

SHARP SOBOLEV TYPE EMBEDDINGS ON THE ENTIRE EUCLIDEAN SPACE

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ABSTRACT. A comprehensive approach to Sobolev-type embeddings, involving arbitrary rearrangement-invariant norms on the entire Euclidean space \mathbb{R}^n , is offered. In particular, the optimal target space in any such embedding is exhibited. Crucial in our analysis is a new reduction principle for the relevant embeddings, showing their equivalence to a couple of considerably simpler one-dimensional inequalities. Applications to the classes of the Orlicz-Sobolev and the Lorentz-Sobolev spaces are also presented. These contributions fill in a gap in the existing literature, where sharp results in such a general setting are only available for domains of finite measure.

1. INTRODUCTION

An embedding theorem of Sobolev type amounts to a statement asserting that a certain degree of integrability of the (weak) derivatives of a function entails extra integrability of the function itself. A basic formulation concerns the space of weakly differentiable functions $W^{1,p}(\mathbb{R}^n)$, endowed with the norm $\|u\|_{W^{1,p}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}$. Membership of u in $W^{1,p}(\mathbb{R}^n)$ guarantees that $u \in L^{\frac{np}{n-p}}(\mathbb{R}^n)$ if $1 \leq p < n$, or $u \in L^\infty(\mathbb{R}^n)$, if $p > n$. Since $\frac{np}{n-p} > p$, this is locally a stronger property than the a priori assumption that $u \in L^p(\mathbb{R}^n)$, but it is weaker, and hence does not add any further information, near infinity.

The same phenomenon occurs in a higher-order version of this result for $W^{m,p}(\mathbb{R}^n)$, the Sobolev space of those m -times weakly differentiable functions u such that the norm

$$(1.1) \quad \|u\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{k=0}^m \|\nabla^k u\|_{L^p(\mathbb{R}^n)}$$

is finite. Here, $\nabla^k u$ denotes the vector of all derivatives of u of order k , and, in particular, $\nabla^1 u$ stands for ∇u and $\nabla^0 u$ for u . Indeed, one has that

$$(1.2) \quad W^{m,p}(\mathbb{R}^n) \rightarrow \begin{cases} L^{\frac{np}{n-mp}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) & \text{if } 1 \leq m < n \text{ and } 1 \leq p < \frac{n}{m}, \\ L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) & \text{if either } m \geq n, \text{ or } 1 \leq m < n \text{ and } p > \frac{n}{m}. \end{cases}$$

Embedding (1.2) is still valid if \mathbb{R}^n is replaced with an open subset Ω with finite measure and regular boundary. The intersection with $L^p(\Omega)$ is irrelevant in the target spaces of the resulting embedding, and $L^{\frac{np}{n-mp}}(\Omega)$, or $L^\infty(\Omega)$ are optimal among all Lebesgue target spaces. By contrast, no optimal Lebesgue target space exists in (1.2).

The existence of optimal target spaces in the Sobolev embedding for $W^{m,p}(\mathbb{R}^n)$ can be restored if the class of admissible targets is enlarged, for instance, to all Orlicz spaces. This class allows to describe different degrees of integrability – not necessarily of power type – locally and near infinity. The use of Orlicz spaces naturally emerges in the borderline missing case in (1.2), corresponding to

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the exponents $1 < p = \frac{n}{m}$, and enables one to cover the full range of exponent m and p . The resulting embedding takes the form

$$(1.3) \quad W^{m,p}(\mathbb{R}^n) \rightarrow L^B(\mathbb{R}^n),$$

where $L^B(\mathbb{R}^n)$ is the Orlicz space associated with a Young function B obeying

$$B(t) \approx \begin{cases} t^{\frac{np}{n-mp}} & \text{if } 1 \leq p < \frac{n}{m} \\ e^{t^{\frac{n}{m}}} & \text{if } 1 < p = \frac{n}{m} \\ \infty & \text{otherwise} \end{cases} \quad \text{near infinity,} \quad B(t) \approx t^p \quad \text{near zero.}$$

Here, \approx denotes equivalence in the sense of Young functions – see Section 2.

Actually, embedding (1.3) is equivalent to (1.2) if either $1 \leq p < \frac{n}{m}$, or $m \geq n$, or $1 \leq m < n$ and $p > \frac{n}{m}$. The case when $1 < p = \frac{n}{m}$ is a consequence of a result of [32, 35, 39], which, for $m = 1$, is also contained in [38].

