ISOMORPHISMS OF SPACES OF AFFINE CONTINUOUS COMPLEX FUNCTIONS

JAKUB RONDOŠ AND JIŘÍ SPURNÝ

Abstract. Let X and Y be compact convex sets such that their each extreme point is a weak peak point. We show that $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$ provided there exists a small-bound isomorphism of the space $\mathfrak{A}(X,\mathbb{C})$ of continuous affine complex functions on X onto $\mathfrak{A}(Y,\mathbb{C})$. Further, we generalize results of Cengiz and Jarosz to the context of compact convex sets.

1. Introduction

We work within the framework of real or complex vector spaces and write $\mathbb{F}$ for the respective field $\mathbb{R}$ or $\mathbb{C}$. Further we write $\mathbb{T}$ for the set $\{\lambda \in \mathbb{C};\ |\lambda| = 1\}$.

First we recall several notions. If X is a compact convex set in a locally convex space, we write $\mathfrak{A}(X,\mathbb{F})$ for the space of all affine continuous $\mathbb{F}$-valued functions on X endowed with the sup-norm. Let $\operatorname{ext} X$ stand for the set of all extreme points of X. If K is compact (Hausdorff) topological space, let $\mathcal{C}(K,\mathbb{F})$ stand for the space of all continuous $\mathbb{F}$-valued functions on K endowed with the sup-norm.

We identify the dual space $(\mathcal{C}(K,\mathbb{F}))^*$ with the space $\mathcal{M}(K,\mathbb{F})$ of all Radon measures on K. We write $\mathcal{M}^+(K)$ for positive Radon measures and $\mathcal{M}^1(K)$ for probability Radon measures on K.

A point x in a compact convex set X is called a *weak peak point* if

(1.1) given $\varepsilon \in (0,1)$ and an open set $U \subset X$ containing x, there exists a in the unit ball $B_{\mathfrak{A}(X,\mathbb{C})}$ of $\mathfrak{A}(X,\mathbb{C})$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1-\varepsilon$.

For any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$ such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X,\mathbb{C})$, see [2, Proposition I.2.1]. We call $r(\mu)$ the *barycenter* of μ. A function $f\colon X \to \mathbb{F}$ satisfies the *barycentric formula* (or is called *strongly affine*) if $\mu(f) = f(r(\mu))$, $\mu \in \mathcal{M}^1(X)$.

If $\mu,\nu \in \mathcal{M}^+(X)$, then $\mu \prec \nu$ if $\mu(k) \le \nu(k)$ for each convex continuous function k on X. A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is $\prec$-maximal. A measure $\mu \in \mathcal{M}(X,\mathbb{F})$ is called *boundary* if its total variation $|\mu|$ is maximal.

By the Choquet–Bishop–de-Leeuw representation theorem (see [2, Theorem I.4.8]), for each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$. If this measure is uniquely determined for each $x \in X$, the set X is called a *simplex*. It is called a *Bauer simplex* if, moreover, the set $\operatorname{ext} X$ is closed. In this case, the space $\mathfrak{A}(X,\mathbb{F})$ is isometric to the space $\mathcal{C}(\operatorname{ext} X,\mathbb{F})$ (see [2, Theorem II.4.3]).

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On the other hand, given a space $\mathcal{C}(K,\mathbb{F})$, it is isometric to the space $\mathfrak{A}(\mathcal{M}^1(K),\mathbb{F})$ ([2, Corollary II.4.2]).

The classical Banach-Stone theorem asserts that, given a pair of compact spaces K and L, they are homeomorphic provided $\mathcal{C}(K)$ is isometric to $\mathcal{C}(L)$ (see [13, Theorem 3.117]).

This can be reformulated in the framework of compact convex sets as follows: If X, Y are Bauer simplices and $\mathfrak{A}(X,\mathbb{F})$ is isometric to $\mathfrak{A}(Y,\mathbb{F})$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

By a result of Lazar in [24], for simplices X, Y, the spaces $\mathfrak{A}(X,\mathbb{R})$ and $\mathfrak{A}(Y,\mathbb{R})$ are isometric only if X is affinely homeomorphic to Y.

A result of Rao (see [28]) precisely describes isometries of $\mathfrak{A}(X,\mathbb{C})$ for a simplex X.

A remarkable generalization of the Banach-Stone theorem was given by Amir [3] and Cambern [6]. They showed that compact spaces K, L are homeomorphic if there exists an isomorphism $T\colon \mathcal{C}(K,\mathbb{F})\to\mathcal{C}(L,\mathbb{F})$ with $\|T\|\cdot\left\|T^{-1}\right\|<2$. Alternative proofs were given by Cohen [10] and Drewnowski [12].

A reformulation of this result for simplices reads as follows: Given Bauer simplices X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{F})\to\mathfrak{A}(Y,\mathbb{F})$ with $\|T\|\cdot\left\|T^{-1}\right\|<2$.

This theorem was improved by Chu and Cohen in [8], who proved that for compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{R})\to\mathfrak{A}(Y,\mathbb{R})$ with $\|T\|\cdot\left\|T^{-1}\right\|<2$ and one of the following conditions hold:

(i) X and Y are simplices such that their extreme points are weak peak points;

(ii) X and Y are metrizable and their extreme points are weak peak points;

(iii) $\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and extreme points of X and Y are split faces.

In [25], it was showed that extreme points of X and Y are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{R})\to\mathfrak{A}(Y,\mathbb{R})$ with $\|T\|\cdot\left\|T^{-1}\right\|<2$, extreme points are weak peak points and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf sets.

In [11] the same result is proved without the assumption of the Lindelöf property.

It turns out that this result is in a sense optimal since the bound 2 cannot be improved (see [9]) and the assumption on weak peak points cannot be omitted (see [17]).

The first main result of our paper is a variant of [11, Theorem 2.1] for complex spaces. It reads as follows.

Theorem 1.1. *Let X, Y be compact convex sets and let $T\colon \mathfrak{A}(X,\mathbb{C})\to\mathfrak{A}(Y,\mathbb{C})$ be an isomorphism satisfying $\|T\|\cdot\left\|T^{-1}\right\|<2$.*

If each point of $\operatorname{ext} X$ and $\operatorname{ext} Y$ is a weak peak point, $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

2. Isomorphisms with a small bound

Before embarking on the proof of Theorem 1.1 we recall several notions.

If X is a compact convex set, a face $F\subset X$ is called a *split face* if the complementary set F' (i.e., the union of all faces disjoint from F) is convex and X is a direct convex sum of F and F', i.e., every point in X can be uniquely represented as a convex combination of a point in F and a point in F' (see [2, p. 133]). If F is a closed split face, then the *upper envelope* of the characteristic function χ_F defined as

$$\chi_F^*(x) = \inf\{a(x);\, a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_F\}, \quad x \in X,$$

is upper semicontinuous and affine, $F = (\chi_F^*)^{-1}(1)$ and $F' = (\chi_F^*)^{-1}(0)$, see [2, Proposition II.6.5 and Proposition II.6.9]. Moreover, the family $\{a \in \mathfrak{A}(X,\mathbb{R});\, \chi_F < a\}$ is downward directed.

In what follows, we consider the weak*-topology on $(\mathfrak{A}(X,\mathbb{C}))^*$, and we understand X as a subset of $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$ via the evaluation mapping, i.e., $x(f) = f(x)$, $f \in \mathfrak{A}(X,\mathbb{C})$, $x \in X$.

The proof of Theorem 1.1 follows the strategy of the proof of [8], but we use [11, Corollary 1.3(b)] as the main tool. Thus we need the key Lemma 2.4 which allows us to represent upper semicontinuous affine function on X as elements of $(\mathfrak{A}(X,\mathbb{C}))^{**}$.

We start with a lemma describing extreme points of $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$, see also Lemma 1 in [19].

Lemma 2.1. *Let X be a compact convex set. Then*

$$\operatorname{ext} B_{(\mathfrak{A}(X,\mathbb{C}))^*} = \mathbb{T}\cdot \operatorname{ext} X.$$

Proof. We denote $K = B_{(\mathfrak{A}(X,\mathbb{C}))^*}$. First we prove that $\operatorname{ext} K \subset \mathbb{T}\cdot X$. Since $\mathbb{T}\cdot X$ is compact, by the Milman theorem it is enough to show that $\overline{\operatorname{co}}\,(\mathbb{T}\cdot X) = K$.

Assuming the contrary, there exist $s \in K \setminus \overline{\operatorname{co}}\,(\mathbb{T}\cdot X)$, $\alpha \in \mathbb{R}$, and $f \in \mathfrak{A}(X,\mathbb{C})$ such that

$$\operatorname{Re} s(f) > \alpha > \sup\left\{\operatorname{Re} r(f);\, r \in \overline{\operatorname{co}}\,(\mathbb{T}\cdot X)\right\}.$$

Let $\mu \in B_{\mathcal{M}(X,\mathbb{C})}$ be a Hahn-Banach extension of s. Since

$$\operatorname{Re} tf(x) < \alpha, \quad t \in \mathbb{T}, x \in X,$$

we obtain $|f(x)| < \alpha$, $x \in X$. Then

$$\alpha < \operatorname{Re} s(f) = \operatorname{Re}\mu(f) \le \left|\int_X f \,\mathrm{d}\mu\right| \le \int_X |f| \,\mathrm{d}\,|\mu| < \int_X \alpha \,\mathrm{d}\,|\mu| = \alpha$$

gives a contradiction. Thus $\operatorname{ext} K \subset \mathbb{T}\cdot X$.

