SMALL-BOUND ISOMORPHISMS OF FUNCTION SPACES

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ABSTRACT. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $i = 1, 2$, let K_i be a locally compact (Hausdorff) topological space and let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ such that each point of the Choquet boundary $\operatorname{Ch} K_i$ of $\mathcal{H}_i$ is a weak peak point. We show that if there exists an isomorphism $T: \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{Ch} K_1$ is homeomorphic to $\operatorname{Ch} K_2$. Next we provide a one-sided version of this result. Finally we prove that under the assumption on weak peak points the Choquet boundaries have the same cardinality provided $\mathcal{H}_1$ is isomorphic to $\mathcal{H}_2$.

1. Introduction

We work within the framework of real or complex vector spaces and write $\mathbb{F}$ for the respective field $\mathbb{R}$ or $\mathbb{C}$. Further, let $S_{\mathbb{F}}$ stand for the set $\{\lambda \in \mathbb{F}; |\lambda| = 1\}$. For a compact (Hausdorff) space K, let $\mathcal{C}(K, \mathbb{F})$ stand for the space of all continuous $\mathbb{F}$-valued functions on K, and for a locally compact (Hausdorff) space K, let $\mathcal{C}_0(K, \mathbb{F})$ denote the space of all continuous $\mathbb{F}$-valued functions vanishing at infinity. We consider both these spaces endowed with the sup-norm. For a compact space K, we identify the dual space $(\mathcal{C}(K, \mathbb{F}))^*$ with the space $\mathcal{M}(K, \mathbb{F})$ of all $\mathbb{F}$-valued Radon measures on K. Unless stated otherwise, we consider $\mathcal{M}(K, \mathbb{F})$ endowed with the weak* topology given by this duality. We further write $\mathcal{M}^1(K)$ for probability Radon measures on K and $\mathcal{M}^+(K)$ for positive Radon measures on K. Let ε_x stand for the Dirac measure at the point $x \in K$.

We start with the classical Banach-Stone theorem asserting that, given a pair of compact spaces K and L, they are homeomorphic provided $\mathcal{C}(K, \mathbb{F})$ is isometric to $\mathcal{C}(L, \mathbb{F})$ (see [15, Theorem 3.117]).

A remarkable generalization of the Banach-Stone theorem was given by Amir [3] and Cambern [7]. They showed that compact spaces K and L are homeomorphic if there exists an isomorphism $T: \mathcal{C}(K, \mathbb{F}) \to \mathcal{C}(L, \mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$. Alternative proofs were given by Cohen [12] and Drewnowski [14].

Chu and Cohen provided in [10] a very nice generalization of these results in the context of affine continuous functions on compact convex sets. In order to explain their results we need a bit of terminology. By a compact convex set we mean a compact convex subset of a locally convex (Hausdorff) space. Let $\mathfrak{A}(X, \mathbb{F})$ be the space of all continuous $\mathbb{F}$-valued affine functions on a compact convex set X endowed with the sup-norm. If X is a compact convex set, for any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$ such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X, \mathbb{C})$, see [2, Proposition I.2.1]. We call $r(\mu)$ the *barycenter* of μ. If $\mu, \nu \in \mathcal{M}^+(X)$, then $\mu \prec \nu$

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if $\mu(k) \le \nu(k)$ for each convex continuous function k on X. A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is $\prec$-maximal.

By the Choquet–Bishop–de-Leeuw representation theorem (see [2, Theorem I.4.8]), for each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$. If this measure is uniquely determined for each $x \in X$, the set X is called a *simplex*. It is called a *Bauer simplex* if, moreover, the set $\operatorname{ext} X$ of extreme points of X is closed. In this case, the space $\mathfrak{A}(X, \mathbb{F})$ is isometric to the space $\mathcal{C}(\operatorname{ext} X, \mathbb{F})$ (see [2, Theorem II.4.3]). On the other hand, given a space $\mathcal{C}(K, \mathbb{F})$, it is isometric to the space $\mathfrak{A}(\mathcal{M}^1(K), \mathbb{F})$ ([2, Corollary II.4.2]).

A reformulation of the result of Amir and Cambern for simplices reads as follows: Given Bauer simplices X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X, \mathbb{F}) \to \mathfrak{A}(Y, \mathbb{F})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$.

The aforementioned Chu and Cohen proved in [10] that for compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathfrak{A}(X, \mathbb{R}) \to \mathfrak{A}(Y, \mathbb{R})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$ and one of the following conditions hold:

(i) X and Y are simplices such that their extreme points are weak peak points;
(ii) X and Y are metrizable and their extreme points are weak peak points;
(iii) $\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and extreme points of X and Y are split faces.

A point $x \in X$ is a *weak peak point* if given $\varepsilon \in (0,1)$ and an open set $U \subset X$ containing x, there exists a in the unit ball $B_{\mathfrak{A}(X,\mathbb{R})}$ of $\mathfrak{A}(X, \mathbb{R})$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1 - \varepsilon$, see [10, p. 73].

In [28], it was showed that extreme points of X and Y are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X, \mathbb{R}) \to \mathfrak{A}(Y, \mathbb{R})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$, extreme points are weak peak points and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf sets.

In [13] the same result is proved without the assumption of the Lindelöf property and paper [30] provides the analogous result for the case of complex functions. It turns out that this result is in a sense optimal since the bound 2 cannot be improved (see [11], where a pair of nonhomeomorphic compact spaces K_1, K_2 for which there exists an isomorphism $T\colon \mathcal{C}(K_1, \mathbb{R}) \to \mathcal{C}(K_2, \mathbb{R})$ with $\|T\| \cdot \left\|T^{-1}\right\| = 2$ is constructed) and the assumption on weak peak points cannot be omitted (see [19], where the author constructs for each $\varepsilon \in (0,1)$ a pair of simplices X_1, X_2 such that $\operatorname{ext} X_1$ is not homeomorphic to $\operatorname{ext} X_2$ but there is an isomorphism $T\colon \mathfrak{A}(X_1, \mathbb{R}) \to \mathfrak{A}(X_2, \mathbb{R})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 1 + \varepsilon$).

As a corollary of the theorems for affine functions results on selfadjoint function spaces were obtained in [30]. More precisely, given a pair of selfadjoint closed spaces $\mathcal{H}_i \subset \mathcal{C}(K, \mathbb{C})$ containing constant functions and separating points of K_i, $i = 1, 2$, their Choquet boundaries are homeomorphic provided points in the Choquet boundaries are weak peak points and there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$ (see [30, Theorem 5.3]).

The aim of the present paper is to extend this result to the case of locally compact spaces and general function spaces. So we need only to assume that, for $i = 1, 2$, $\mathcal{H}_i$ is a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i such that each point of the Choquet boundary $\operatorname{Ch} K_i$ is a weak peak point. We recall that $x \in K_i$ is a *weak peak point* if for a given $\varepsilon \in (0,1)$ and a neighborhood U of x there exists a function $h \in B_{\mathcal{H}_i}$ such that $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch} K_i \setminus U$.

We show that if there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$, the Choquet boundaries of K_1 and K_2 are homeomorphic. (We recall that, given a closed subspace $\mathcal{H} \subset \mathcal{C}_0(K, \mathbb{F})$ for some locally compact space K, a point $x \in K$ is in the Choquet boundary $\operatorname{Ch} K$ if the point evaluation functional $\phi(x)$ defined as $\phi(x)(h) = h(x)$, $h \in \mathcal{H}$, is an extreme point of $B_{\mathcal{H}^*}$.)

