ is a condition on j_ℓ and h.

Remark 4.6 (*Thermodynamics in special case of equal mobilities.*) As already observed in [29, 32], the special case (4.10) gives the heat sources $\sum_{\ell=1}^L f_\ell\cdot j_\ell$ in the form

$$
\sum_{\ell=1}^L f_\ell\cdot j_\ell = m\rho\nabla q\cdot\nabla\phi + \sum_{\ell=1}^L mc_\ell z_\ell^2|\nabla\phi|^2 - mq^2|\nabla\phi|^2. \tag{4.22}
$$

The meaning of these terms is the following: The first term $m\rho\nabla q\cdot\nabla\phi$ is the power (per unit volume) of the electric current arising by the diffusion flux, which can create local cooling effects, which is related with the *Peltier effect* mentioned already in Remark 2.1. This cooling effect may seemingly violate the entropy production law but, at least in equilibrium situations (i.e. here spatially isothermal cases when $\theta(t,\cdot)$ is constant) the overall entropy production due to this term on Ω is nonnegative: indeed, by using Green's formula, one gets

$$
\int_\Omega \nabla q\cdot\nabla\phi\,\mathrm{d}x = -\int_\Omega \varepsilon\nabla(\Delta\phi)\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega \varepsilon|\Delta\phi|^2\mathrm{d}x \geq 0. \tag{4.23}
$$

In the anisothermal case, we would get the non-negative entropy production if the coefficient m in (4.10) would be proportional to the absolute temperature θ, as it is really considered e.g. in the kinetic theory of gases and known as Einstein's law. Such dependence would, however, make derivation of the a-priori estimates (3.22e) difficult because $\inf\theta > 0$ would have to be proved. The second term $\sum_{\ell=1}^L mc_\ell z_\ell^2|\nabla\phi|^2$ in (4.22) is the power of hypothetical *Joule's heat* produced by the electric currents j_ℓ in ideally diluted water solutions. The third term $-mq^2|\nabla\phi|^2 = -mf_\mathrm{R}^2$ reduces it and represents the rate of cooling by the force which balances the volume-additivity constraint. Besides, the total actual Joule's heat is always non-negative because the second term in (4.22) always dominates the third one thanks to the algebraic inequality

$$
\sum_{\ell=1}^L c_\ell z_\ell^2 \geq \Big(\sum_{\ell=1}^L c_\ell z_\ell\Big)^2 \tag{4.24}
$$

if (2.2) holds, cf. [32, Remark 2.2].

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