Now we show that $\operatorname{ext} K = \mathbb{T}\cdot \operatorname{ext} X$. Let $s \in \operatorname{ext} K$ be given. Then $s = tx$ for some $t \in \mathbb{T}$ and $x \in X$. If $x = \frac{1}{2}(x_1 + x_2)$ for some distinct points $x_1, x_2 \in X$, then

$$s = \frac{1}{2}(tx_1 + tx_2),$$

where $tx_1 \ne tx_2$. Thus $s \notin \operatorname{ext} K$, which is impossible. This proves "$\subset$".

On the other hand, let $tx = \frac{1}{2}(s_1 + s_2)$ for some $t \in \mathbb{T}$, $x \in \operatorname{ext} X$ and $s_1, s_2 \in K$. Then $x = \frac{1}{2}\left(t^{-1}s_1 + t^{-1}s_2\right)$. Let $\mu_1, \mu_2 \in B_{\mathcal{M}(X,\mathbb{C})}$ be Hahn-Banach extensions of $t^{-1}s_1$ and $t^{-1}s_2$, respectively. Then

$$f(x) = \frac{1}{2}(\mu_1(f) + \mu_2(f)), \quad f \in \mathfrak{A}(X,\mathbb{C}).$$

If $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, then $\mu(1) = 1 = \|\mu\|$, and thus $\mu \in \mathcal{M}^1(X)$. Further, x is the barycenter of μ. Since $x \in \operatorname{ext} X$, we obtain that $\mu = \varepsilon_x$, the Dirac measure centered at the point x. Since

$$\mu_1(1) = \mu_2(1) = \|\mu_1\| = \|\mu_2\|,$$

it follows that $\mu_1 = \mu_2 = \varepsilon_x$, and thus $t^{-1}s_1 = t^{-1}s_2 = x$. Thus $s_1 = s_2 = tx$ and $tx \in \operatorname{ext} K$.

This proves "$\supset$" and finishes the proof. $\square$

Lemma 2.2. *Let $\mu \in \mathcal{M}^1(K)$ be a probability measure on a compact space K and let $\{f_j\}_{j\in J}$ be a bounded downward directed net of functions in $\mathcal{C}(K,\mathbb{R})$ converging to a function f. Then for any $g \in \mathcal{C}(K,\mathbb{C})$ it holds*

$$\lim_{j\in J} \int_K g f_j \, \mathrm{d}\mu = \int_K g f \, \mathrm{d}\mu.$$

Proof. It is known (see [15, Corollary 414B]), that for a bounded downward directed net $\{h_j\}_{j\in J}$ in $\mathcal{C}(K,\mathbb{R})$ pointwise converging to h we have $\mu(h) = \lim_{j\in J} \mu(h_j)$. We decompose $g = \sum_{k=0}^{3} i^k g_k$, where $g_0, \dots, g_3 \in \mathcal{C}(K,\mathbb{R})$ positive, and apply this fact to the bounded downward directed nets $\{g_k f_j\}_{j\in J}$, $k = 0, \dots, 3$, and obtain

$$\mu(gf) = \sum_{k=0}^{3} i^k \mu(g_k f) = \sum_{k=0}^{3} i^k \lim_{j\in J} \mu(g_k f_j) = \lim_{j\in J} \mu(g f_j).$$

$\square$

If K is a compact topological space and T is a topological space, a function $f\colon K \to T$ is said to be *of the first Borel class* if, for each $U \subset T$ open, the set $f^{-1}(U)$ can be written as a countable union of differences of closed sets in K, see [30, Definition 3.2]. In case of $T = \mathbb{R}$, any semicontinuous function is of the first Borel class.

Lemma 2.3. *Let K be a compact topological space. If $f_1, f_2 : K \to \mathbb{C}$ are two functions of the first Borel class, then their product*

$$h : x \in K \mapsto f_1(x) f_2(x) \in \mathbb{C}$$

is of the first Borel class as well.

Proof. Consider the mappings $f : x \in K \mapsto (f_1(x), f_2(x)) \in \mathbb{C}^2$ and $\phi : (y,z) \in \mathbb{C}^2 \mapsto yz \in \mathbb{C}$. Since ϕ is continuous and $h = \phi \circ f$, it is enough to show that the mapping f is of the first Borel class. To this end, let $\mathcal{U}$ be a countable basis of $\mathbb{C}$ and $U_1, U_2 \in \mathcal{U}$. Write

$$f_i^{-1}(U_i) = \bigcup_{n\in\mathbb{N}} (H_i^n \setminus F_i^n), \quad i = 1, 2,$$

where $F_i^n, H_i^n \subseteq K$ are closed sets. Then

$$f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2) = (\bigcup_{n\in\mathbb{N}} (H_1^n \setminus F_1^n)) \cap (\bigcup_{m\in\mathbb{N}} (H_2^m \setminus F_2^m)) =$$
$$= \bigcup_{n,m\in\mathbb{N}} (H_1^n \setminus F_1^n) \cap (H_2^m \setminus F_2^m) = \bigcup_{n,m\in\mathbb{N}} (H_1^n \cup H_2^m) \setminus (F_1^n \cup F_2^m).$$

Further, any open subset V of $\mathbb{C}^2$ may be written as a countable union of sets R_i of the form $R_i = U_i^1 \times U_i^2$, where $U_i^1, U_i^2 \in \mathcal{U}$. Thus we have that $f^{-1}(V)$ can be written as a countable union of differences of closed sets in K, i.e., f is of the first Borel class. $\square$

Lemma 2.4. *Let X be a compact convex set and $f\colon X \to \mathbb{R}$ be an upper semicontinuous affine function. Then the following assertions hold.*

(a) *There exists an element $a^{**} \in (\mathfrak{A}(X,\mathbb{C}))^{**}$ such that $a^{**}(x) = f(x)$, $x \in X$.*

(b) *If* $\{a_j\}_{j\in J}$ *is a bounded downward directed net in* $\mathfrak{A}(X,\mathbb{R})$ *satisfying*

$$f(x)=\lim_{j\in J}a_j(x)=\inf_{j\in J}a_j(x),\quad x\in X,$$

then $a_j\to a^{**}$ *weak**.

(c) *The function* a^{**} *is of the first Borel class on* $\mathbb{T}\cdot X$.

(d) *For each* $\mu\in\mathcal{M}^1(\mathbb{T}\cdot X)$, *it holds* $\mu(a^{**})=a^{**}(r(\mu))$.

(e) *The function* a^{**} *is of the first Borel class on any* $rB_{(\mathfrak{A}(X,\mathbb{C}))^*}$, $r>0$.

(f) *There is only one element* $a^{**}\in(\mathfrak{A}(X,\mathbb{C}))^{**}$ *extending* f.

Proof. (a) Since f is upper semicontinuous and affine, it is bounded (see e.g. [27, Lemma 4.20 and Theorem 4.21]). Hence we may assume that $-M<f<M$ for some constant $M>0$. By [27, Proposition 4.12],

$$f(x)=\inf\left\{a(x);\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\right\},\quad x\in X,$$

in other words, the downward directed net $\{a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\}$ converges pointwise to f. We consider the family

$$\{a\in\mathfrak{A}(X,\mathbb{R});\,f<a\le M\}$$

as a net of elements in $(\mathfrak{A}(X,\mathbb{C}))^{**}$. We claim that this net converges weak* to an element $a^{**}\in(\mathfrak{A}(X,\mathbb{C}))^{**}$.

Indeed, let $s\in\mathfrak{A}(X,\mathbb{C})^*$ be given. We extend s by the Hahn-Banach theorem to an element $\mu\in\mathcal{M}(X,\mathbb{C})$ with $\|\mu\|=\|s\|$ and write $\mu=\sum_{k=0}^3 i^k c_k\mu_k$, where $c_k\ge 0$ and $\mu_k\in\mathcal{M}^1(X)$, $k=0,\dots,3$. Let x_k be the barycenter of μ_k, $k=0,\dots,3$. Then

$$s(a)=\mu(a)=\sum_{k=0}^3 i^k c_k\mu_k(a)=\sum_{k=0}^3 i^k c_k a(x_k),\quad a\in\mathfrak{A}(X,\mathbb{C}).$$

Since the net $\{a(x_k);\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\}$ converges to $f(x_k)$ for each $k=0,\dots,3$, the net $\{s(a);\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\}$ converges.

By setting

$$a^{**}(s)=\lim\left\{s(a);\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\right\},\quad s\in(\mathfrak{A}(X,\mathbb{C}))^*,$$

we obtain a linear functional on $(\mathfrak{A}(X,\mathbb{C}))^*$.

To conclude the proof it is enough to show that it is bounded. Considering $s\in(\mathfrak{A}(X,\mathbb{C}))^*$ as above, let $\mu\in\mathcal{M}(X,\mathbb{C})$ be again a Hahn-Banach extension of s, i.e., $\|\mu\|=\|s\|$. Then we can decompose μ as $\mu=\sum_{k=0}^3 i^k c_k\mu_k$, where $c_k\ge 0$ and $\mu_k\in\mathcal{M}^1(X)$, $k=0,\dots,3$, and, moreover, $c_0+c_1+c_2+c_3\le 2\,\|\mu\|$.