Thus the main result of our paper is the following theorem.

Theorem 1.1. *For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\operatorname{Ch} K_i$ is a weak peak point and let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then $\operatorname{Ch} K_1$ is homeomorphic to $\operatorname{Ch} K_2$.*

We refer the reader to [26] and [24] for results on function algebras in the spirit of the above theorem. The case of vector-valued Banach-Stone type theorem is treated e.g. in [5], [17], [25], [8], [6] or [1].

The next result considers isomorphisms that are not generally surjective. Jarosz showed in [23] that if K_1, K_2 are locally compact spaces, $A \subset \mathcal{C}_0(K_1, \mathbb{C})$ is an extremely regular closed subspace and a not necessarily surjective isomorphism $T\colon A \to \mathcal{C}_0(K_2, \mathbb{C})$ satisfies $\|T\| \cdot \|T^{-1}\| < 2$, then K_1 is a continuous image of a subset of K_2. An analogous result for function spaces reads as follows.

Theorem 1.2. *For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\operatorname{Ch} K_1$ is a weak peak point and let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an into isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then there exists a set $L \subset \operatorname{Ch} K_2$ and a continuous surjective mapping $\varphi\colon L \to \operatorname{Ch} K_1$.*

In [9], it is proved that locally compact spaces K_1, K_2 have the same cardinality provided $\mathcal{C}_0(K_1, \mathbb{F})$ is isomorphic to $\mathcal{C}_0(K_2, \mathbb{F})$. We generalize this result to the context of function spaces by proving the following theorem.

Theorem 1.3. *For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\operatorname{Ch} K_i$ is a weak peak point and let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism. Then the cardinality of $\operatorname{Ch} K_1$ is equal to the cardinality of $\operatorname{Ch} K_2$.*

It has turned out that we can use the basic strategy of the proofs in [30] (which in turn are adapted from [10]), however, some adjustments have to be made. The outcome of these adjustments is even some simplification of the methods used in [30]. More precisely, we work directly with functions in $\mathcal{M}(K, \mathbb{F})^*$ instead of $\mathcal{H}^{**}$, see Lemmas 2.6 and 2.7. This allows to use the standard decomposition of measures and thus we can avoid e.g. [30, Lemma 2.5]. Also, the construction of a "peaking" function a_x^{**} in Lemma 2.6 is simpler then the one in [30, Lemma 2.4].

2. Auxiliary results

This section collects auxiliary lemmas needed for the proofs of the main results.

Lemma 2.1. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{C}(K, \mathbb{F})$ for some compact space K and $\phi\colon K \to B_{\mathcal{H}^*}$ be the evaluation mapping. Then*

$$\operatorname{ext} B_{\mathcal{H}^*} \subset S_{\mathbb{F}} \cdot \phi(\operatorname{Ch} K).$$

Proof. By [16, Corollary 2.3.6],

$$\operatorname{ext} B_{\mathcal{H}^*} \subset \{\lambda \phi(x);\ |\lambda| = 1, x \in K\}.$$

Let $s \in \operatorname{ext} B_{\mathcal{H}^*}$ be given. Then $s = \lambda\phi(x)$ for some $\lambda \in \mathbb{F}$ with $|\lambda| = 1$ and $x \in K$. We want to prove that $\phi(x) \in \operatorname{ext} B_{\mathcal{H}^*}$. Assuming the contrary, there exist distinct points $s_1, s_2 \in B_{\mathcal{H}^*}$ such that $\phi(x) = \frac{1}{2}(s_1 + s_2)$. Then $\lambda s_1 \neq \lambda s_2$ and

$$s = \lambda\phi(x) = \frac{1}{2}(\lambda s_1 + \lambda s_2)$$

is not an extreme point of $B_{\mathcal{H}^*}$. This contradiction finishes the proof. □

The following lemma is a very particular result on representing functionals by means of measures carried by the Choquet boundary. We refer the reader to [22], [18], [20], [32], [4] or [31] for related results.

Lemma 2.2. *Let $\mathcal{H}$ be a subspace of $\mathcal{C}(K, \mathbb{F})$ for some compact space K. Then for any $s \in \mathcal{H}^*$ there exists a measure $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K}, \mathbb{F})$ such that $\mu = s$ on $\mathcal{H}$ and $\|\mu\| = \|s\|$.*

Proof. Let $s \in \mathcal{H}^*$ be given. We write $\mathcal{A} \subset \mathcal{C}(\overline{\operatorname{Ch} K}, \mathbb{F})$ for the space $\{h|_{\overline{\operatorname{Ch} K}}; h \in \mathcal{H}\}$. By [16, Theorem 2.3.8], for each $h \in \mathcal{H}$ there exists $x \in \operatorname{Ch} K$ such that $|h(x)| = \|h\|$. Thus the restriction mapping $r\colon \mathcal{H} \to \mathcal{A}$ given by $r(h) = h|_{\overline{\operatorname{Ch} K}}$ is an isometric isomorphism and we can define $t \in \mathcal{A}^*$ be the formula

$$t(a) = s(h), \quad h \in \mathcal{H} \text{ satisfies } h|_{\overline{\operatorname{Ch} K}} = a, \quad a \in \mathcal{A}.$$

Then $\|t\| = \|s\|$. Using the Hahn-Banach theorem we find a measure

$$\mu \in (\mathcal{C}(\overline{\operatorname{Ch} K}, \mathbb{F}))^* = \mathcal{M}(\overline{\operatorname{Ch} K}, \mathbb{F})$$

such that $\|\mu\| = \|t\|$ and $t = \mu$ on $\mathcal{A}$. Then $\|\mu\| = \|s\|$ and

$$\mu(h) = \int_{\overline{\operatorname{Ch} K}} h \,\mathrm{d}\mu = t(h|_{\overline{\operatorname{Ch} K}}) = s(h), \quad h \in \mathcal{H}.$$

This finishes the proof. □

The important topological notion is that of a function of the first Borel class. Thus we recall that, given a pair of topological spaces K, L, a function $f\colon K \to L$ is *of the first Borel class* if $f^{-1}(U)$ is a countable union od differences of closed sets in K for any $U \subset L$ open (see [33] or [29, Definition 5.13]). We just mention that, if $L = \mathbb{R}$, any semicontinuous function $f\colon K \to \mathbb{R}$ is of the first Borel class.

Lemma 2.3. *Let $L_1, \dots, L_n$ be compact convex sets in a locally convex space and $L = \operatorname{co}(L_1 \cup \dots \cup L_n)$. Let $f\colon L \to \mathbb{F}$ be an affine function such that $f|_{L_i}$ is of the first Borel class for each $i = 1, \dots, n$. Then f is of the first Borel class on L.*

Proof. Let

$$\Delta = \left\{ \lambda \in [0, \infty)^n; \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

and

$$H = L_1 \times \dots \times L_n \times \Delta.$$

Let further $g\colon H \to \mathbb{F}$ be defined as

$$g(x_1, \dots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i f(x_i), \quad (x_1, \dots, x_n, \lambda) \in H.$$

By the proof of [29, Theorem 5.10], g is of the first Borel class on H. We consider a continuous surjection $\varphi\colon H \to L$ defined as

$$\varphi(x_1, \dots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i x_i, \quad (x_1, \dots, x_n, \lambda) \in H.$$

Since f is affine on L, we obtain $f \circ \varphi = g$. By [29, Theorem 5.16], f is of the first Borel class on L. $\square$

Lemma 2.4. *Let K be a compact space and $f\colon K \to \mathbb{F}$ be a bounded function of the first Borel class. Then $\widehat{f}\colon \mathcal{M}(K, \mathbb{F}) \to \mathbb{F}$ defined as*

$$\widehat{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}(K, \mathbb{F}),$$

is of the first Borel class on any ball $rB_{\mathcal{M}(K,\mathbb{F})}$, $r > 0$.

