A REMARK ON FUNCTIONS CONTINUOUS ON ALL LINES

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Abstract. We prove that each linearly continuous function f on $\mathbb{R}^n$ (i.e., each function continuous on all lines) belongs to the first Baire class, which answers a problem formulated by K.C. Ciesielski and D. Miller (2016). The same result holds also for f on an arbitrary Banach space X, if f has moreover the Baire property. We also prove (extending a known finite-dimensional result) that such f on a separable X is continuous at all points outside a first category set which is also null in any usual sense.

1. Introduction

Separately continuous functions on $\mathbb{R}^n$ (i.e., functions continuous on all lines parallel to an coordinate axis) and also linearly continuous functions (i.e., functions continuous on all lines) were investigated in a number of articles, see the survey [1].

Recall here Lebesgue result of [4] which asserts that

(1.1) each separately continuous function on $\mathbb{R}^n$ belongs to the $(n-1)$-th Baire class.

We prove (see Theorem 3.5 below) that each linearly continuous function f with the Baire property on a Banach space X belongs to the first Baire class. Of course, if X is infinite-dimensional, then there exists an (everywhere) discontinuous linear functional f on X (which is linearly continuous), which shows that, in Theorem 3.5, it is not possible to omit the assumption that f has the Baire property. However, using Lebesgue result (1.1), we obtain that each linearly continuous function f on $\mathbb{R}^n$ belongs to to the first Baire class, which answers [1, Problem 2, p. 12].

The natural question how big can be the set $D(f)$ of all discontinuity points of a separately (resp. linearly) continuous function were considered in several works, see [1].

A complete characterization of sets $D(f)$ for separately continuous functions in $\mathbb{R}^n$ was given in [2] (and independently in [8]), cf. [1]. This characterization, in particular, shows that $D(f)$ is a first category set, but it can have positive Lebesgue measure (even its complement can be Lebesgue null).

Slobodnik proved in [8] that, for each linearly continuous f on $\mathbb{R}^n$,

(1.2) $D(f)$ is contained in a countable union of Lipschitz hypersurfaces,

in particular, the Hausdorff dimension of $D(f)$ is at most $(n-1)$ (and so $D(f)$ is Lebesgue null). We show that (1.2) holds also in each separable Banach space X under the additional assumption that f has the Baire property. Consequently $D(f)$ is null in any usual sense, in particular it is Aronszajn null and Γ-null.

1991 *Mathematics Subject Classification.* Primary: 26B05; Secondary: 46B99.

Key words and phrases. linear continuity, Baire class one, discontinuity sets, Banach space.

The research was supported by the grant GAČR P201/15-08218S.

2. Preliminaries

In the following, by a Banach space we mean a real Banach space. If X is a Banach space, we set $S_X := \{x \in X : \|x\| = 1\}$. The symbol $B(x,r)$ will denote the open ball with center x and radius r. The oscillation of a function f at a point x will be denoted by $\operatorname{osc}(f,x)$.

Let X be a Banach space, $\emptyset \neq G \subset X$ an open set and $f : G \to \mathbb{R}$ a function. Then we say that f is *linearly continuous*, if the restriction $f \restriction_{L \cap G}$ is continuous for each line $L \subset X$ intersecting G.

We will essentially use the following well-known characterization of Baire class one functions (see e.g. [5, Theorem 2.12]).

Lemma 2.1. *Let X be a strong Baire metric space and $f : X \to \mathbb{R}$ a function. Then the following conditions are equivalent. h*

(i) *f is a Baire class one function.*

(ii) *For every nonempty closed set $F \subset X$ and for every real numbers $\alpha < \beta$, the sets $\{z \in F : f(z) \leq \alpha\}$ and $\{z \in F : f(z) \geq \beta\}$ cannot be dense in F simultaneously.*

Recall that X is called strong Baire if every closed subspace of X is a Baire space. Thus each topologically complete metric space (and so each G_δ subspace of a complete space) is strong Baire.

We will use the classical Baire terminology concerning his category theory. So complements of first category sets (= meager sets) are called residual (= comeager) sets and sets of the second category are those which are not of the first category. We will need the following well-known fact which follows e.g. from [3, § 10, (7) and (11)] (cf. the text below (11)).