Then the inequalities

$$\begin{aligned}|a^{**}(s)|&=\left|\lim\left\{s(a);\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\right\}\right|\\&=\left|\lim\left\{\sum_{k=0}^3 i^k c_k a(r(\mu_k));\,a\in\mathfrak{A}(X,\mathbb{R}),f<a\le M\right\}\right|\\&\le\sum_{k=0}^3 c_k M\le 2M\,\|\mu\|=2M\,\|s\|\end{aligned}$$

imply the boundedness of a^{**}, i.e., $a^{**}\in(\mathfrak{A}(X,\mathbb{C}))^{**}$.

(b) Let $\{a_j\}_{j\in J}$ be a bounded downward directed net in $\mathfrak{A}(X,\mathbb{R})$ pointwise converging to f on X. Given $s\in(\mathfrak{A}(X,\mathbb{C}))^*$, we extend s to $\mu\in\mathcal{M}(X,\mathbb{C})$ as above and write $\mu = \sum_{k=0}^{3} i^k c_k \mu_k$, where $c_k \geq 0$, $\mu_k \in \mathcal{M}^1(X)$. Then

$$a^{**}(s) = \lim \{s(a);\, a \in \mathfrak{A}(X,\mathbb{R}), f < a \leq M\} = \sum_{k=0}^{3} i^k c_k f(r(\mu_k))$$
$$= \sum_{k=0}^{3} i^k c_k \lim_{j\in J} a_j(r(\mu_k)) = \lim_{j\in J} \left(\sum_{k=0}^{3} i^k c_k r(\mu_k)\right)(a_j) = \lim_{j\in J} s(a_j).$$

(c) We remind that we understand X as a subset of $(\mathfrak{A}(X,\mathbb{C})^*$, and we consider a homeomorphic mapping $\varphi\colon \mathbb{T}\times X \to \mathbb{T}\cdot X$ defined by $\varphi(t,x) = tx$, $(t,x)\in \mathbb{T}\times X$.

Then the function $h\colon \mathbb{T}\times X \to \mathbb{C}$ defined as $h(t,x) = tf(x)$, $(t,x)\in\mathbb{T}\times X$, as a product of a continuous function and a function of the first Borel class, is of the first Borel class as well by Lemma 2.3.

Thus $a^{**} = h\circ\varphi^{-1}$ is of the first Borel class on $\mathbb{T}\cdot X$ also.

(d) Let $\mu \in \mathcal{M}^1(\mathbb{T}\cdot X)$ be given. Then its barycenter $r(\mu)$ belongs to the set $B_{\mathfrak{A}(X,\mathbb{C})^*}$. We denote $\nu = \varphi^{-1}\mu \in \mathcal{M}^1(\mathbb{T}\times X)$ and pick a bounded downward directed net $\{a_j\}_{j\in J}$ in $\mathfrak{A}(X,\mathbb{R})$ converging to f.

Then we have using Lemma 2.2 and (b)

$$\mu(a^{**}) = (\varphi\nu)(a^{**}) = \nu(a^{**}\circ\varphi) = \nu(h) = \lim_{j\in J}\int_{\mathbb{T}\times X} t a_j(x)\,\mathrm{d}\nu(t,x)$$
$$= \lim_{j\in J}\int_{\mathbb{T}\cdot X} a_j\,\mathrm{d}\mu = \lim_{j\in J} a_j(r(\mu)) = a^{**}(r(\mu)).$$

(e) We first show that a^{**} is of the first Borel class on $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$. To this end we recall that

$$\overline{\operatorname{ext}} B_{(\mathfrak{A}(X,\mathbb{C}))^*} \subset \mathbb{T}\cdot X.$$

We know from (d) that the function a^{**} satisfies the barycentric formula for each $\mu\in\mathcal{M}^1(\overline{\operatorname{ext}} B_{(\mathfrak{A}(X,\mathbb{C}))^*})$. By [29, Theorem 3.3], a^{**} is strongly affine on $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$. Since a^{**} is of the first Borel class on $\overline{\operatorname{ext}} B_{(\mathfrak{A}(X,\mathbb{C}))^*}$, [26, Theorem 3.5] implies that a^{**} is of the first Borel class on $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$.

If $r > 0$ is arbitrary, we realize that $rB_{(\mathfrak{A}(X,\mathbb{C}))^*}$ is affinely homeomorphic to $B_{(\mathfrak{A}(X,\mathbb{C}))^*}$ and a^{**} is linear. Hence a^{**} is of the first Borel class on $rB_{(\mathfrak{A}(X,\mathbb{C}))^*}$.

(f) It is enough to show that, given $a^{**}\in(\mathfrak{A}(X,\mathbb{C}))^{**}$, $a^{**}=0$ provided $a^{**}=0$ on X. Let $s\in(\mathfrak{A}(X,\mathbb{C}))^*$ be arbitrary. We extend s to an element $\mu\in\mathcal{M}(X,\mathbb{C})$ and write $\mu = \sum_{k=0}^{3} i^k c_k\mu_k$, where $c_k\geq 0$, $\mu_k\in\mathcal{M}^1(X)$, $k=0,\dots,3$. Then $s = \sum_{k=0}^{3} i^k c_k r(\mu_k)$, and thus

$$a^{**}(s) = \sum_{k=0}^{3} i^k c_k a^{**}(r(\mu_k)) = 0.$$

Hence $a^{**}=0$ as needed. □

Next we need a decomposition lemma which is well known for real spaces $\mathfrak{A}(X,\mathbb{R})$.

Lemma 2.5. *Let X be a compact convex set and $F\subset X$ be a closed split face. Let F' be the complementary face of F. Then $(\mathfrak{A}(X,\mathbb{C}))^* = \operatorname{span} F \oplus_{\ell_1} \operatorname{span} F'$.*

Proof. Let $s\in(\mathfrak{A}(X,\mathbb{C}))^*$ be given. We extend s to $\mu\in\mathcal{M}(X,\mathbb{C})$ which is boundary (see [19, Theorem], [18] and [16, Theorem 1.2]). Then $|\mu|(\chi_F) = |\mu|(\chi_F^*)$ (see [2, Proposition I.4.5 and the subsequent Remark]), and thus μ is carried by the set $\{\chi_F = \chi_F^*\} = F\cup F'$. We write $\mu|_F = \sum_{k=0}^{3} i^k c_k\mu_k$ and $\mu|_{F'} = \sum_{k=0}^{3} i^k d_k\nu_k$, where $c_k, d_k \geq 0$, $\mu_k \in \mathcal{M}^1(F)$ and $\nu_k \in \mathcal{M}^1(F')$, $k = 0, \dots, 3$. Let $x_k = r(\mu_k)$, $y_k = r(\nu_k)$, $k = 0, \dots, 3$. By [2, Corollary II.6.11], $x_k \in F$ and $y_k \in F'$. Thus

$$s_F = \sum_{k=0}^{3} i^k c_k x_k \in \operatorname{span} F, \quad s_{F'} = \sum_{k=0}^{3} i^k d_k y_k \in \operatorname{span} F'$$

and for $a \in \mathfrak{A}(X, \mathbb{C})$ we have

$$s(a) = \mu(a) = \mu|_F(a) + \mu|_{F'}(a) = \sum_{k=0}^{3} i^k c_k a(x_k) + \sum_{k=0}^{3} i^k d_k a(y_k) = s_F(a) + s_{F'}(a).$$

Thus $s = s_F + s_{F'} \in \operatorname{span} F + \operatorname{span} F'$.

Further, $\|s_F\| \leq \|\mu|_F\|$ and $\|s_{F'}\| \leq \|\mu|_{F'}\|$, and thus

$$\|s\| = \|\mu\| = \|\mu|_F\| + \|\mu|_{F'}\| \geq \|s_F\| + \|s_{F'}\|.$$

Hence $\|s\| = \|s_F\| + \|s_{F'}\|$.

Let $s \in \operatorname{span} F \cap \operatorname{span} F'$. Then there exists $c_k \geq 0$, $d_k \geq 0$, $x_k \in F$, $y_k \in F'$, $k = 0, \dots, 3$, such that

$$s = (c_0 x_0 - c_1 x_1) + i(c_2 x_2 - c_3 x_3) = (d_0 y_0 - d_1 y_1) + i(d_2 y_2 - d_3 y_3).$$

If we apply s to an arbitrary $a \in \mathfrak{A}(X, \mathbb{R})$, we obtain

$$(c_0 x_0 - c_1 x_1)(a) = (d_0 y_0 - d_1 y_1)(a), \quad (c_2 x_2 - c_3 x_3)(a) = (d_2 y_2 - d_3 y_3)(a).$$

Thus

$$c_0 x_0 - c_1 x_1 = d_0 y_0 - d_1 y_1, \quad c_2 x_2 - c_3 x_3 = d_2 y_2 - d_3 y_3.$$

An application of s to a constant function 1 yields

$$c = c_0 + d_1 = d_0 + c_1, \quad d = c_2 + d_3 = d_2 + c_3.$$

If $c = 0$, then $c_0 = c_1 = d_0 = d_1 = 0$, and thus $c_0 x_0 - c_1 x_1 = 0$. Otherwise we have the following equality

$$\frac{c_0}{c} x_0 + \frac{d_1}{c} y_1 = \frac{d_0}{c} y_0 + \frac{c_1}{c} x_1.$$

Since X is a direct convex sum of F and F', these convex combinations must be equal, i.e.,

$$c_0 x_0 = c_1 x_1, \quad d_1 y_1 = d_0 y_0.$$

Hence $c_0 x_0 - c_1 x_1 = 0$.