Proof. We provide a proof for the case of complex measures, the easier case of real measures would be done similarly. Assume first that f is real. By [29, Proposition 5.30], $\widehat{f}$ is of the first Borel class on $\mathcal{M}^1(K)$. Let $L_1 = 2\mathcal{M}^1(K)$, $L_2 = -2\mathcal{M}^1(K)$, $L_3 = 2i\mathcal{M}^1(K)$ and $L_4 = -2i\mathcal{M}^1(K)$. Since L_i is affinely homeomorphic to $\mathcal{M}^1(K)$ and $\widehat{f}$ is linear, $\widehat{f}$ is of the first Borel class on each L_i, $i = 1, \dots, 4$. By the decomposition of a complex measure we obtain

$$B_{\mathcal{M}(K,\mathbb{F})} \subset L = \operatorname{co}\left(L_1 \cup L_2 \cup L_3 \cup L_4\right).$$

By linearity it is enough to prove that $\widehat{f}$ is of the first Borel class on L. But this follows from Lemma 2.3.

If $f\colon K \to \mathbb{C}$, we have $f = f_1 + if_2$, where f_1, f_2 are real bounded functions of the first Borel class. Then the function

$$\widehat{f}(\mu) = \mu(f) = \mu(f_1) + i\mu(f_2) = \widehat{f_1}(\mu) + i\widehat{f_2}(\mu), \quad \mu \in B_{\mathcal{M}(K,\mathbb{F})},$$

is of the first Borel class as well. (Indeed, since the functions $\mu \mapsto \widehat{f_1}$ and $\mu \mapsto i\widehat{f_2}(\mu)$ are of the first Borel class, their sum is easily seen to be of the first Borel class as well, see e.g. the proof of [29, Theorem 5.10].) $\square$

Lemma 2.5. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{C}(K, \mathbb{F})$ for some compact space K. Let $x \in K$ be a weak peak point. Then for any*

$$\mu \in \mathcal{M}(\overline{\operatorname{Ch} K}, \mathbb{F}) \cap \mathcal{H}^{\perp}$$

holds $\mu(\chi_{\{x\}}) = 0$.

Proof. Let $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K}, \mathbb{F}) \cap \mathcal{H}^{\perp}$ be arbitrary and $\varepsilon > 0$ be given. We write $\mu = \lambda\varepsilon_x + \nu$, where $\nu(\{x\}) = 0$. Let U be a closed neighborhood of x such that $|\nu|(U) < \varepsilon$. Using the assumption we find $h \in B_{\mathcal{H}}$ with $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch} K \setminus U$. Then $|h| \leq \varepsilon$ on

$$\overline{\operatorname{Ch} K} \setminus U \subset \overline{\operatorname{Ch} K \setminus U},$$

and thus

$$\begin{aligned}
\left|\mu(\chi_{\{x\}})\right| &= \left|\mu(\chi_{\{x\}}) - \mu(h)\right| \leq \left|\lambda\varepsilon_x(\chi_{\{x\}} - h)\right| + |\nu(h)| \\
&\leq |\lambda|\,(1 - h(x)) + \int_{\overline{\operatorname{Ch} K} \cap U} |h| \,\mathrm{d}\,|\nu| + \int_{\overline{\operatorname{Ch} K} \setminus U} |h| \,\mathrm{d}\,|\nu| \\
&\leq |\lambda|\,\varepsilon + \varepsilon + \varepsilon\,\|\nu\| \leq \varepsilon(1 + \|\mu\|).
\end{aligned}$$

Hence $\mu(\chi_{\{x\}}) = 0$. $\square$

Lemma 2.6. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{C}(K,\mathbb{F})$ for some compact space K and let $\pi\colon \mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}) \to \mathcal{H}^*$ be the restriction mapping. Let $x \in K$ be a weak peak point. For each $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F})$ we define $\widehat{\chi_{\{x\}}}(\mu) = \mu(\chi_{\{x\}})$. Then there exists $a_x^{**} \in \mathcal{H}^{**}$ such that $a_x^{**} \circ \pi = \widehat{\chi_{\{x\}}}$.*

Proof. The element $\widehat{\chi_{\{x\}}}$ is contained in $(\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}))^*$. In order to find the required element $a_x^{**} \in \mathcal{H}^{**}$ it is enough to realize that for any $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}) \cap \mathcal{H}^{\perp}$ we have $\mu(\chi_{\{x\}}) = 0$ (see Lemma 2.5). Thus we can define

$$a_x^{**}(s) = \widehat{\chi_{\{x\}}}(\mu), \quad \mu \in \pi^{-1}(s), s \in \mathcal{H}^*.$$

This finishes the proof. □

Given a Banach space E with its dual E^*, we write $\langle \cdot,\cdot \rangle\colon E \times E^* \to \mathbb{F}$ for the duality mapping.

Lemma 2.7. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{C}(K,\mathbb{F})$ for some compact space K and $\pi\colon \mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}) \to \mathcal{H}^*$ be the restriction mapping. Let $\widehat{f} \in (\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}))^*$ and $a^{**} \in \mathcal{H}^{**}$ satisfy $\widehat{f} = a^{**} \circ \pi$.*

(a) *Then for any $s \in \mathcal{H}^*$ and $\mu \in \pi^{-1}(s)$ holds*

$$\langle s, a^{**} \rangle = \langle \mu, \widehat{f} \rangle.$$

(b) *For any $r > 0$, if $\widehat{f}$ is of the first Borel class on $rB_{\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F})}$, then a^{**} is of the first Borel class on $(rB_{\mathcal{H}^*}, weak^*)$.*

Proof. (a) Given $s \in \mathcal{H}^*$ and $\mu \in \pi^{-1}(s)$, we realize that the dual mapping $\pi^*\colon \mathcal{H}^{**} \to (\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F}))^*$ satisfies $\pi^*(a^{**}) = a^{**} \circ \pi = \widehat{f}$. Thus

$$\langle s, a^{**} \rangle = \langle \pi(\mu), a^{**} \rangle = \langle \mu, \pi^*(a^{**}) \rangle = \langle \mu, \widehat{f} \rangle.$$

(b) For any $r > 0$, the mapping $\pi\colon rB_{\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F})} \to rB_{\mathcal{H}^*}$ is a weak*-weak* continuous surjection (see Lemma 2.2). By [21, Theorem 10] (see also [29, Theorem 5.26(d)]), if $\widehat{f}$ is of the first Borel class on $rB_{\mathcal{M}(\overline{\operatorname{Ch} K},\mathbb{F})}$, a^{**} is of the first Borel class on $rB_{\mathcal{H}^*}$. □

Lemma 2.8. *Let $\mathcal{A}$ be a closed subspace of $\mathcal{C}_0(K,\mathbb{F})$ for some locally compact space K. Let $J = K \cup \{\alpha\}$ be the one-point compactification of K, where α is the point at infinity. Let*

$$\mathcal{B} = \{g \in \mathcal{C}(J,\mathbb{F});\ g|_K \in \mathcal{A}\ \&\ g(\alpha) = 0\}.$$

Then $\mathcal{B}$ is a closed subspace of $\mathcal{C}(J,\mathbb{F})$ isometric to $\mathcal{A}$ such that $\operatorname{Ch} K = \operatorname{Ch} J$.