Lemma 2.2. *If M is a second category subset of a metric space X, then there exists an open set $\emptyset \neq U \subset X$ such that $M \cap V$ is of the second category for each open $\emptyset \neq V \subset U$.*

In a metric space (X,ρ), the system of all sets with the Baire property is the smallest σ-algebra containing all open sets and all first category sets. We will say that a *mapping $f : (X,\rho_1) \to (Y,\rho_2)$ has the Baire property* if f is measurable with respect to the σ-algebra of all sets with the Baire property. In other words, f has the Baire property, if and only if $f^{-1}(B)$ has the Baire property for all Borel sets $B \subset Y$ (see [3, § 32]). We will need the following fact (see e.g. [3, § 32, II]).

Lemma 2.3. *If Y is separable, then f has the Baire property, if and only there exists a residual set R in X such that the restriction $f \restriction_R$ is continuous.*

Let X be a Banach space, $x \in X$, $v \in S_X$ and $\delta > 0$. Then we define the open cone $C(x,v,\delta)$ as the set of all $y \neq x$ for which $\|v - \frac{y-x}{\|y-x\|}\| < \delta$.

The following easy inequality is well known (see e.g. [6, Lemma 5.1]):

$$\text{if } v,w \in X \setminus \{0\}, \text{ then } \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{2}{\|v\|}\, \|v-w\|. \tag{2.1}$$

We will need the following special case of [7, Lemma 2.4]. It can be proved by the Kuratowski-Ulam theorem (as is noted in [7]), but the proof given in [7] is more direct.

Lemma 2.4. *Let U be an open subset of a Banach space X. Let $M \subset U$ be a set residual in U and $z \in U$. Then there exists a line $L \subset X$ such that z is a point of accumulation of $M \cap L$.*

3. Baire class one

Lemma 3.1. *Let X be a Banach space, $\emptyset \neq G \subset X$ an open set and let $f : G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then for each $\eta > 0$ there exist $u \in S_X$, $\delta > 0$ and $p \in \mathbb{N}$ such that*

$$|f(y) - f(x)| \leq \eta \text{ whenever } y \in C(x, u, \delta) \cap B(x, 1/p). \tag{3.1}$$

Proof. Let $x \in X$ and $\eta > 0$ be given; we can and will suppose that $x = 0$. For each $k \in \mathbb{N}$, set

$$S_k := \{v \in S_X : |f(x + tv) - f(x)| \leq \eta \quad \text{for each} \quad 0 < t < 1/k.\}$$

Since S_X is clearly covered by all sets S_k, by the Baire theorem (in S_X) we can choose $p \in \mathbb{N}$ such that S_p is a second category set (in S_X). So Lemma 2.2 implies that we can find $u \in S_X$ and $\delta > 0$ such that $S_p \cap V$ is of the second category in S_X whenever $\emptyset \neq V \subset S_X \cap B(u, \delta)$ is an open subset in S_X. Set

$$U := C(0, u, \delta) \cap B(0, 1/p) \quad \text{and} \quad M := \{y \in U : \ |f(y) - f(x)| \leq \eta\}.$$

Then (3.1) is equivalent to the equality $M = U$.

We will first prove that M is residual in U. To this end consider the product metric space

$$U^* := (S_X \cap B(u, \delta)) \times (0, 1/p)$$

and the mapping

$$\varphi : U^* \to U, \qquad \varphi((v, t)) := tv.$$

Then φ is clearly a homeomorphism (with $\varphi^{-1}(z) = (z/\|z\|, \|z\|)$ for $z \in U$). Since f has the Baire property, we obtain that M has the Baire property in G (and consequently also in U). Therefore $M^* := \varphi^{-1}(M)$ has the Baire property in U^*. Consequently (cf. e.g. [3, § 11, IV, Corollary 2]), to prove that M^* is residual in U^*, it is sufficient to prove that $M^* \cap (V \times G)$ is of the second category in U^* whenever $\emptyset \neq V \subset S_X \cap B(u, \delta)$ is an open subset of S_X and $\emptyset \neq G \subset (0, 1/p)$ is open. To prove this last statement, observe that the definition of S_p implies that

$$(S_p \cap V) \times G \subset M^* \cap (V \times G).$$

Further, since $S_p \cap V$ is of the second category in $S_X \cap B(u, \delta)$ and G is of the second category in $(0, 1/p)$, we obtain (see e.g. [3, § 22, V, Corollary 1b]) that $M^* \cap (V \times G)$ is of the second category in U^*.