Similarly we handle the second term $c_2 x_2 - c_3 x_3 = d_2 y_2 - d_3 y_3$ and obtain $s = 0$.

Thus $(\mathfrak{A}(X, \mathbb{C}))^* = \operatorname{span} F \oplus_{\ell_1} \operatorname{span} F'$.

□

Lemma 2.6. *Let X be a compact convex set and $f \colon X \to \mathbb{C}$ be an affine function of the first Borel class. Then*

$$\sup_{x \in X} |f(x)| = \sup_{x \in \operatorname{ext} X} |f(x)|.$$

Proof. It is proved in [23, Theorem 2.3] that every complex function of the first Borel class on a compact space has the point of continuity property. For the rest of the proof see [11, Corollary 1.5(b)].

□

Lemma 2.7. *Let x be a weak peak point of a compact convex set X. Then $\{x\}$ is a split face of X.*

Proof. Suppose that x is a weak peak point. First we prove that x is an extreme point. To this end, let $\mu \in \mathcal{M}^1(X)$ be a maximal measure representing x. For the proof that x is extreme it is enough to show that $\mu = \varepsilon_x$, the Dirac measure centered at the point x. We fix an arbitrary closed neighborhood U of x and $\varepsilon > 0$. Then there is a function $a \in B_{\mathfrak{A}(X,\mathbb{C})}$ satisfying

$$a(x) > 1 - \varepsilon \text{ and } |a| < \varepsilon \text{ on } \operatorname{ext} X \setminus U.$$

Since a is continuous and U is closed, it even holds that $|a| \leq \varepsilon$ on the set $\overline{\operatorname{ext} X} \setminus U \subseteq \overline{\operatorname{ext} X \setminus U}$. So, since μ is maximal measure, we have by [2, Proposition I.4.6]

$$1 - \varepsilon < a(x) \leq \int_X |a| d\mu = \int_{\overline{\operatorname{ext} X}} |a| d\mu = \int_U |a| d\mu + \int_{\overline{\operatorname{ext} X} \setminus U} |a| \, d\mu \leq \mu(U) + \varepsilon.$$

In other words, $\mu(U) > 1 - 2\varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily, we have that $\mu(U) = 1$. Hence $\mu(V) = 1$ for each closed neighborhood V of x. From this it easily follows that $\mu = \varepsilon_x$.

For the fact that x is actually a split face it is enough to follow the proof of [8, Proposition 1].

□

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. We write $\langle \cdot, \cdot \rangle$ for the duality mapping. We write $A = \mathfrak{A}(X, \mathbb{C})$ and $B = \mathfrak{A}(Y, \mathbb{C})$.

We assume that there exist $c' \in \mathbb{R}$ such that $1 < c' < 2$ and $\|T\| < 2$ and $\|Ta\| > c' \|a\|$ for all $a \in A \setminus \{0\}$ (otherwise we would find $1 < c' < 2$ such that $\|T\| \cdot \|T^{-1}\| < \frac{2}{c'} < 2$ and consider the mapping $c' \|T^{-1}\| T$; see [8, p. 76]). We fix $c \in \mathbb{R}$ satisfying $1 < c < c'$.

Claim 1.: For any $a^{**} \in A^{**} \setminus \{0\}$ and $b^{**} \in B^{**} \setminus \{0\}$ we have $\|T^{**}a^{**}\| > c \|a^{**}\|$ and $\|(T^{-1})^{**}b^{**}\| > \frac{1}{2} \|b^{**}\|$.

Indeed, for $a^{**} \in A^{**} \setminus \{0\}$ we have

$$\|a^{**}\| = \|(T^{-1})^{**}T^{**}a^{**}\| \leq (c')^{-1} \|T^{**}a^{**}\| < c^{-1} \|T^{**}a^{**}\|.$$

The second inequality is analogous.

For each $x \in \operatorname{ext} X$ we consider the function $f_x = \chi^*_{\{x\}}$. Since $\{x\}$ is a split face, f_x is an upper semicontinuous affine function on X. We extend f_x using Lemma 2.4 to an element $a_x^{**} \in A^{**}$. By Lemma 2.4(e), a_x^{**} is of the first Borel class on any ball in A^*.

Analogously we define for $y \in \operatorname{ext} Y$ the function g_y and the element $b_y^{**} \in B^{**}$.

We define mappings ρ_X and ρ_Y as follows:

$$\rho_X(x) = \left\{ y \in \operatorname{ext} Y;\ \left|\langle x, (T^{-1})^{**} b_y^{**} \rangle\right| > \frac{1}{2} \right\}, \quad x \in \operatorname{ext} X, \quad \text{and} \tag{2.1}$$
$$\rho_Y(y) = \left\{ x \in \operatorname{ext} X;\ |\langle y, T^{**} a_x^{**} \rangle| > c \right\}, \quad y \in \operatorname{ext} Y.$$

Claim 2. ρ_X and ρ_Y are mappings.

Let $x \in \operatorname{ext} X$ be such that there exist distinct points $y_1, y_2 \in \operatorname{ext} Y$ with

$$\left|\langle (T^{-1})^* x, b_{y_i}^{**} \rangle\right| = \left|\langle x, (T^{-1})^{**} b_{y_i}^{**} \rangle\right| > \frac{1}{2}, \quad i = 1, 2.$$

Using Lemma 2.5 we write

$$(T^{-1})^* x = \lambda_1 y_1 + \mu_1 = \lambda_2 y_2 + \mu_2,$$

where $\lambda_1,\lambda_2\in\mathbb{C}$, $\mu_1\in\operatorname{span}\{y_1\}'$ and $\mu_2\in\operatorname{span}\{y_2\}'$. Then

$$\frac{1}{2}<\left|\langle (T^{-1})^*x, b_{y_i}^{**}\rangle\right|=\left|\langle \lambda_i y_i, b_{y_i}^{**}\rangle+\langle \mu_i, b_{y_i}^{**}\rangle\right|=\left|\lambda_i\right|,\quad i=1,2.$$

Since

$$1\ge\left\|(T^{-1})^*x\right\|=|\lambda_i|+\|\mu_i\|>\frac{1}{2}+\|\mu_i\|,\quad i=1,2,$$

we obtain

$$1>\|\mu_1\|+\|\mu_2\|\ge\|\mu_1-\mu_2\|=\|\lambda_1y_1-\lambda_2y_2\|=|\lambda_1|+|\lambda_2|>1,$$

i.e., a contradiction.

Analogously we show that $\rho_Y(y)$ is at most single-valued.

Let $\widehat{X}$ and $\widehat{Y}$ denote the domain of ρ_X and ρ_Y, respectively.

Claim 3.: The mappings $\rho_X\colon\widehat{X}\to\operatorname{ext}Y$ and $\rho_Y\colon\widehat{Y}\to\operatorname{ext}X$ are surjective.

Let $y\in\operatorname{ext}Y$ be given. We assume that $\left|\langle x,(T^{-1})^{**}b_y^{**}\rangle\right|\le\frac{1}{2}$ for each $x\in\operatorname{ext}X$ and seek a contradiction.

We show that the element $(T^{-1})^{**}b_y^{**}\in A^{**}$ is of the first Borel class on B_{A^*}.

Indeed, we know that b_y^{**} is of the first Borel class on any ball in B^*, in particular on $2B_{B^*}$. Since $(T^{-1})^*$ is a weak*-weak* homeomorphism, $(T^{-1})^*(B_{A^*})\subset 2B_{B^*}$ and $(T^{-1})^{**}b_y^{**}=b_y^{**}\circ(T^{-1})^*$, it follows that $(T^{-1})^{**}b_y^{**}$ is of the first Borel class on B_{A^*} as well.

By Lemma 2.6,

$$\begin{aligned}\frac{1}{2}&\le\frac{1}{2}\left\|b_y^{**}\right\|<\left\|(T^{-1})^{**}b_y^{**}\right\|=\sup_{a^*\in\operatorname{ext}B_{A^*}}\left|\langle a^*,(T^{-1})^{**}b_y^{**}\rangle\right|\\&=\sup_{a^*\in\mathbb{T}\cdot\operatorname{ext}X}\left|\langle a^*,(T^{-1})^{**}b_y^{**}\rangle\right|=\sup_{x\in\operatorname{ext}X}\left|\langle x,(T^{-1})^{**}b_y^{**}\rangle\right|\\&\le\frac{1}{2}.\end{aligned}$$

This contradiction implies that ρ_X is surjective.

Analogously we check that ρ_Y is surjective.

Claim 4.: We have $\widehat{X}=\operatorname{ext}X$ and $\widehat{Y}=\operatorname{ext}Y$ and $\rho_Y(\rho_X(x))=x$, $x\in\operatorname{ext}X$, and $\rho_X(\rho_Y(y))=y$, $y\in\operatorname{ext}Y$.