Proof. Clearly, any function $a \in \mathcal{A}$ has a unique extension $b_a \in \mathcal{B}$ and the mapping $a \mapsto b_a$ is an isometric isomorphism. Thus $\mathcal{B}$ is a closed subspace of $\mathcal{C}(J,\mathbb{F})$.

Let $\phi_K\colon K \to \mathcal{A}^*$ and $\phi_J\colon J \to \mathcal{B}^*$ be the evaluation mappings. Let $x \in \operatorname{Ch} K$ be given. Assume that $\phi_J(x) = \frac{1}{2}(t_1 + t_2)$, where $t_1, t_2 \in B_{\mathcal{B}^*}$ are distinct. Then $s_i \in \mathcal{A}^*$ defined as $s_i(a) = t_i(b_a)$, $i = 1, 2$, are of norm 1, distinct, and satisfy

$$\frac{1}{2}(s_1 + s_2)(a) = \frac{1}{2}(t_1 + t_2)(b_a) = \phi_J(x)(b_a) = b_a(x) = a(x) = \phi_K(x), \quad a \in \mathcal{A}.$$

Thus $\phi_K(x) \notin \operatorname{ext} B_{\mathcal{A}^*}$, a contradiction.

Conversely, let $x \in \operatorname{Ch} J$ be given. Since

$$\phi_J(\alpha)(b) = b(\alpha) = 0, \quad b \in \mathcal{B},$$

we have $\phi_J(\alpha) \notin \operatorname{ext} B_{\mathcal{B}^*}$. Hence $x \in K$. If $\phi_K(x) = \frac{1}{2}(s_1+s_2)$ for some $s_1, s_2 \in B_{\mathcal{A}^*}$ distinct, then $t_i \in B_{\mathcal{B}^*}$ defined as $t_i(b) = s_i(b|_K)$, $i = 1, 2$, satisfy

$$\frac{1}{2}(t_1 + t_2)(b) = \frac{1}{2}(s_1 + s_2)(b|_K) = \phi_K(x)(b|_K) = b(x) = \phi_J(x)(b), \quad b \in \mathcal{B}.$$

Since t_1, t_2 are distinct, $x \notin \operatorname{Ch} J$, which is a contradiction. Hence the assertion follows. □

Lemma 2.9. *Let $f: X \to \mathbb{F}$ be an affine function of the first Borel class on a compact convex set. Then*

$$\sup_{x \in X} |f(x)| = \sup_{x \in \operatorname{ext} X} |f(x)| .$$

Proof. The assertion follows from [13, Corollary 1.5] since any function of the first Borel class has the point of continuity property (see [27, Theorem 2.3]). □

3. Proof of Theorem 1.1

The main strategy of the proof is originated by the proofs in [10], however, we need to add some extra arguments provided by the previous lemmas. In particular, Lemma 2.7 allows us to work directly with measures.

Proof of Theorem 1.1. We first assume that the spaces K_1, K_2 are compact. Secondly, we suppose that there exists $c' \in \mathbb{R}$ such that $1 < c' < 2$ and $\|T\| < 2$ and $\|Th\| > c' \|h\|$ for all $h \in \mathcal{H}_1 \setminus \{0\}$ (otherwise we would find $1 < c' < 2$ such that $\|T\| \cdot \|T^{-1}\| < \frac{2}{c'} < 2$ and consider the mapping $c' \|T^{-1}\| T$; see [10, p. 76]). We fix $c \in \mathbb{R}$ satisfying $1 < c < c'$.

Claim 1.: For any $a^{**} \in \mathcal{H}_1^{**} \setminus \{0\}$ and $b^{**} \in \mathcal{H}_2^{**} \setminus \{0\}$ we have $\|T^{**}a^{**}\| > c \|a^{**}\|$ and $\|(T^{-1})^{**}b^{**}\| > \frac{1}{2} \|b^{**}\|$.

For the proof see [30, Lemma 4.2]

For $i = 1, 2$, let $\pi_i: \mathcal{M}(\overline{\operatorname{Ch} K_i}, \mathbb{F}) \to \mathcal{H}_i^*$ be the restriction mapping and let $\phi_i: K_i \to B_{\mathcal{H}_i^*}$ be the evaluation mapping. For each $x \in \operatorname{Ch} K_1$ we consider the function $\chi_{\{x\}}$. Let $\widehat{\chi_{\{x\}}}: \mathcal{M}(\overline{\operatorname{Ch} K_1}) \to \mathbb{F}$ be defined as in Lemma 2.4 and let $a_x^{**} \in \mathcal{H}_1^{**}$ satisfies $a_x^{**} \circ \pi_1 = \widehat{\chi_{\{x\}}}$ (see Lemma 2.6). Then a_x^{**} is of the first Borel class on $rB_{\mathcal{H}_1^*}$ for any $r > 0$, see Lemma 2.4 and 2.7(b). Analogously we define for $y \in \operatorname{Ch} K_2$ the function $\chi_{\{y\}}$ and the element $b_y^{**} \in \mathcal{H}_2^{**}$.

We define mappings ρ_1 and ρ_2 as follows:

$$\begin{aligned} \rho_1(x) &= \left\{ y \in \operatorname{Ch} K_2;\ \left|\langle \phi_1(x), (T^{-1})^{**} b_y^{**} \rangle\right| > \frac{1}{2} \right\}, \quad x \in \operatorname{Ch} K_1, \\ \rho_2(y) &= \left\{ x \in \operatorname{Ch} K_1;\ |\langle \phi_2(y), T^{**} a_x^{**} \rangle| > c \right\}, \quad y \in \operatorname{Ch} K_2. \end{aligned} \tag{3.1}$$

Claim 2. ρ_1 and ρ_2 are mappings.

Let $x \in \operatorname{Ch} K_1$ be such that there exist distinct points $y_1, y_2 \in \operatorname{Ch} K_2$ with

$$\left|\langle (T^{-1})^* \phi_1(x), b_{y_i}^{**} \rangle\right| = \left|\langle \phi_1(x), (T^{-1})^{**} b_{y_i}^{**} \rangle\right| > \frac{1}{2}, \quad i = 1, 2.$$

We find a measure $\nu_x \in \mathcal{M}(\overline{\operatorname{Ch} K_2}, \mathbb{F})$ with $\pi_2(\nu_x) = (T^{-1})^* \phi_1(x)$ and $\|\nu_x\| = \|(T^{-1})^* \phi_1(x)\|$. We write

$$\nu_x = \lambda_1 \varepsilon_{y_1} + \mu_1 = \lambda_2 \varepsilon_{y_2} + \mu_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{F}$, $\mu_1(\{y_1\}) = \mu_2(\{y_2\}) = 0$. Using Lemma 2.7(a) we obtain

$$\frac{1}{2} < \left|\langle (T^{-1})^* \phi_1(x), b_{y_i}^{**}\rangle\right| = \left|\langle \nu_x, \widehat{\chi_{\{y_i\}}}\rangle\right| = \left|\nu_x(\chi_{\{y_i\}})\right| =$$
$$= \left|(\lambda_i \varepsilon_{y_i} + \mu_i)(\chi_{\{y_i\}})\right| = |\lambda_i|, \quad i = 1, 2.$$

Since $y_1 \neq y_2$, we obtain

$$1 \geq \left\|(T^{-1})^* \phi_1(x)\right\| = \|\nu_x\| \geq |\lambda_1| + |\lambda_2| > 1,$$

i.e., a contradiction.