The space $L^B(\mathbb{R}^n)$ is the optimal (i.e. smallest) target in (1.3) among all Orlicz spaces. However, embedding (1.3) can still be improved when either $1 \leq p < \frac{n}{m}$, or $1 < p = \frac{n}{m}$, provided that the class of admissible target spaces is further broadened. For instance, if $1 \leq p < \frac{n}{m}$, then

$$(1.4) \quad W^{m,p}(\mathbb{R}^n) \rightarrow L^{\frac{np}{n-mp},p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

where $L^{\frac{np}{n-mp},p}(\mathbb{R}^n)$ is a Lorentz space. The intersection space in (1.4) is the best possible among all rearrangement-invariant spaces. A parallel result can be shown to hold in the limiting situation corresponding to $1 < p = \frac{n}{m}$, and involves spaces of Lorentz-Zygmund type.

This follows as a special case of the results of the present paper, whose purpose is to address the problem of optimal embeddings, in the whole of \mathbb{R}^n , for Sobolev spaces built upon arbitrary rearrangement-invariant spaces. Precisely, given any rearrangement-invariant space $X(\mathbb{R}^n)$, we find the smallest rearrangement-invariant space $Y(\mathbb{R}^n)$ which renders the embedding

$$(1.5) \quad W^m X(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n)$$

true. Here, $W^m X(\mathbb{R}^n)$ denotes the m -th order Sobolev type space built upon $X(\mathbb{R}^n)$, and equipped with the norm defined as in (1.1), with $L^p(\mathbb{R}^n)$ replaced with $X(\mathbb{R}^n)$. A rearrangement-invariant space is, in a sense, a space of measurable functions endowed with a norm depending only on the measure of the level sets of the functions. A precise definition will be recalled in the next section.

Questions of this kind have been investigated in the literature in the case when \mathbb{R}^n is replaced with a domain of finite measure. The optimal target problem has been solved in full generality for these domains in [18, 24]. Apart from the examples mentioned above, few special instances are known in the entire \mathbb{R}^n . In this connection, let us mention that first-order sharp Orlicz-Sobolev embeddings in \mathbb{R}^n are established in [12], and that the paper [40] contains the borderline case corresponding to the choice $m = 1$ and $p = n$, that is excluded in (1.4).

We also warn that the available results concerning more classical inequalities for compactly supported functions in \mathbb{R}^n , which involve norms depending only on the highest-order derivatives of trial functions, should not be confused with ours. Indeed, a major novelty of our work is to deal with the substantially different situation when full Sobolev norms come into play.

A key result in our approach is what we call a reduction principle, asserting the equivalence between a Sobolev embedding of the form (1.5), and a couple of one-dimensional inequalities involving the representation norms of $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ on $(0, \infty)$. This is the content of Theorem 3.3.

The optimal rearrangement-invariant target space $Y(\mathbb{R}^n)$ for $W^m X(\mathbb{R}^n)$ in (1.5) is exhibited in Theorem 3.1. The conclusion shows that the phenomenon recalled above for the standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ carries over to any space $W^m X(\mathbb{R}^n)$: no higher integrability of a function near infinity follows from membership of its derivatives in $X(\mathbb{R}^n)$, whatever rearrangement-invariant space is $X(\mathbb{R}^n)$. Loosely speaking, the norm in the optimal target space behaves locally like the optimal target norm for embeddings of the space $W^m X(\mathbb{R}^n)$ with \mathbb{R}^n replaced by a bounded subset, and like

the norm of $X(\mathbb{R}^n)$ near infinity. We stress that, since the norm in $X(\mathbb{R}^n)$ is not necessarily of integral type, a precise definition of the norm of the optimal target space is not straightforward, and the proofs of Theorems 3.1 and 3.3 call for new ingredients compared to the finite-measure framework.

Theorems 3.1 and 3.3 can be applied to derive optimal embeddings for customary and unconventional spaces of Sobolev type. In particular, we are able to identify the optimal Orlicz target and the optimal rearrangement-invariant target in Orlicz-Sobolev embeddings (Theorems 3.5 and 3.9), and the optimal rearrangement-invariant target in Lorentz-Sobolev embeddings (Theorem 3.11).