Thus we have proved that M^* is residual in U^* and consequently M is residual in U. Now consider an arbitrary $z \in U$. By Lemma 2.4 there exists a line $L \subset X$ and points $z_n \in M \cap L \cap U$ with $z_n \to z$. Since the restriction of f to $L \cap U$ is continuous, we obtain $f(z_n) \to f(z)$, and consequently $z \in M$. So $M = U$, which implies (3.1). □

Lemma 3.2. *Let X be a Banach space, $u \in S_X$, $0 < \delta \leq 1$ and $0 < \xi < \delta/2$. Then, for each $x, y \in X$ with $\|x - y\| < \delta\xi/4$, we have*

(i) $$z := y + (\xi/2)u \in C(x, u, \delta) \cap B(x, \delta) \quad \textit{and}$$

(ii) $$C(x, u, \delta) \cap B(x, \delta) \cap C(y, u, \delta) \cap B(y, \delta) \neq \emptyset.$$

Proof. Since

$$\|z - x\| \leq \|z - y\| + \|y - x\| \leq \frac{\xi}{2} + \frac{\delta\xi}{4} \leq \frac{\delta}{4} + \frac{\delta}{4} < \delta,$$

we have $z \in B(x, \delta)$. Since

$$\|z - x\| \geq \|z - y\| - \|y - x\| \geq \frac{\xi}{2} - \frac{\xi}{4} > 0,$$

we can apply (2.1) to $v := (\xi/2)u = z - y$ and $w := z - x \neq 0$. Because $\|w - v\| = \|y - x\| < \delta\xi/4$, the inequality (2.1) gives

$$\left\| u - \frac{w}{\|w\|} \right\| = \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| < \frac{2}{\xi/2} \cdot \frac{\delta\xi}{4} = \delta.$$

Consequently $z \in C(x, u, \delta)$ and so (i) follows.

Since $z \in C(y, u, \delta) \cap B(y, \delta)$, (i) implies (ii). □

The following result is not labeled as a theorem, since it will be generalized to all Banach spaces.

Proposition 3.3. *Let X be a separable Banach space, $\emptyset \neq G \subset X$ an open set and let $f : G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then f belongs to the first Baire class.*

Proof. We can suppose $\dim X > 1$. Suppose to the contrary that f is not in the first Baire class. Then by Lemma 2.1 there exists a set $\emptyset \neq F \subset G$ closed in G and reals $\alpha < \beta$ such that the both sets

$$A := \{z \in F : \ f(z) \leq \alpha\} \quad \text{and} \quad B := \{z \in F : \ f(z) \geq \beta\}$$

are dense in F. Set $\eta := (1/7)(\beta - \alpha)$. Now choose a dense sequence $(u_n)_1^\infty$ in S_X and, for each $n \in \mathbb{N}$, set

$$P_n := \{x \in F : \ |f(y) - f(x)| \leq \eta \quad \text{whenever} \quad y \in C(x, u_n, 1/n) \cap B(x, 1/n)\}.$$

Lemma 3.1 implies that $F = \bigcup_1^\infty P_n$. Indeed, for each $x \in F$ we can choose $u \in S_X$, $\delta > 0$ and $p \in \mathbb{N}$ for which (3.1) holds. Further choose $n > p$ such that $1/n < \delta/2$ and $\|u_n - u\| < \delta/2$. Then clearly

$$C(x, u_n, 1/n) \cap B(x, 1/n) \subset C(x, u, \delta) \cap B(x, 1/p)$$

and consequently $x \in P_n$ by (3.1).

Since F is closed in G, the Baire theorem in F holds and thus there exists $k \in \mathbb{N}$ such that P_k is not nowhere dense in F. Therefore there exist $c \in F$ and $0 < r < 1/(32k^2)$ such that P_k is dense in $B(c, r) \cap F$.