Analogously we show that $\rho_2(y)$ is at most single-valued for each $y \in \operatorname{Ch} K_2$.

Let L_1 and L_2 denote the domain of ρ_1 and ρ_2, respectively.

Claim 3.: The mappings $\rho_1 \colon L_1 \to \operatorname{Ch} K_2$ and $\rho_2 \colon L_2 \to \operatorname{Ch} K_1$ are surjective.

Let $y \in \operatorname{Ch} K_2$ be given. We assume that $\left|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle\right| \leq \frac{1}{2}$ for each $x \in \operatorname{Ch} K_1$ and seek a contradiction.

First we show that the element $(T^{-1})^{**} b_y^{**} \in \mathcal{H}_1^{**}$ is of the first Borel class on $B_{\mathcal{H}_1^*}$.

Indeed, we know that b_y^{**} is of the first Borel class on any ball in $\mathcal{H}_2^*$, in particular on $2B_{\mathcal{H}_2^*}$. Since $(T^{-1})^*$ is a weak*-weak* homeomorphism, $(T^{-1})^*(B_{\mathcal{H}_1^*}) \subset 2B_{\mathcal{H}_2^*}$ and $(T^{-1})^{**} b_y^{**} = b_y^{**} \circ (T^{-1})^*$, it follows that $(T^{-1})^{**} b_y^{**}$ is of the first Borel class on $B_{\mathcal{H}_1^*}$ as well.

By Lemma 2.9 and 2.1 we have

$$\frac{1}{2} \leq \frac{1}{2}\left\|b_y^{**}\right\| < \left\|(T^{-1})^{**} b_y^{**}\right\| = \sup_{a^* \in B_{\mathcal{H}_1^*}} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right|$$
$$= \sup_{a^* \in \operatorname{ext} B_{\mathcal{H}_1^*}} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right| \leq \sup_{a^* \in S_{\mathbb{F}} \cdot \phi_1(\operatorname{Ch} K_1)} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right|$$
$$= \sup_{x \in \operatorname{Ch} K_1} \left|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle\right| \leq \frac{1}{2}.$$

This contradiction implies that ρ_1 is surjective.

Analogously we check that ρ_2 is surjective.

Claim 4.: We have $L_1 = \operatorname{Ch} K_1$ and $L_2 = \operatorname{Ch} K_2$ and $\rho_2(\rho_1(x)) = x$, $x \in \operatorname{Ch} K_1$, and $\rho_1(\rho_2(y)) = y$, $y \in \operatorname{Ch} K_2$.

Let $y \in L_2$ be given. We want to show that $\rho_1(\rho_2(y)) = y$, i.e., that

$$\left|\langle \phi_1(\rho_2(y)), (T^{-1})^{**} b_y^{**}\rangle\right| > \frac{1}{2}. \tag{3.2}$$

We have

$$d = \sup_{x \in \operatorname{Ch} K_1} \left|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle\right| = \sup_{a^* \in S_{\mathbb{F}} \cdot \phi_1(\operatorname{Ch} K_1)} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right| =$$
$$\geq \sup_{a^* \in \operatorname{ext} B_{\mathcal{H}_1^*}} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right| = \sup_{a^* \in B_{\mathcal{H}_1^*}} \left|\langle a^*, (T^{-1})^{**} b_y^{**}\rangle\right|$$
$$= \left\|(T^{-1})^{**} b_y^{**}\right\| > \frac{1}{2}\left\|b_y^{**}\right\| \geq \frac{1}{2}.$$

Since $c > 1$, we have $d > \max\{\frac{d}{c}, \frac{1}{2}\}$. Hence there exists $x \in \operatorname{Ch} K_1$ such that

$$\left|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle\right| > \max\left\{\frac{d}{c}, \frac{1}{2}\right\} \geq \frac{1}{2}.$$

Thus $y = \rho_1(x)$.

Assume that (3.2) does not hold. Then $\rho_2(y) \neq x$. By Claim 3 there exists $\widehat{y} \in L_2$ such that $\rho_2(\widehat{y}) = x$. Then $\widehat{y} \neq y$. We find $\mu_{\widehat{y}} \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ such that $\|\mu_{\widehat{y}}\| = \|T^*\phi_2(\widehat{y})\|$ and $\pi_1(\mu_{\widehat{y}}) = T^*\phi_2(\widehat{y})$. We write

$$\mu_{\widehat{y}} = \lambda\varepsilon_x + \mu, \quad \lambda \in \mathbb{F}, \mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F}) \text{ with } \mu(\{x\}) = 0.$$

By Lemma 2.7(a),

$$\begin{aligned} \langle \lambda\phi_1(x) + \pi_1(\mu), (T^{-1})^{**}b_y^{**}\rangle &= \langle \pi_1(\mu_{\widehat{y}}), (T^{-1})^{**}b_y^{**}\rangle = \langle T^*\phi_2(\widehat{y}), (T^{-1})^{**}b_y^{**}\rangle \\ &= \langle \phi_2(\widehat{y}), T^{**}(T^{-1})^{**}b_y^{**}\rangle = \langle \phi_2(\widehat{y}), b_y^{**}\rangle = \langle \varepsilon_{\widehat{y}}, \widehat{\chi_y}\rangle \\ &= \chi_y(\widehat{y}) = 0. \end{aligned}$$

Since $x = \rho_2(\widehat{y})$, we have

$$c < |\langle \phi_2(\widehat{y}), T^{**}a_x^{**}\rangle| = |\langle T^*\phi_2(\widehat{y}), a_x^{**}\rangle| = |\langle \pi_1(\mu_{\widehat{y}}), a_x^{**}\rangle| = |\langle \lambda\varepsilon_x + \mu, \widehat{\chi_x}\rangle| = |\lambda|\,.$$

Since

$$\|\mu\| + |\lambda| = \|\mu_{\widehat{y}}\| = \|T^*\phi_2(\widehat{y})\| < 2\,\|\phi_2(\widehat{y})\| = 2,$$

we obtain $\|\mu\| < 2 - c$. By putting everything together we get

$$\begin{aligned} d < |\lambda|\,\frac{d}{c} &< |\lambda|\left|\langle \phi_1(x), (T^{-1})^{**}b_y^{**}\rangle\right| = \left|\langle \lambda\phi_1(x), (T^{-1})^{**}b_y^{**}\rangle\right| \\ &= \left|\langle \pi_1(\mu), (T^{-1})^{**}b_y^{**}\rangle\right| \le d\,\|\mu\| \le d(2-c) < d, \end{aligned}$$

a contradiction. Thus (3.2) holds, which means that $\rho_1(\rho_2(y)) = y$, $y \in L_2$.

Now, let $x \in \operatorname{Ch} K_1$ be given. Then there exists $y \in L_2$ such that $\rho_2(y) = x$. Then $y = \rho_1(\rho_2(y)) = \rho_1(x)$, which means that $x \in L_1$.

Let $y \in \operatorname{Ch} K_2$ be given. Then we can find $x \in L_1 = \operatorname{Ch} K_1$ with $\rho_1(x) = y$ and further we can select $\widehat{y} \in L_2$ such that $\rho_2(\widehat{y}) = x$. Then

$$y = \rho_1(x) = \rho_1(\rho_2(\widehat{y})) = \widehat{y} \in L_2.$$

Hence $L_2 = \operatorname{Ch} K_2$.