Let $y\in\widehat{Y}$ be given. We want to show that $\rho_X(\rho_Y(y))=y$, i.e., that

$$\left|\langle\rho_Y(y),(T^{-1})^{**}b_y^{**}\rangle\right|>\frac{1}{2}.\tag{2.2}$$

We have

$$\begin{aligned}d=\sup_{x\in\operatorname{ext}X}\left|\langle x,(T^{-1})^{**}b_y^{**}\rangle\right|&=\sup_{s\in\mathbb{T}\cdot\operatorname{ext}X}\left|\langle s,(T^{-1})^{**}b_y^{**}\rangle\right|=\\&=\left\|(T^{-1})^{**}b_y^{**}\right\|>\frac{1}{2}\left\|b_y^{**}\right\|\ge\frac{1}{2}.\end{aligned}$$

Since $c>1$, we have $d>\max\{\frac{d}{c},\frac{1}{2}\}$. Hence there exists $x\in\operatorname{ext}X$ such that

$$\left|\langle x,(T^{-1})^{**}b_y^{**}\rangle\right|>\max\left\{\frac{d}{c},\frac{1}{2}\right\}\ge\frac{1}{2}.$$

Thus $y=\rho_X(x)$.

Assume that (2.2) does not hold. Then $\rho_Y(y)\neq x$. By Claim 3 there exists $\widehat{y}\in\widehat{Y}$ such that $\rho_Y(\widehat{y})=x$. Then $\widehat{y}\in\{y\}'$, and thus $\langle\widehat{y},b_y^{**}\rangle=0$. We write

$$T^*\widehat{y}=\lambda x+\mu,\quad\lambda\in\mathbb{C},\mu\in\operatorname{span}\{x\}'.$$

Then

$$\begin{aligned}0 &= \langle \widehat{y}, b_y^{**}\rangle = \langle \widehat{y}, T^{**}(T^{-1})^{**}b_y^{**}\rangle = \langle T^*\widehat{y}, (T^{-1})^{**}b_y^{**}\rangle\\ &= \langle \lambda x, (T^{-1})^{**}b_y^{**}\rangle + \langle \mu, (T^{-1})^{**}b_y^{**}\rangle.\end{aligned}$$

Since $x = \rho_Y(\widehat{y})$, we have

$$c < |\langle \widehat{y}, T^{**}a_x^{**}\rangle| = |\langle T^*\widehat{y}, a_x^{**}\rangle| = |\langle \lambda x + \mu, a_x^{**}\rangle| = |\lambda|\,.$$

Since

$$\|\mu\| + |\lambda| = \|T^*\widehat{y}\| < 2\,\|\widehat{y}\| = 2,$$

we obtain $\|\mu\| < 2 - c$. By putting everything together we get

$$\begin{aligned}d &< |\lambda|\,\frac{d}{c} < |\lambda|\left|\langle x, (T^{-1})^{**}b_y^{**}\rangle\right| = \left|\langle \lambda x, (T^{-1})^{**}b_y^{**}\rangle\right|\\ &= \left|\langle \mu, (T^{-1})^{**}b_y^{**}\rangle\right| \le d\,\|\mu\| \le d(2-c) < d,\end{aligned}$$

a contradiction. Thus (2.2) holds, which means that $\rho_X(\rho_Y(y)) = y$, $y \in \widehat{Y}$.

Now, let $x \in \operatorname{ext} X$ be given. Then there exists $y \in \widehat{Y}$ such that $\rho_Y(y) = x$. Then $y = \rho_X(\rho_Y(y)) = \rho_X(x)$, which means that $x \in \widehat{X}$.

Let $y \in \operatorname{ext} Y$ be given. Then we can find $x \in \widehat{X} = \operatorname{ext} X$ with $\rho_X(x) = y$ and further we can select $\widehat{y} \in \widehat{Y}$ such that $\rho_Y(\widehat{y}) = x$. Then

$$y = \rho_X(x) = \rho_X(\rho_Y(\widehat{y})) = \widehat{y} \in \widehat{Y}.$$

Hence $\widehat{Y} = \operatorname{ext} Y$.

Finally, if $x \in \operatorname{ext} X$, we find $y \in \operatorname{ext} Y$ with $\rho_Y(y) = x$ and obtain

$$\rho_Y(\rho_X(x)) = \rho_Y(\rho_X(\rho_Y(y))) = \rho_Y(y) = x.$$

Till now we have proved that $\rho_X \colon \operatorname{ext} X \to \operatorname{ext} Y$ is a bijection with ρ_Y being its inverse. Now we use the assumption on weak peak points to check that ρ_X is a homeomorphism. To this end it is enough to follow the proof of [8, Theorem 7], see also the proof of Theorem 4.1. □

3. Cardinality of extreme points

The second result of our paper generalizes a theorem of Cengiz [7] who proved that a pair of locally compact spaces K, L have the same cardinality provided $\mathcal{C}_0(X, \mathbb{F})$ is isomorphic to $\mathcal{C}_0(Y, \mathbb{F})$.

We show in Theorem 3.2 the same result in the framework of compact convex sets. Before its proof we need the following lemma on finite-dimensional compact convex sets.

Lemma 3.1. *Let X be a compact convex set in a finite-dimensional space and let each point of $\operatorname{ext} X$ be a split face. Then the set $\operatorname{ext} X$ is finite and X is a simplex.*

Proof. We identify X with a subset of $\mathbb{R}^m$ for a suitable $m \in \mathbb{N}$.

First we show that the set $\operatorname{ext} X$ is finite. Assuming the contrary, there is a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in $\operatorname{ext} X$ converging to a point $x \in X$. By the Minkowski theorem (see e.g. [2, Corollary I. 6.13] or [27, Theorem 2.11]), x belongs to the convex hull of $\operatorname{ext} X$, thus there exist finite sequences $\{\lambda_i\}_{i=1}^{k}$ in $(0, 1]$ and $\{z_i\}_{i=1}^{k}$ in $\operatorname{ext} X$ such that

$$\sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad x = \sum_{i=1}^{k} \lambda_i z_i.$$

Now, the function $\chi^*_{z_1}$ is affine, and hence continuous. Since $\{z_1\}$ is a closed split face and $\chi^*_{z_1} = 0$ on $\operatorname{ext} X \setminus \{z_1\}$, the sequence of real numbers $\{\chi^*_{z_1}(x_n)\}_{n=n_0}^{\infty}$ is identically zero for some suitable $n_0 \in \mathbb{N}$. So, by the continuity of $\chi^*_{z_1}$ we have $\chi^*_{z_1}(x) = 0$. On the other hand, it holds by the affinity of $\chi^*_{z_1}$ that

$$\chi^*_{z_1}(x) = \chi^*_{z_1}\left(\sum_{i=1}^{k} \lambda_i z_i\right) \geq \lambda_1 \chi^*_{z_1}(z_1) = \lambda_1 > 0,$$

which gives a contradiction. Thus $\operatorname{ext} X$ is a finite set.

Now we show that X is a simplex. We write $\operatorname{ext} X = \{x_i\}_{i=1}^{k}$. We fix an element $x \in X \setminus \operatorname{ext} X$ and assume that there are two convex combinations

$$x = \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \mu_i x_i,$$

where $\lambda_i, \mu_i \in [0, 1)$, $i = 1, \ldots, n$. Fix arbitrary $j \in \{1, \ldots, n\}$. By the assumption, $\{x_j\}$ is a split face. Since $\{x_i;\, i \neq j\}$ is contained in the complementary face $\{x_j\}'$ and

$$x = \lambda_j x_j + (1 - \lambda_j) \sum_{i \neq j} \frac{\lambda_i}{1 - \lambda_j} x_j = \mu_j x_j + (1 - \mu_j) \sum_{i \neq j} \frac{\mu_i}{1 - \mu_j} x_j,$$

from the uniqueness of the decomposition of X to $\{x_j\}$ and $\{x_j\}'$ we obtain $\lambda_j = \mu_j$.

Thus x is a unique convex combination of extreme points of X, from which it follows that X is a simplex. This finishes the proof. □

Theorem 3.2. *Let X, Y be compact convex sets such that $\mathfrak{A}(X, \mathbb{C})$, $\mathfrak{A}(Y, \mathbb{C})$ are isomorphic. If each point of $\operatorname{ext} X$ and $\operatorname{ext} Y$ is a split face, then the cardinality of $\operatorname{ext} X$ is equal to the cardinality of $\operatorname{ext} Y$.*

Proof. First we suppose that the space $\mathfrak{A}(X, \mathbb{C})$ is finite-dimensional. Then also $\mathfrak{A}(Y, \mathbb{C})$ is finite-dimensional, with $\dim(\mathfrak{A}(Y, \mathbb{C})) = \dim(\mathfrak{A}(X, \mathbb{C}))$, and also the sets X and Y are finite-dimensional as well. By Lemma 3.1 we have that X is a Bauer simplex with finitely many extreme points, and so it holds that

$$\mathfrak{A}(X, \mathbb{C}) = \mathcal{C}(\operatorname{ext} X, \mathbb{C}) = \ell^\infty(\operatorname{ext} X, \mathbb{C}),$$

and the same holds for Y. Thus

$$|\operatorname{ext} X| = \dim(\ell^\infty(\operatorname{ext} X, \mathbb{C})) = \dim(\ell^\infty(\operatorname{ext} Y, \mathbb{C})) = |\operatorname{ext} Y|\,.$$

Now suppose that the space $\mathfrak{A}(X, \mathbb{C})$ (and hence also the space $\mathfrak{A}(Y, \mathbb{C})$) is infinite-dimensional. Let $T\colon \mathfrak{A}(X, \mathbb{C}) \to \mathfrak{A}(Y, \mathbb{C})$ be an isomorphism. We will show that $|\operatorname{ext} X| \leq |\operatorname{ext} Y|$.