Now choose $y \in A \cap B(c, r)$ and $y^* \in B \cap B(c, r)$. Since f is linearly continuous, we can choose $0 < \xi < 1/(2k)$ such that

$$f(z) \leq \alpha + \eta \quad \text{for} \quad z := y + (\xi/2)u_k. \tag{3.2}$$

Further choose $x \in P_k \cap B(c, r)$ with $\|y - x\| < \xi/(4k)$. Applying Lemma 3.2 (i) with $u := u_k$ and $\delta := 1/k$ we obtain that $z \in C(x, u_k, 1/k) \cap B(x, 1/k)$, and consequently $|f(z) - f(x)| \leq \eta$ since $x \in P_k$. So (3.2) gives $f(x) \leq \alpha + 2\eta$.

Proceeding quite analogously as above (working now with y^* and B instead of y and A) we find $x^* \in P_k \cap B(c, r)$ with $f(x^*) \geq \beta - 2\eta$. Since $0 < r < 1/(32k^2)$, we have $\|x - x^*\| < 1/(16k^2)$. So we can apply Lemma 3.2 (ii) with $u := u_k$, $\delta := 1/k$, $\xi := 1/(4k)$, x and $y := x^*$ to find a point

$$b \in C(x, u_k, 1/k) \cap B(x, 1/k) \cap C(x^*, u_k, 1/k) \cap B(x^*, 1/k).$$

Since $x, x^* \in P_k$, we have $|f(b) - f(x)| \leq \eta$, $|f(b) - f(x^*)| \leq \eta$, and therefore $\beta - 3\eta \leq f(b) \leq \alpha + 3\eta$. Consequently $\beta - \alpha \leq 6\eta$ which contradicts the choice of η. □

Since each function from $(n-1)$-th Baire class has the Baire property, Lebesgue result (1.1) and Proposition 3.3 give the following main result of the present note which answers [1, Problem 2].

Theorem 3.4. *Each linearly continuous function on $\mathbb{R}^n$ belongs to the first Baire class.*

Using easy "separable reduction" arguments, we obtain that the assumption of separability of X in Proposition 3.3 can be deleted.

Theorem 3.5. *Let X be an arbitrary Banach space, $\emptyset \neq G \subset X$ an open set and let $f : G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then f belongs to the first Baire class.*

Proof. Suppose to the contrary that f is not in the first Baire class. Then by Lemma 2.1 there exist a set $\emptyset \neq F \subset G$ closed in G and reals $\alpha < \beta$ such that the both sets

$$A := \{z \in F : \ f(z) \leq \alpha\} \quad \text{and} \quad B := \{z \in F : \ f(z) \geq \beta\}$$

are dense in F.

Now we will define inductively a nondecreasing sequence $(M_n)_{n=1}^{\infty}$ of countable subsets of F. We set $M_1 := \{a\}$, where $a \in F$ is an arbitrarily chosen point. If $n > 1$ and a countable set M_{n-1} is defined, we choose for each point $\mu \in M_{n-1}$ sequences $(a_k^{\mu})_{k=1}^{\infty}$, $(b_k^{\mu})_{k=1}^{\infty}$ converging to μ with $a_k^{\mu} \in A$ and $b_k^{\mu} \in B$, $k \in \mathbb{N}$. Then we set

$$M_n := M_{n-1} \cup \bigcup_{\mu \in M_{n-1}} \bigcup_{k \in \mathbb{N}} \{a_k^{\mu}, b_k^{\mu}\}.$$

Setting

$$\tilde{F} := \overline{\bigcup_{n \in \mathbb{N}} M_n} \cap G,$$

we easily see that $\tilde{F}$ is a separable subset of F which is closed in F and

$$\text{both } A \cap \tilde{F} \text{ and } B \cap \tilde{F} \text{ are dense in } \tilde{F}. \tag{3.3}$$

Denote by X_1 the closure of the linear span of $\tilde{F}$. Then X_1 is a closed separable subspace of X. By Lemma 2.3 there exists a residual set R in G such that the restriction $f\restriction_R$ is continuous. [11, Lemma 4.6] implies that there exists a separable closed subspace X_2 of X such that $X_2 \supset X_1$ and $R \cap X_2$ is residual in X_2. Consequently the function $g := f\restriction_{X_2 \cap G}$ has the Baire property. Since g is linearly continuous on $X_2 \cap G$, Proposition 3.3 implies that g is in the first Baire class. But this contradicts Lemma 2.1, since $X_2 \cap G$ is a strong Baire space (even a topologically complete space), $\tilde{F}$ is closed in $X_2 \cap G$ and (3.3) holds. □

4. Set of discontinuity points

In this short section we will show that Lemma 3.1 easily implies a Slobodnik's result of [8] (Corollary 4.3 below) and its analogues in infinite-dimensional Banach spaces. First we recall some definitions and facts.