Finally, if $x \in \operatorname{Ch} K_1$, we find $y \in \operatorname{Ch} K_2$ with $\rho_2(y) = x$ and obtain

$$\rho_2(\rho_1(x)) = \rho_2(\rho_1(\rho_2(y))) = \rho_2(y) = x.$$

Till now we have proved that $\rho_1\colon \operatorname{Ch} K_1 \to \operatorname{Ch} K_2$ is a bijection with ρ_2 being its inverse. Now we use the assumption on weak peak points to check that ρ_1 is a homeomorphism.

Claim 5.: The mapping ρ_2 is continuous.

Let $F \subset \operatorname{Ch} K_1$ be a nonempty closed set and let $F = \operatorname{Ch} K_1 \cap H$ for some closed set $H \subset K_1$. Obviously we may assume that $F \neq \operatorname{Ch} K_1$. We want to prove that $\rho_2^{-1}(F)$ is closed in $\operatorname{Ch} K_2$.

To this end, we construct for each $x \in \operatorname{Ch} K_1 \setminus F$ a function $h_x \in \mathcal{H}_1$ as follows. Fix $x \in \operatorname{Ch} K_1 \setminus F$ and $y \in \operatorname{Ch} K_2$ with $\rho_2(y) = x$. Let V be a closed neighborhood of x with $V \cap H = \emptyset$. We write $T^*\phi_2(y) = \pi_1(\mu_x)$, where $\mu_x \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\|\mu_x\| = \|T^*\phi_2(y)\|$. Let $\mu_x = \lambda\varepsilon_x + \nu$, where $\lambda \in \mathbb{F}$ and $\nu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\nu(\{x\}) = 0$. Since $\rho_2(y) = x$,

$$c < |\langle \phi_2(y), T^{**}a_x^{**}\rangle| = |T^*\phi_2(y), a_x^{**}\rangle| = |\langle \pi_1(\mu_x), a_x^{**}\rangle| = \left|\mu_x(\chi_{\{x\}})\right| = |\lambda|\,.$$

Let $\varepsilon > 0$ satisfy

$$\varepsilon < \min\left\{\frac{|\lambda| - c}{|\lambda| + 3}, c - 1\right\}.$$

We select a compact set $A \subset \overline{\operatorname{Ch} K_1}$ such that $x \notin A$ and $|\nu| (\overline{\operatorname{Ch} K_1} \setminus A) < \varepsilon$ and let $U \subset V$ be a closed neighborhood of x satisfying $U \cap A = \emptyset$. Using the assumption on weak peak points we select $h_x \in B_{\mathcal{H}_1}$ with $h_x(x) > 1 - \varepsilon$ and $|h_x| < \varepsilon$ on $\operatorname{Ch} K_1 \setminus U$. Then $|h_x| \leq \varepsilon$ on $F \cup A$ since $|h_x| \leq \varepsilon$ on

$$\overline{\operatorname{Ch} K_1 \setminus U} \supset \overline{\operatorname{Ch} K_1} \setminus U \supset F \cup A.$$

Now we claim that

$$\rho_2^{-1}(F) = \bigcap_{x \in \operatorname{Ch} K_1} \{z \in \operatorname{Ch} K_2; \, |Th_x(z)| \leq c\} . \tag{3.3}$$

Indeed, let $y \in \operatorname{Ch} K_2 \setminus \rho_2^{-1}(F)$ be given. Then we consider the function h_x, where $x = \rho_2(y) \in \operatorname{Ch} K_1 \setminus F$. Using the inequality $\|\mu_x\| \leq 2$ we then have

$$\begin{aligned} |Th_x(y)| &= |\langle Th_x, \phi_2(y)\rangle| = |\langle h_x, T^*\phi_2(y)\rangle| = |\mu_x(h_x)| = |\lambda h_x(x) + \nu(h_x)| \\ &\geq |\lambda| (1-\varepsilon) - \int_{\overline{\operatorname{Ch} K_1}} |h_x| \,\mathrm{d}\, |\nu| \\ &= |\lambda| (1-\varepsilon) - \int_{\overline{\operatorname{Ch} K_1} \setminus A} |h_x| \,\mathrm{d}\, |\nu| - \int_A |h_x| \,\mathrm{d}\, |\nu| \\ &\geq |\lambda| (1-\varepsilon) - \varepsilon - \varepsilon \|\nu\| \geq |\lambda| (1-\varepsilon) - 3\varepsilon > c, \end{aligned}$$

which shows the inclusion "$\supset$" in (3.3).

For the proof of the reverse inclusion we select $z \in \rho_2^{-1}(F)$ and let $x \in \operatorname{Ch} K_1 \setminus F$ be arbitrary. Then $\rho_2(z) \in F$ and, by the definition of ρ_2,

$$c < \left| \langle \phi_2(z), T^{**} a^{**}_{\rho_2(z)} \rangle \right| = \left| \langle T^*\phi_2(z), a^{**}_{\rho_2(z)} \rangle \right| .$$

Let $T^*\phi_2(z) = \pi_1(\mu_z)$, where $\mu_z \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\|\mu_z\| = \|T^*\phi_2(z)\|$. We write $\mu_z = \lambda \varepsilon_{\rho_2(z)} + \mu$, where $\lambda \in \mathbb{F}$ and $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\mu(\{\rho_2(z)\}) = 0$. Then

$$c < \left| \langle \pi_1(\mu_z), a^{**}_{\rho_2(z)} \rangle \right| = \left| \mu_z(\chi_{\{\rho_2(z)\}}) \right| = |\lambda| ,$$

and thus

$$2 > \|T^*\phi_2(z)\| = \|\mu_z\| = |\lambda| + \|\mu\| > c + \|\mu\| .$$

From these estimates it follows

$$\begin{aligned} |Th_x(z)| &= |\langle h_x, T^*\phi_2(z)\rangle| = \left| (\lambda \varepsilon_{\rho_2(z)} + \mu)(h_x)\rangle \right| \\ &\leq |\lambda| \varepsilon + (2-c) \leq 2\varepsilon + (2-c) < c. \end{aligned}$$

Hence

$$z \in \{u \in \operatorname{Ch} K_2; \, |Th_x(u)| \leq c\}$$

and (3.3) is verified.

By (3.3), $\rho_2^{-1}(F)$ is a closed subset of $\operatorname{Ch} K_2$, and thus ρ_2 is continuous.

Analogously we would verify that ρ_1 is continuous.

This finishes the proof for the compact case. Now we assume that K_1, K_2 are locally compact and consider their one-point compactifications $J_i = K_i \cup \{\alpha_i\}$, where, for $i = 1, 2$, α_i is the point representing infinity. The spaces $\mathcal{H}_i$ are then closed subspaces of $\mathcal{C}(J_i, \mathbb{F})$ satisfying $h(\alpha_i) = 0$, $h \in \mathcal{H}_i$. Since $\operatorname{Ch} K_i = \operatorname{Ch} J_i$ by Lemma 2.8, the assumption on weak peak points for $\operatorname{Ch} J_i$ is satisfied. Thus the compact case implies the existence of a homeomorphism between $\operatorname{Ch} J_1$ and $\operatorname{Ch} J_2$. Hence the theorem follows. $\square$

4. Proof of Theorem 1.2

Proof of Theorem 1.2. We follow the proof of Theorem 1.1. Again we may assume that the spaces K_1, K_2 are compact, see Lemma 2.8. We consider $1 < c < c' < 2$ and T such that $\|T\| < 2$ and $\|Th\| \geq c' \|h\|$, $h \in \mathcal{H}_1$.