To this end, let $y \in \operatorname{ext} Y$ be fixed. For each $x \in \operatorname{ext} X$ we consider the upper semicontinuous affine function $f_x = \chi^*_{\{x\}}$ and its extension $a^{**}_x \in (\mathfrak{A}(X, \mathbb{C}))^{**}$, see Lemma 2.4. Let $\lambda_y(x) = \langle T^* y, a^{**}_x \rangle$. We claim that the set

$$X_y = \{x \in \operatorname{ext} X;\, \lambda_y(x) \neq 0\}$$

is at most countable. Indeed, let $s = T^* y$ and $\mu \in \mathcal{M}(X, \mathbb{C})$ be a boundary measure extending s. Let $x \in \operatorname{ext} X$ be arbitrary. Let $\{a_j\}_{j \in J}$ be a bounded downward directed net of functions in $\mathfrak{A}(X, \mathbb{R})$ converging to $f_x = \chi^*_{\{x\}}$. Then we have

$$\begin{aligned}\mu(\{x\}) &= \mu(\chi_{\{x\}}) = \mu(\chi^*_{\{x\}}) = \lim_{j \in J} \mu(a_j) = \lim_{j \in J} \langle s, a_j \rangle \\ &= \lim_{j \in J} \langle T^* y, a_j \rangle = \langle T^* y, a^{**}_x \rangle = \lambda_y(x).\end{aligned}$$

Since $\|\mu\| < \infty$, $\mu(\{x\}) \neq 0$ for at most countably many $x \in \operatorname{ext} X$.

Now we prove that for each $x \in \operatorname{ext} X$ there exists $y \in \operatorname{ext} Y$ such that $x \in X_y$. To this end, we assume the contrary. Let $x \in \operatorname{ext} X$ be such that

$$\langle T^* y, a_x^{**} \rangle = 0, \quad y \in \operatorname{ext} Y.$$

Using the same argument as in the proof of Theorem 1.1, Lemma 2.6 yields

$$\begin{aligned} 0 &= \sup_{y \in \operatorname{ext} Y} |\langle T^* y, a_x^{**} \rangle| = \sup_{y \in \operatorname{ext} Y} |\langle y, T^{**} a_x^{**} \rangle| = \sup_{s \in \mathbb{T} \cdot \operatorname{ext} Y} |\langle s, T^{**} a_x^{**} \rangle| \\ &= \sup_{s \in B_{(\mathfrak{A}(Y,\mathbb{C}))^*}} |\langle s, T^{**} a_x^{**} \rangle| = \|T^{**} a_x^{**}\| \neq 0, \end{aligned}$$

i.e., a contradiction.

Now both the spaces $\mathfrak{A}(X,\mathbb{C})$ and $\mathfrak{A}(Y,\mathbb{C})$ are infinite-dimensional, and thus the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are infinite. Indeed, if $\operatorname{ext} X$ were finite, by the minimum principle we would obtain that the space $\mathfrak{A}(X,\mathbb{C}) \subset \ell^\infty(\operatorname{ext} X, \mathbb{C})$ is finite-dimensional.

Now, since we have $\operatorname{ext} X = \bigcup_{y \in \operatorname{ext} Y} X_y$, we get $|\operatorname{ext} X| \leq |\operatorname{ext} Y|$.

By reversing the role of X and Y we obtain the converse inequality, which concludes the proof. □

4. Continuous images of extreme points

Our next result deals with isomorphisms that are not generally surjective. The starting point is a result of Jarosz [20] who proved that if K, L are locally compact spaces, $A \subset \mathcal{C}_0(K,\mathbb{C})$ is an extremely regular closed subspace and $T\colon A \to \mathcal{C}_0(L,\mathbb{C})$ satisfies $\|T\| \cdot \|T^{-1}\| < 2$, then K is a continuous image of a subset of L. (The assumption of the extreme regularity of A reminds the definition of a weak peak point, see [20]).

Theorem 4.1. *Let X, Y be compact convex sets such that each point of $\operatorname{ext} X$ is a weak peak point. If $T\colon \mathfrak{A}(X,\mathbb{C}) \to \mathfrak{A}(Y,\mathbb{C})$ is an into isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then there exists a set $\widehat{Y} \subset \operatorname{ext} Y$ and a continuous surjective mapping $\varphi\colon \widehat{Y} \to \operatorname{ext} X$.*

Lemma 4.2. *Let A, B be Banach spaces and $T\colon A \to B$ be a bounded operator satisfying for some $c > 0$ estimate $\|Ta\| \geq c\|a\|$, $a \in A$. Then*

$$\|T^{**} a^{**}\| \geq c \|a^{**}\|, \quad a^{**} \in A^{**}.$$

Proof. Since

$$\|T^{**} a^{**}\| = \sup_{b^* \in B_{B^*}} |\langle b^*, T^{**} a^{**} \rangle| = \sup_{b^* \in B_{B^*}} |\langle T^* b^*, a^{**} \rangle|,$$

it is enough to show that $T^*(B_{B^*}) \supset c B_{A^*}$.

Let $a^* \in c B_{A^*}$ be given. Then the functional $c^* \in (\operatorname{Rng} T)^*$ defined as $\langle Ta, c^* \rangle = \langle a, a^* \rangle$, $a \in A$, is well defined and is of norm 1. Indeed, if $\|Ta\| \leq 1$, then $\|a\| \leq \frac{1}{c}$, and thus

$$|\langle Ta, c^* \rangle| = |\langle a, a^* \rangle| \leq \|a^*\| \|a\| \leq c \frac{1}{c} = 1.$$

Let $b^* \in B_{B^*}$ be a Hahn-Banach extension of c^*. Then $T^* b^* = a^*$, and thus $a^* \in T^*(B_{B^*})$. This finishes the proof. □

Proof of Theorem 4.1. We follow the proof of Theorem 1.1. We write $A = \mathfrak{A}(X, \mathbb{C})$ and $B = \mathfrak{A}(Y, \mathbb{C})$. Again we consider $1 < c < c' < 2$ and T such that $\|T\| < 2$ and $\|Ta\| \geq c' \|a\|$, $a \in A$.

By Lemma 4.2 we have

$$\|T^{**}a^{**}\| > c\,\|a^{**}\|, \quad a^{**} \in A^{**} \setminus \{0\}.$$

Let f_x and a_x^{**} be as in the proof of Theorem 1.1. Again we define

$$\rho(y) = \{x \in \operatorname{ext} X;\, |\langle y, T^{**}a_x^{**}\rangle| > c\}, \quad y \in \operatorname{ext} Y.$$

Claim 1. ρ is a mapping. Indeed, let $x_1, x_2 \in \operatorname{ext} X$ be such that $\left|\langle y, T^{**}a_{x_i}^{**}\rangle\right| > c$ for some $y \in \operatorname{ext} Y$. We write

$$T^*y = \lambda_i x_i + \mu_i,$$

where $\lambda_i \in \mathbb{C}$ and $\mu_i \in \operatorname{span}\{x_i\}'$, $i = 1, 2$. Then

$$c < \left|\langle T^*y, a_{x_i}^{**}\rangle\right| = \left|\langle \lambda_i x_i, a_{x_i}^{**}\rangle\right| = |\lambda_i|, \quad i = 1, 2,$$

and

$$2 > \|T^*y\| = |\lambda_i| + \|\mu_i\| > c + \|\mu_i\|, \quad i = 1, 2,$$

yield

$$2(2-c) > \|\mu_1\| + \|\mu_2\| \geq \|\mu_1 - \mu_2\| = \|\lambda_1 x_1 - \lambda_2 x_2\| = |\lambda_1| + |\lambda_2| > 2c.$$

But this contradicts the inequality $c > 1$.

Let $\widehat{Y}$ denote the domain of ρ.

Claim 2. ρ is surjective. Assume that for some $x \in \operatorname{ext} X$ we have $c \geq |\langle y, T^{**}a_x^{**}\rangle|$, $y \in \operatorname{ext} Y$. Then we have as in the proof of Theorem 1.1

$$\begin{aligned} c \geq \sup_{y \in \operatorname{ext} Y} |\langle y, T^{**}a_x^{**}\rangle| &= \sup_{s \in \mathbb{T}\cdot \operatorname{ext} Y} |\langle s, T^{**}a_x^{**}\rangle| \\ &= \sup_{s \in B_{B^*}} |\langle s, T^{**}a_x^{**}\rangle| = \|T^{**}a_x^{**}\| > c\,\|a_x^{**}\| \geq c, \end{aligned}$$

i.e., a contradiction.

Claim 3. $\rho\colon \widehat{Y} \to \operatorname{ext} X$ is continuous. We modify the proof of [8, Theorem 7]. Let $F \subset \operatorname{ext} X$ be a closed set and let $F = \operatorname{ext} X \cap H$ for some closed set $H \subset X$. We want to prove that $\rho^{-1}(F)$ is closed in $\widehat{Y}$.