2. BACKGROUND

Let $\mathcal{E} \subset \mathbb{R}^n$ be a Lebesgue measurable set. We denote by $\mathcal{M}(\mathcal{E})$ the set of all Lebesgue measurable functions $u : \mathcal{E} \rightarrow [-\infty, \infty]$. Here, vertical bars $|\cdot|$ stand for Lebesgue measure. We also define $\mathcal{M}_+(\mathcal{E}) = \{u \in \mathcal{M}(\mathcal{E}) : u \geq 0\}$, and $\mathcal{M}_0(\mathcal{E}) = \{u \in \mathcal{M}(\mathcal{E}) : u \text{ is finite a.e. in } \mathcal{E}\}$. The *non-increasing rearrangement* $u^* : [0, \infty) \rightarrow [0, \infty)$ of a function $u \in \mathcal{M}(\mathcal{E})$ is defined as

$$u^*(s) = \inf\{t \geq 0 : |\{x \in \mathcal{E} : |u(x)| > t\}| \leq s\} \quad \text{for } s \in [0, \infty).$$

We also define $u^{**} : (0, \infty) \rightarrow [0, \infty)$ as

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(r) dr \quad \text{for } s \in (0, \infty).$$

The operation of rearrangement is neither linear, nor sublinear. However,

$$(2.1) \quad (u + v)^*(s) \leq u^*(s/2) + v^*(s/2) \quad \text{for } s > 0,$$

for every $u, v \in \mathcal{M}_+(\mathcal{E})$.

Let $L \in (0, \infty]$. We say that a functional $\|\cdot\|_{X(0,L)} : \mathcal{M}_+(0,L) \rightarrow [0, \infty]$ is a *function norm* if, for every f, g and $\{f_k\}$ in $\mathcal{M}_+(0,L)$, and every $\lambda \geq 0$, the following properties hold:

- (P1) $\|f\|_{X(0,L)} = 0$ if and only if $f = 0$ a.e.; $\|\lambda f\|_{X(0,L)} = \lambda \|f\|_{X(0,L)}$;
- $\|f + g\|_{X(0,L)} \leq \|f\|_{X(0,L)} + \|g\|_{X(0,L)}$;
- (P2) $f \leq g$ a.e. implies $\|f\|_{X(0,L)} \leq \|g\|_{X(0,L)}$;
- (P3) $f_k \nearrow f$ a.e. implies $\|f_k\|_{X(0,L)} \nearrow \|f\|_{X(0,L)}$;
- (P4) $\|\chi_{\mathcal{F}}\|_{X(0,L)} < \infty$ for every set $\mathcal{F} \subset (0, L)$ of finite measure;
- (P5) for every set $\mathcal{F} \subset (0, L)$ of finite measure there exists a positive constant $C_{\mathcal{F}}$ such that $\int_{\mathcal{F}} f(s) ds \leq C_{\mathcal{F}} \|f\|_{X(0,L)}$ for every $f \in \mathcal{M}_+(0, L)$.

Here, and in what follows, $\chi_{\mathcal{F}}$ denotes the characteristic function of a set \mathcal{F} .

If, in addition,

$$(P6) \quad \|f\|_{X(0,L)} = \|g\|_{X(0,L)} \text{ whenever } f^* = g^*,$$

we say that $\|\cdot\|_{X(0,L)}$ is a *rearrangement-invariant function norm*.

The fundamental function $\varphi_X : (0, L) \rightarrow [0, \infty)$ of a rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$ is defined as

$$(2.2) \quad \varphi_X(s) = \|\chi_{(0,s)}\|_{X(0,L)} \quad \text{for } s \in (0, L).$$

If $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ are rearrangement-invariant function norms, then the functional $\|\cdot\|_{(X \cap Y)(0,L)}$, defined by

$$\|f\|_{(X \cap Y)(0,L)} = \|f\|_{X(0,L)} + \|f\|_{Y(0,L)},$$

is also a rearrangement-invariant function norm.

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$ is associated another functional on $\mathcal{M}_+(0, L)$, denoted by $\|\cdot\|_{X'(0,L)}$, and defined as

$$\|g\|_{X'(0,L)} = \sup_{\substack{f \in \mathcal{M}_+(0,L) \\ \|f\|_{X(0,L)} \leq 1}} \int_0^L f(s)g(s) ds$$

for $g \in \mathcal{M}_+(0, L)$. It turns out that $\|\cdot\|_{X'(0,L)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,L)}$. Also,

$$\|f\|_{X(0,L)} = \sup_{\substack{g \in \mathcal{M}_+(0,L) \\ \|g\|_{X'(0,L)} \leq 1}} \int_0^L f(s)g(s) ds = \|f\|_{X''(0,L)}$$

for every $f \in \mathcal{M}_+(0, L)$.

The generalized *Hölder inequality*

$$\int_0^L f(s)g(s) ds \leq \|f\|_{X(0,L)} \|g\|_{X'(0,L)}$$

holds for every $f \in \mathcal{M}_+(0, L)$ and $g \in \mathcal{M}_+(0, L)$.