By [30, Lemma 4.2] we have

$$\|T^{**}h^{**}\| > c \|h^{**}\|, \quad h^{**} \in \mathcal{H}_1^{**} \setminus \{0\}.$$

For $x \in \operatorname{Ch} K_1$, let $\chi_{\{x\}}$ and a_x^{**} be as in the proof of Theorem 1.1. Again we define

$$\rho(y) = \{x \in \operatorname{Ch} K_1;\, |\langle \phi_2(y), T^{**}a_x^{**}\rangle| > c\}, \quad y \in \operatorname{Ch} K_2.$$

Claim 1. ρ is a mapping. Indeed, let $x_1, x_2 \in \operatorname{Ch} K_1$ be distinct such that $\left|\langle \phi_2(y), T^{**}a_{x_i}^{**}\rangle\right| > c$ for some $y \in \operatorname{Ch} K_2$, $i = 1, 2$. We find $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ with $\pi_1(\mu) = T^*\phi_2(y)$ and $\|\mu\| = \|T^*\phi_2(y)\|$. We write

$$\mu = \lambda_i \varepsilon_{x_i} + \mu_i,$$

where $\lambda_i \in \mathbb{F}$ and $\mu_i(\{x_i\}) = 0$, $i = 1, 2$. By Lemma 2.7,

$$c < \left|\langle T^*\phi_2(y), a_{x_i}^{**}\rangle\right| = \left|\langle \mu, \chi_{\{x_i\}}\rangle\right| = |\lambda_i|, \quad i = 1, 2.$$

Thus

$$2 > \|T^*\phi_2(y)\| = \|\mu\| \geq |\lambda_1| + |\lambda_2| > 2c$$

yields a contradiction with the inequality $c > 1$.

Let L denote the domain of ρ.

Claim 2. ρ is surjective. Assume that for some $x \in \operatorname{Ch} K_1$ we have $c \geq |\langle \phi_2(y), T^{**}a_x^{**}\rangle|$, $y \in \operatorname{Ch} K_2$. Then we have as in the proof of Theorem 1.1

$$\begin{aligned} c &\geq \sup_{y \in \operatorname{Ch} K_2} |\langle \phi_2(y), T^{**}a_x^{**}\rangle| = \sup_{s \in S_{\mathbb{F}} \cdot \phi_2(\operatorname{Ch} K_2))} |\langle s, T^{**}a_x^{**}\rangle| \\ &= \sup_{s \in B_{\mathcal{H}_2^*}} |\langle s, T^{**}a_x^{**}\rangle| = \|T^{**}a_x^{**}\| > c \|a_x^{**}\| \geq c, \end{aligned}$$

i.e., a contradiction.

Claim 3. $\rho\colon L \to \operatorname{Ch} K_1$ is continuous. Let $F \subset \operatorname{Ch} K_1$ be a closed set and let $F = \operatorname{Ch} K_1 \cap H$ for some closed set $H \subset K_1$. We may assume that $F \neq \operatorname{Ch} K_1$. We want to prove that $\rho^{-1}(F)$ is closed in L.

To this end, we construct for each $x \in \operatorname{Ch} K_1 \setminus F$ and $y \in \rho^{-1}(x)$ a function $h_{x,y} \in \mathcal{H}_1$ as follows. Let V be a closed neighborhood of x with $V \cap H = \emptyset$. We write

$$T^*\phi_2(y) = \pi_1(\mu) \quad \text{and} \quad \mu = \lambda\varepsilon_x + \nu,$$

where $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\|\mu\| = \|T^*\phi_2(y)\|$, $\lambda \in \mathbb{F}$ and $\nu(\{x\}) = 0$. Then $|\lambda| > c$. Let $\varepsilon > 0$ satisfy

$$\varepsilon < \min\left\{\frac{|\lambda| - c}{3 + |\lambda|}, c - 1\right\}.$$

Let $A \subset \overline{\operatorname{Ch} K_1} \setminus \{x\}$ be a compact set satisfying $|\nu|\,(\overline{\operatorname{Ch} K_1} \setminus A) < \varepsilon$. Then there exists a function $h_{x,y} \in \mathcal{H}_1$ such that

$$\|h_{x,y}\| \leq 1, \quad h_{x,y}(x) > 1 - \varepsilon \quad \text{and} \quad |h_{x,y}| \leq \varepsilon \text{ on } F \cup A.$$

Now we claim that

$$\rho^{-1}(F) = \bigcap_{x \in \operatorname{Ch} K_1 \setminus F} \; \bigcap_{y \in \rho^{-1}(x)} \{z \in L;\, |Th_{x,y}(z)| \leq c\}. \tag{4.1}$$

Indeed, if $y \in L \setminus \rho^{-1}(F)$, then we consider the function $h_{x,y}$, where $x = \rho(y) \in \operatorname{Ch} K_1 \setminus F$. Then we write as above

$$T^* y = \pi_1(\mu) \quad \text{and} \quad \mu = \lambda \varepsilon_x + \nu.$$

By the choice of the function $h_{x,y}$ we have

$$\begin{aligned} |T h_{x,y}(y)| &= |\langle h_{x,y}, T^* \phi_2(y) \rangle| = |\langle h_{x,y}, \lambda \varepsilon_x + \nu \rangle| \\ &\geq |\lambda| \, |1 - \varepsilon| - \int_{\overline{\operatorname{Ch} K_1} \setminus A} |h_{x,y}| \, \mathrm{d}\,|\nu| - \int_A |h_{x,y}| \, \mathrm{d}\,|\nu| \\ &\geq |\lambda| \, (1 - \varepsilon) - \varepsilon - \varepsilon \, \|\nu\| > c. \end{aligned}$$

Hence

$$y \notin \bigcap_{x \in \operatorname{Ch} K_1 \setminus F} \; \bigcap_{u \in \rho^{-1}(x)} \left\{ z \in L; \, |T h_{x,u}(z)| \leq c \right\},$$

which shows the inclusion "$\supset$" in (4.1).

For the proof of the reverse inclusion we select $z \in \rho^{-1}(F)$ and let $x \in \operatorname{Ch} K_1 \setminus F$ and $y \in \rho^{-1}(x)$ be arbitrary. Then $\rho(z) \in F$ and, by the definition of ρ,

$$c < \left| \langle \phi_2(z), T^{**} a^{**}_{\rho(z)} \rangle \right| = \left| \langle T^* \phi_2(z), a^{**}_{\rho(z)} \rangle \right|.$$

Let

$$T^* \phi_2(z) = \pi_1(\mu) \quad \text{and} \quad \mu = \lambda \varepsilon_{\rho(z)} + \nu,$$

where $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ satisfies $\|\mu\| = \|T^* \phi_2(z)\|$, $\lambda \in \mathbb{F}$ and $\nu(\{\rho(z)\}) = 0$. Then

$$c < \left| \langle \pi_1(\mu), a^{**}_{\rho(z)} \rangle \right| = \left| \langle \lambda \varepsilon_{\rho(z)} + \nu, \chi_{\{\rho(z)\}} \rangle \right| = |\lambda|,$$

and thus

$$2 > \|T^* \phi_2(z)\| = \|\mu\| = |\lambda| + \|\nu\| > c + \|\nu\|.$$

From these estimates it follows

$$\begin{aligned} |T h_{x,y}(z)| = |\langle h_{x,y}, T^* \phi_2(z) \rangle| &= \left| \int_{\overline{\operatorname{Ch} K_1}} h_{x,y} \, \mathrm{d}\mu \right| \\ &\leq |\lambda| \, \varepsilon + (2 - c) \leq 2\varepsilon + (2 - c) < c. \end{aligned}$$

Hence

$$z \in \{u \in L; \, |T h_{x,y}(u)| \leq c\}$$

and (4.1) is verified.