To this end, we construct for each $x \in \operatorname{ext} X \setminus F$ and $y \in \rho^{-1}(x)$ a function $h_{x,y} \in \mathfrak{A}(X, \mathbb{C})$ as follows. Fix $x \in \operatorname{ext} X \setminus F$ and $y \in \widehat{Y}$ with $\rho(y) = x$. Let V be a closed neighborhood of x with $V \cap H = \emptyset$. We write $T^*y = \lambda x + \mu$, where $\lambda \in \mathbb{C}$ and $\mu \in \operatorname{span}\{x\}'$. Let $\mu = \sum_{j=1}^n r_j x_j$, where $r_j \in \mathbb{C}$ and $x_j \in \{x\}'$. Let $r = \sum_{j=1}^n |r_j|$ and μ_j be a maximal measure representing x_j, $j = 1, \dots, n$. Let $\varepsilon > 0$ satisfy

$$\varepsilon < \min\left\{\frac{|\lambda| - c}{r + |\lambda|}, c - 1\right\}.$$

By [8, Proposition 1], there are closed neighborhoods U_j of x such that $\mu_j(U_j) < \frac{\varepsilon}{2}$, $j = 1, \dots, n$. Let $U = V \cap \bigcap_{j=1}^n U_j$. Since $F \subset \operatorname{ext} X \setminus U$, by the proof of [8, Proposition 1] there exists a function $h_{x,y} \in \mathfrak{A}(X, \mathbb{C})$ such that

$$\|h_{x,y}\| \leq 1, \quad h_{x,y}(x) > 1 - \varepsilon \quad \text{and} \quad |h_{x,y}| \leq \varepsilon \text{ on } F \cup \{x_1, \dots, x_n\}.$$

Now we claim that

$$\rho^{-1}(F) = \bigcap_{x \in \operatorname{ext} X \setminus F}\ \bigcap_{y \in \rho^{-1}(x)} \left\{z \in \widehat{Y};\, |\langle Th_{x,y}, z\rangle| \leq c\right\}. \tag{4.1}$$

Indeed, if $y \in \widehat{Y} \setminus \rho^{-1}(F)$, then we consider the function $h_{x,y}$, where $x = \rho(y) \in \operatorname{ext} X \setminus F$. Then we write as above

$$T^* y = \lambda x + \mu = \lambda x + \sum_{j=1}^{n} r_j x_j.$$

By the choice of the function $h_{x,y}$ we have

$$\begin{aligned} |\langle Th_{x,y}, y\rangle| = |\langle h_{x,y}, T^* y\rangle| &= \left| \langle h_{x,y}, \lambda x + \sum_{j=1}^{n} r_j x_j \rangle \right| \\ &\geq |\lambda| \, |1-\varepsilon| - \sum_{j=1}^{n} |r_j| \, |\langle h_{x,y}, x_j\rangle| \\ &\geq |\lambda| \, |1-\varepsilon| - r\varepsilon > c. \end{aligned}$$

Hence

$$y \notin \bigcap_{x \in \operatorname{ext} X \setminus F} \bigcap_{y \in \rho^{-1}(x)} \left\{ z \in \widehat{Y};\, |\langle Th_{x,y}, z\rangle| \leq c \right\},$$

which shows inclusion "$\supset$" in (4.1).

For the proof of the reverse inclusion we select $z \in \rho^{-1}(F)$ and let $x \in \operatorname{ext} X \setminus F$ and $y \in \rho^{-1}(x)$ be arbitrary. Then $\rho(z) \in F$ and, by the definition of ρ,

$$c < \left| \langle z, T^{**} a^{**}_{\rho(z)} \rangle \right| = \left| \langle T^* z, a^{**}_{\rho(z)} \rangle \right|.$$

Let

$$T^* z = \lambda \rho(z) + \mu,$$

where $\lambda \in \mathbb{C}$ and $\mu \in \operatorname{span}\{\rho(z)\}'$. Then

$$c < \left| \langle \lambda \rho(z) + \mu, a^{**}_{\rho(z)} \rangle \right| = |\lambda|,$$

and thus

$$2 > \|T^* z\| = |\lambda| + \|\mu\| > c + \|\mu\|.$$

From these estimates it follows

$$\begin{aligned} |\langle Th_{x,y}, z\rangle| = |\langle h_{x,y}, T^* z\rangle| &= |\langle h_{x,y}, \lambda \rho(z) + \mu\rangle| \\ &\leq |\lambda| \, \varepsilon + (2-c) \leq 2\varepsilon + (2-c) < c. \end{aligned}$$

Hence

$$z \in \left\{ u \in \widehat{Y};\, |\langle Th_{x,y}, u\rangle| \leq c \right\}$$

and (4.1) is verified.

By (4.1), $\rho^{-1}(F)$ is a closed subset of $\widehat{Y}$, and thus ρ is continuous. This finishes the proof. □

5. Isomorphisms of complex function spaces

This section uses the results of the previous sections to deduce analogous theorems on selfadjoint function spaces. Throughout this section we consider a compact (Hausdorff) space K and a closed subspace $\mathcal{H} \subset \mathcal{C}(K, \mathbb{C})$ which contains constants and separates points of K. By $\mathbf{S}(\mathcal{H})$ we denote the *state space* of $\mathcal{H}$, i.e., the set

$$\mathbf{S}(\mathcal{H}) = \{ s \in \mathcal{H}^*;\, \|s\| = s(1) = 1 \}$$

endowed with the weak* topology. Let $\phi\colon K \to \mathbf{S}(\mathcal{H})$ be the evaluation mapping, then ϕ homeomorphically embeds K into the compact convex set $\mathbf{S}(\mathcal{H})$. The *Choquet boundary* $\operatorname{Ch}_{\mathcal{H}} K$ of $\mathcal{H}$ is defined as

$$\operatorname{Ch}_{\mathcal{H}} K = \{x \in K;\ \phi(x) \in \operatorname{ext} \mathbf{S}(\mathcal{H})\}\,.$$

By [5, Theorem 2.2.8], $\operatorname{ext} \mathbf{S}(\mathcal{H}) = \phi(\operatorname{Ch}_{\mathcal{H}} K)$. Let $\Phi\colon \mathcal{H} \to \mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$ be defined as $\Phi(h)(s) = s(h)$, $s \in \mathbf{S}(\mathcal{H})$, $h \in \mathcal{H}$. Then we have the following identification.

Lemma 5.1. *Let $\mathcal{H}$ be a selfadjoint closed subspace of $\mathcal{C}(K, \mathbb{C})$ for some compact space K such that $\mathcal{H}$ contains constants and separates points of K. Then the mapping Φ is a isometric isomorphism of $\mathcal{H}$ onto $\mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$.*

Proof. Clearly, Φ is linear and of norm 1. Since

$$\|h\| \ge \|\Phi(h)\|_{\mathfrak{A}(\mathbf{S}(\mathcal{H}),\mathbb{C})} = \sup_{s \in \mathbf{S}(\mathcal{H})} |s(h)| \ge \sup_{x \in K} |h(x)| = \|h\|\,,$$

Φ is an isometry. It remains to show that Φ is onto.

To this end, let $f \in \mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$ be given. Any $s \in \mathcal{H}^*$ can be written as $s = \sum_{k=0}^3 i^k a_k s_k$, where $a_k \ge 0$, $s_k \in \mathbf{S}(\mathcal{H})$, $k = 0, \dots, 3$. We define $\widetilde{f} : \mathcal{H}^* \to \mathbb{C}$ as

$$\widetilde{f}(s) = \sum_{k=0}^{3} i^k a_k f(s_k), \quad s = \sum_{k=0}^{3} i^k a_k s_k, a_k \ge 0, s_k \in \mathbf{S}(\mathcal{H}), k = 0, \dots, 3. \tag{5.1}$$

We have to check that this definition is correct, i.e., that

$$\sum_{k=0}^{3} i^k a_k f(s_k) = \sum_{k=0}^{3} i^k b_k f(t_k),$$

whenever $\sum_{k=0}^3 i^k a_k s_k = \sum_{k=0}^3 i^k b_k t_k$, $a_k, b_k \ge 0$, $s_k, t_k \in \mathbf{S}(\mathcal{H})$, $k = 0, \dots, 3$.

So let

$$(a_0 s_0 - a_1 s_1) + i(a_2 s_2 - a_3 s_3) = (b_0 t_0 - b_1 t_1) + i(b_2 t_2 - b_3 t_3). \tag{5.2}$$

Since any $s \in \mathbf{S}(\mathcal{H})$ can be extended by the Hahn-Banach theorem to a measure $\mu \in \mathcal{M}^1(K)$, $s(\operatorname{Re} h) \in \mathbb{R}$ for each $h \in \mathcal{H}$. (We remind that $\operatorname{Re} h, \operatorname{Im} h \in \mathcal{H}$ for each $h \in \mathcal{H}$ since $\mathcal{H}$ is selfadjoint.) An application of (5.2) to the constant function 1 yields

$$a = a_0 + b_1 = b_0 + a_1, \quad b = a_2 + b_3 = b_2 + a_3.$$

If $a = 0$, $a_0 = a_1 = b_0 = b_1 = 0$, and thus $a_0 f(s_0) - a_1 f(s_1) = b_0 f(t_0) - b_1 f(t_1)$. Otherwise we have for each $h \in \mathcal{H}$ equality

$$((a_0 s_0 - a_1 s_1) + i(a_2 s_2 - a_3 s_3))\,(\operatorname{Re} h) = ((b_0 t_0 - b_1 t_1) + i(b_2 t_2 - b_3 t_3))\,(\operatorname{Re} h),$$

which implies

$$(a_0 s_0 - a_1 s_1)(\operatorname{Re} h) = (b_0 t_0 - b_1 t_1)(\operatorname{Re} h), \quad h \in \mathcal{H}.$$

In other words,

$$a\left(\frac{a_0}{a} s_0 + \frac{b_1}{a} t_1\right)(\operatorname{Re} h) = a\left(\frac{b_0}{a} t_0 + \frac{a_1}{a} s_1\right)(\operatorname{Re} h), \quad h \in \mathcal{H}.$$

Since $\operatorname{Im} h \in \mathcal{H}$ and $\operatorname{Re}(\operatorname{Im} h) = \operatorname{Im} h$ for each $h \in \mathcal{H}$,

$$\frac{a_0}{a} s_0 + \frac{b_1}{a} t_1 = \frac{b_0}{a} t_0 + \frac{a_1}{a} s_1.$$

Since f is affine, we obtain

$$\frac{a_0}{a}f(s_0)+\frac{b_1}{a}f(t_1)=f\left(\frac{a_0}{a}s_0+\frac{b_1}{a}t_1\right)=f\left(\frac{b_0}{a}t_0+\frac{a_1}{a}s_1\right)=\frac{b_0}{a}f(t_0)+\frac{a_1}{a}f(s_1),$$

i.e.,

$$a_0f(s_0)-a_1f(s_1)=b_0f(t_0)-b_1f(t_1).$$

Similarly we get

$$a_2f(s_2)-a_3f(s_3)=b_2f(t_2)-b_3f(t_3),$$

which shows that $\widetilde{f}$ is by (5.1) well defined.