Let X be a Banach space. We say that $A \subset X$ is a Lipschitz hypersurface if there exists a 1-dimensional linear space $F \subset X$, its topological complement E and a Lipschitz mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : \ x \in E\}$.

Recall (see [10, 4C]) that if X is separable, then each $M \subset X$ which can be covered by countably many Lipschitz hypersurfaces (note that such sets are sometimes called "sparse", see [10]) is not only a first category set but is also Aronszajn ($\equiv$ Gauss) null and Γ-null (in Lindenstrauss-Preiss sense).

A natural generalization of "sparse sets" to arbitrary (nonseparable) spaces are σ-cone supported sets. Their definition (see e.g. [10, Definition 4.4]) works with cones defined by a slightly different way than the cones $C(x,v,\delta)$ in Preliminaries; namely with cones $A(v,c) := \bigcup_{\lambda>0} \lambda \cdot B(v,c)$, where $\|v\| = 1$ and $0 < c < 1$. However, for such v and c, obviously $C(0,v,c) \subset A(v,c)$ and (2.1) easily implies $A(v,c/2) \subset C(0,v,c)$. Consequently [10, Definition 4.4] can be equivalently rewritten as follows:

We say that a subset M of a Banach space X is *cone supported* if for each $x \in M$ there exist $v \in S_X$, $\delta > 0$ and $r > 0$ such that $M \cap C(x,v,\delta) \cap B(x,r) = \emptyset$. A set is called *$\sigma$-cone supported* if it is a countable union of cone supported sets.

Recall that [9, Lemma 1] easily implies that if X is separable, then

$$M \subset X \ \text{ is } \sigma - \text{cone supported if and only if} \\ \text{it can be covered by countably many Lipschitz hypersurfaces.} \tag{4.1}$$

Theorem 4.1. *Let X be an arbitrary Banach space, $\emptyset \neq G \subset X$ an open set and let $f: G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then the set $D(f)$ of all discontinuity points of f is σ-cone supported.*

Proof. Denote $D_n := \{x \in G : \operatorname{osc}(f,x) \geq 1/n\}$, $n \in \mathbb{N}$. Since $D(f) = \bigcup_{n=1}^{\infty} D_n$, it is sufficient to prove that each D_n is a cone supported set. To this end fix an arbitrary $n \in \mathbb{N}$ and consider an arbitrary point $x \in D_n$. By Lemma 3.1 there exist $v \in S_X$, $\delta > 0$ and $r > 0$ such that

$$|f(y) - f(x)| \leq \frac{1}{3n} \quad \text{whenever} \quad y \in C(x,v,\delta) \cap B(x,r).$$

Consequently the oscillation of f on the open set $C(x,v,\delta) \cap B(x,r)$ is at most $2/(3n)$ and therefore $D_n \cap C(x,v,\delta) \cap B(x,r) = \emptyset$. So we have proved that D_n is cone supported. $\square$

Using (4.1), we obtain the following corollary.

Corollary 4.2. *Let X be a separable Banach space, $\emptyset \neq G \subset X$ an open set and let $f: G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then the set $D(f)$ of all discontinuity points of f can be covered by countably many Lipschitz hypersurfaces. In particular, $D(f)$ is a first category set which is Aronszajn null and also Γ-null.*

We obtain also the following result which was proved by S.G. Slobodnik in [8] by an essentially different way.

Corollary 4.3. *Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open set and let $f: G \to \mathbb{R}$ be a linearly continuous function. Then the set $D(f)$ of all discontinuity points of f can be covered by countably many Lipschitz hypersurfaces.*

Proof. If $G = \mathbb{R}^n$, it is sufficient to use Theorem 4.1 together with (1.1). If G is an open interval we can use instead of (1.1) its generalization [3, § 31, V, Theorem 2]. Using this special case, we easily obtain the general one, if we write $G = \bigcup_{n \in \mathbb{N}} I_n$, where I_n are open intervals. $\square$

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