By (4.1), $\rho^{-1}(F)$ is a closed subset of L, and thus ρ is continuous. This finishes the proof. □

5. Proof of Theorem 1.3

We start the proof of Theorem 1.3 by the following result imitating [30, Lemma 3.1] for the case of function spaces.

Lemma 5.1. *Let K be a compact space, $\mathcal{H}$ be a subspace of $\mathcal{C}(K, \mathbb{F})$ of finite dimension and let each point of $\operatorname{Ch} K$ be a weak peak point. Then $\operatorname{Ch} K$ is finite and $\mathcal{H}$ is isometrically isomorphic to $\ell^\infty(\operatorname{Ch} K, \mathbb{F})$.*

Proof. Let $x \in \operatorname{Ch} K$ be given. We show that there exists a function $h_x \in B_{\mathcal{H}}$ such that $h_x(x) = 1$ and $h_x = 0$ on $\operatorname{Ch} K \setminus \{x\}$. To this end we consider net $\{h_{U,\varepsilon}\}$ in $B_{\mathcal{H}}$, where U is a neighborhood of x, $\varepsilon \in (0,1)$ and $h_{U,\varepsilon}$ is a function satisfying $h_{U,\varepsilon}(x) > 1-\varepsilon$ and $|h_{U,\varepsilon}| < \varepsilon$ on $\operatorname{Ch} K \setminus U$. We consider the partial order on the set of pairs (U,ε) given by $(U_1,\varepsilon_1) \le (U_2,\varepsilon_2)$ provided $U_2 \subset U_1$ and $\varepsilon_2 < \varepsilon_1$. Since $B_{\mathcal{H}}$ is compact in the norm topology, the net $\{h_{U,\varepsilon}\}$ possesses a cluster point $h_x \in B_{\mathcal{H}}$. Then $h_x(x) = 1$ and $h_x = 0$ on $\operatorname{Ch} K \setminus \{x\}$.

From this observation now follows that $\operatorname{Ch} K$ is finite. Indeed, assuming that $\operatorname{Ch} K$ contains infinite set $\{x_n;\, n \in \mathbb{N}\}$, the functions $\{h_{x_n};\, n \in \mathbb{N}\}$ are linearly independent, which contradicts the assumption.

It remains to prove that the restriction mapping $r\colon \mathcal{H} \to \ell^\infty(\operatorname{Ch} K, \mathbb{F})$ is a surjective isometry. It is an isometry by the maximum principle [16, Theorem 2.3.8]. It is surjective, since any $f \in \ell^\infty(\operatorname{Ch} K, \mathbb{F})$ can be written as

$$f = \sum_{x \in \operatorname{Ch} K} f(x) h_x|_{\operatorname{Ch} K},$$

and thus $h = \sum_{x \in \operatorname{Ch} K} f(x) h_x$ satisfies $r(h) = f$. □

Proof of Theorem 1.3. Using Lemma 2.8 we may assume that K_1, K_2 are compact. If $\mathcal{H}_1$ (and thus also $\mathcal{H}_2$) is finite-dimensional, we obtain from Lemma 5.1 the equality

$$\begin{aligned} |\operatorname{Ch} K_1| &= \dim(\ell^\infty(\operatorname{Ch} K_1, \mathbb{F})) = \dim \mathcal{H}_1 = \dim \mathcal{H}_2 \\ &= \dim(\ell^\infty(\operatorname{Ch} K_2, \mathbb{F})) = |\operatorname{Ch} K_2|. \end{aligned}$$

From now on we may thus assume that both spaces are infinite-dimensional.

Let $\pi_1\colon \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F}) \to \mathcal{H}_1^*$ be the restriction mapping. For each $x \in \operatorname{Ch} K_1$ we consider the function $\chi_{\{x\}}$. Let $\widehat{\chi_{\{x\}}}\colon \mathcal{M}(\overline{\operatorname{Ch} K_1}) \to \mathbb{F}$ be defined as in Lemma 2.4 and let $a_x^{**} \in \mathcal{H}_1^{**}$ satisfies $a_x^{**} \circ \pi_1 = \widehat{\chi_{\{x\}}}$ (see Lemma 2.6). Then a_x^{**} is of the first Borel class on $rB_{H_1^*}$ for any $r > 0$, see Lemmas 2.4 and 2.7(b).

For a fixed point $y \in \operatorname{Ch} K_2$, let $\lambda_y(x) = \langle T^*\phi_2(y), a_x^{**}\rangle$. We claim that the set

$$X_y = \{x \in \operatorname{Ch} K_1;\, \lambda_y(x) \neq 0\}$$

is at most countable. Indeed, let $s = T^*\phi_2(y)$ and $\mu \in \mathcal{M}(\overline{\operatorname{Ch} K_1}, \mathbb{F})$ extends s. Let $x \in \operatorname{Ch} K_1$ be arbitrary. Using Lemma 2.7 we have

$$\mu(\{x\}) = \mu(\chi_{\{x\}}) = \langle \mu, \widehat{\chi_{\{x\}}}\rangle = \langle T^*\phi_2(y), a_x^{**}\rangle.$$

Since $\|\mu\| < \infty$, $\mu(\{x\}) \neq 0$ for at most countably many $x \in \operatorname{Ch} K_1$.

Now we prove that for each $x \in \operatorname{Ch} K_1$ there exists $y \in \operatorname{Ch} K_2$ such that $x \in X_y$. To this end, we assume the contrary. Let $x \in \operatorname{Ch} K_1$ be such that

$$\langle T^*\phi_2(y), a_x^{**}\rangle = 0, \quad y \in \operatorname{Ch} K_2.$$

Lemma 2.9 then yields

$$\begin{aligned} 0 &= \sup_{y \in \operatorname{Ch} K_2} |\langle T^*\phi_2(y), a_x^{**}\rangle| = \sup_{y \in \operatorname{Ch} K_2} |\langle \phi_2(y), T^{**}a_x^{**}\rangle| = \sup_{s \in S_{\mathbb{F}} \cdot \phi_2(\operatorname{Ch} K_2)} |\langle s, T^{**}a_x^{**}\rangle| \\ &= \sup_{s \in B_{\mathcal{H}_2^*}} |\langle s, T^{**}a_x^{**}\rangle| = \|T^{**}a_x^{**}\| \neq 0, \end{aligned}$$

i.e., a contradiction.

Now both the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite-dimensional, and thus the sets $\operatorname{Ch} K_1$ and $\operatorname{Ch} K_2$ are infinite. Indeed, if $\operatorname{Ch} K_1$ were finite, by the maximum principle (see

[16, Theorem 2.3.8]) we would obtain that the space $\mathcal{H}_1 \subset \ell^\infty(\operatorname{Ch} K_1, \mathbb{F})$ is finite-dimensional.

Now, since we have $\operatorname{Ch} K_1 = \bigcup_{y \in \operatorname{Ch} K_2} X_y$, we get $|\operatorname{Ch} K_1| \leq |\operatorname{Ch} K_2|$.

By reversing the role of K_1 and K_2 we obtain the converse inequality, which concludes the proof. □

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