It follows from (5.1) that $\widetilde{f}\colon \mathcal{H}^*\to\mathbb{C}$ is linear. Indeed, let $s,t\in\mathcal{H}^*$ be given and let

$$s=\sum_{k=0}^{3}i^k a_k s_k,\quad t=\sum_{k=0}^{3}i^k b_k t_k,$$

where $a_k,b_k\geq 0$, $s_k,t_k\in\mathbf{S}(\mathcal{H})$, $k=0,\dots,3$. We select $u\in\mathbf{S}(\mathcal{H})$ and define

$$u_k=\begin{cases}\frac{a_k}{a_k+b_k}s_k+\frac{b_k}{a_k+b_k}t_k, & a_k+b_k>0,\\ u, & a_k=b_k=0,\end{cases}\quad\text{and}\quad c_k=a_k+b_k,\quad k=0,\dots,3.$$

Then $u_k\in\mathbf{S}(\mathcal{H})$ and

$$s+t=\sum_{k=0}^{3}i^k c_k u_k.$$

Since f is affine on $\mathbf{S}(\mathcal{H})$, we obtain

$$\widetilde{f}(s+t)=\sum_{k=0}^{3}i^k c_k f(u_k)=\sum_{k=0}^{3}i^k\left(a_kf(s_k)+b_kf(t_k)\right)=\widetilde{f}(s)+\widetilde{f}(t).$$

It is even more straightforward to verify that $\widetilde{f}(\lambda s)=\lambda\widetilde{f}(s)$, whenever $s\in\mathcal{H}^*$ and $\lambda\geq 0$, $\lambda=-1$, or $\lambda=i$. Thus $\widetilde{f}$ is linear.

To check that $\widetilde{f}$ is given by an element from $\mathcal{H}$ it is enough to verify its weak* continuity on $\mathcal{H}^*$. Since $\widetilde{f}$ is linear, it is enough to check its weak* continuity on $B_{\mathcal{H}^*}$ (see [13, Corollary 3.94]). We assume that this is not the case and seek a contradiction. So let $\{s_j\}_{j\in J}$ be a net in $B_{\mathcal{H}^*}$ weak* converging to $s\in B_{\mathcal{H}^*}$ such that $\left|\widetilde{f}(s_j)-\widetilde{f}(s)\right|\geq\eta$ for some $\eta>0$. Using the Hahn-Banach theorem and the decomposition of a complex measure we write each s_j as $s_j=\sum_{k=0}^{3}i^k a_k^j s_k^j$, where $a_k^j\geq 0$, $s_k^j\in\mathbf{S}(\mathcal{H})$ and $a_0^1+a_1^j+a_2^j+a_3^j\leq 2$. By compactness argument we may assume that $a_k^j\to a_k$ and $s_k^j\to s_k$ in the weak* topology, $k=0,\dots,3$. Then $s=\sum_{k=0}^{3}i^k a_k s_k$. By the continuity of f on $\mathbf{S}(\mathcal{H})$, $f(s_k^j)\to f(s_k)$ for each $k=0,\dots,3$. But then

$$\eta\leq\lim_{j\in J}\left|\widetilde{f}(s_j)-\widetilde{f}(s)\right|=\lim_{j\in J}\left|\sum_{k=0}^{3}i^k a_k^j f(s_k^j)-\sum_{k=0}^{3}i^k a_k f(s_k)\right|=0$$

gives a contradiction. Hence $\widetilde{f}$ is weak* continuous on $B_{\mathcal{H}^*}$, and thus on $\mathcal{H}^*$.

Thus there exists an element $h\in\mathcal{H}$ such that $\widetilde{f}(s)=s(h)$, $s\in\mathcal{H}^*$. In particular, $\Phi(h)=f$. □

As in the first section we say that $x \in K$ is a *weak peak point* if

$$\text{(5.3)} \qquad \begin{aligned} &\text{given } \varepsilon \in (0,1) \text{ and an open set } U \subset K \text{ containing } x, \text{ there exists } f \in B_{\mathcal{H}} \\ &\text{such that } |f| < \varepsilon \text{ on } \operatorname{Ch}_{\mathcal{H}} K \setminus U \text{ and } f(x) > 1 - \varepsilon. \end{aligned}$$

Lemma 5.2. *Let $x \in K$ be a weak peak point in the sense of* (5.3)*. Then $\phi(x)$ is a weak peak point of $\mathbf{S}(\mathcal{H})$ in the sense of* (1.1)*.*

Proof. Suppose that $x \in K$ is a weak peak point in the sense of (5.3), and that we are given $\varepsilon > 0$ and an open neighborhood V of $\phi(x)$ in $\mathbf{S}(\mathcal{H})$. Then we have that $U = \phi^{-1}(V)$ is an open neighborhood of x. So there exists $f \in B_{\mathcal{H}}$ such that $|f| < \varepsilon$ on the set $\operatorname{Ch}_{\mathcal{H}} K \setminus U$ and $f(x) > 1 - \varepsilon$. We denote $a = \Phi(f) \in B_{\mathfrak{A}(\mathbf{S}(\mathcal{H}),\mathbb{C})}$ and we show that a is witnessing the fact that $\phi(x)$ is a weak peak point of $\mathbf{S}(\mathcal{H})$. Firstly, we have that

$$a(\phi(x)) = \Phi(f)(\phi(x)) = \phi(x)(f) = f(x) > 1 - \varepsilon.$$

Now, suppose that $s \in \operatorname{ext} \mathbf{S}(\mathcal{H}) \setminus V$. There is $y \in \operatorname{Ch}_{\mathcal{H}} K$ such that $s = \phi(y)$. Then $\phi(y) \notin V$, and hence $y \notin U$. Thus

$$|a(s)| = |\Phi(f)(\phi(y))| = |f(y)| < \varepsilon,$$

which concludes the proof. □

Now we can extend the results of the previous sections to the context of function spaces.

Theorem 5.3. *For $i = 1, 2$, let K_i be a compact space and $\mathcal{H}_i$ be a selfadjoint closed subspace of $\mathcal{C}(K_i, \mathbb{C})$ which contains constants and separates points of K_i. Let each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point.*

If there exists an isomorphism $T \colon \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\operatorname{Ch}_{\mathcal{H}_2} K_2$.

Proof. By the identification given by Lemma 5.1, the space $\mathfrak{A}(\mathbf{S}(\mathcal{H}_1), \mathbb{C})$ is isomorphic to $\mathfrak{A}(\mathbf{S}(\mathcal{H}_2), \mathbb{C})$ by an isomorphism T satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Moreover, Lemma 5.2 allows us to use Theorem 1.1 to conclude that $\operatorname{ext} \mathbf{S}(\mathcal{H}_1)$ is homeomorphic to $\operatorname{ext} \mathbf{S}(\mathcal{H}_2)$. Hence the assertion follows. □

The next result is a corollary of Theorem 4.1.

Theorem 5.4. *For $i = 1, 2$, let K_i be a compact space and $\mathcal{H}_i$ be a selfadjoint closed subspace of $\mathcal{C}(K_i, \mathbb{C})$ which contains constants and separates points of K_i. Let each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point.*

If there exists an into isomorphism $T \colon \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ is continuous image of a subset of $\operatorname{Ch}_{\mathcal{H}_2} K_2$.

An application of Theorem 3.2 yields the following result.

Theorem 5.5. *For $i = 1, 2$, let K_i be a compact space and $\mathcal{H}_i$ be a selfadjoint closed subspace of $\mathcal{C}(K_i, \mathbb{C})$ which contains constants and separates points of K_i. Let each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point.*

If there exists an isomorphism $T \colon \mathcal{H}_1 \to \mathcal{H}_2$, then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ has the same cardinality as $\operatorname{Ch}_{\mathcal{H}_2} K_2$.

We refer the reader to [22] and [21] for results on function algebras in the spirit of the above theorems. The case of vector-valued Banach-Stone type theorem is treated e.g. in [4], [14] or [1].

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75, Praha 8, Czech Republic

E-mail address: spurny@karlin.mff.cuni.cz