Given any $t > 0$, the *dilation operator* E_λ is defined at $f \in \mathcal{M}(0, L)$ by

$$(E_\lambda f)(s) = \begin{cases} f(s/\lambda) & \text{if } s \in (0, \lambda L) \\ 0 & \text{if } s \in [\lambda L, L]. \end{cases}$$

The inequality

$$\|E_\lambda f\|_{X(0,L)} \leq \max\{1, 1/\lambda\} \|f\|_{X(0,L)}$$

holds for every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$, and for every $f \in \mathcal{M}_+(0, L)$.

A “localized” notion of a rearrangement-invariant function norm will be needed for our purposes. The localized rearrangement-invariant function norm of $\|\cdot\|_{X(0,\infty)}$ is denoted by $\|\cdot\|_{\tilde{X}(0,1)}$ and defined as follows. Given a function $f \in \mathcal{M}_+(0, 1)$, we call \tilde{f} its continuation to $(0, \infty)$ by 0 outside $(0, 1)$, namely

$$\tilde{f}(t) = \begin{cases} f(s) & \text{if } s \in (0, 1), \\ 0 & \text{if } s \in [1, \infty). \end{cases}$$

Then, we define the functional $\|\cdot\|_{\tilde{X}(0,1)}$ by

$$(2.3) \quad \|f\|_{\tilde{X}(0,1)} = \|\tilde{f}\|_{X(0,\infty)}$$

for $f \in \mathcal{M}_+(0, 1)$. One can verify that $\|\cdot\|_{\tilde{X}(0,1)}$ is actually a rearrangement-invariant function norm.

Given a measurable set $\mathcal{E} \subset \mathbb{R}^n$ and a rearrangement-invariant function norm $\|\cdot\|_{X(0,|\mathcal{E}|)}$, the space $X(\mathcal{E})$ is defined as the collection of all functions $u \in \mathcal{M}(\mathcal{E})$ such that the quantity

$$\|u\|_{X(\mathcal{E})} = \|u^*\|_{X(0,|\mathcal{E}|)}$$

is finite. The functional $\|\cdot\|_{X(\mathcal{E})}$ defines a norm on $X(\mathcal{E})$, and the latter is a Banach space endowed with this norm, called a *rearrangement-invariant space*. Moreover, $X(\mathcal{E}) \subset \mathcal{M}_0(\mathcal{E})$ for any rearrangement-invariant space $X(\mathcal{E})$.

The rearrangement-invariant space $X'(\mathcal{E})$ built upon the function norm $\|\cdot\|_{X'(0,|\mathcal{E}|)}$ is called the *associate space* of $X(\mathcal{E})$.

Given any rearrangement-invariant spaces $X(\mathcal{E})$ and $Y(\mathcal{E})$, one has that

$$(2.4) \quad X(\mathcal{E}) \rightarrow Y(\mathcal{E}) \quad \text{if and only if} \quad Y'(\mathcal{E}) \rightarrow X'(\mathcal{E}),$$

with the same embedding norms.

We refer the reader to [4] for a comprehensive treatment of rearrangement-invariant spaces.

In remaining part of this section, we recall the definition of some rearrangement-invariant function norms and spaces that, besides the Lebesgue spaces, will be called into play in our discussion.

Let us begin with the Orlicz spaces, whose definition makes use of the notion of Young function. A

Young function $A : [0, \infty) \rightarrow [0, \infty]$ is a convex (non trivial), left-continuous function vanishing at 0. Any such function takes the form

$$(2.5) \quad A(t) = \int_0^t a(\tau) d\tau \quad \text{for } t \geq 0,$$

for some non-decreasing, left-continuous function $a : [0, \infty) \rightarrow [0, \infty]$ which is neither identically equal to 0, nor to ∞ . The Orlicz space built upon the Young function A is associated with the *Luxemburg function norm* defined as

$$\|f\|_{L^A(0,L)} = \inf \left\{ \lambda > 0 : \int_0^L A\left(\frac{f(s)}{\lambda}\right) ds \leq 1 \right\}$$

for $f \in \mathcal{M}_+(0, L)$. In particular, given a measurable set $\mathcal{E} \subset \mathbb{R}^n$, one has that $L^A(\mathcal{E}) = L^p(\mathcal{E})$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\mathcal{E}) = L^\infty(\mathcal{E})$ if $A(t) = \infty \chi_{(1, \infty)}(t)$.

A Young function A is said to dominate another Young function B near infinity [resp. near zero] [resp. globally] if there exist positive constants $c > 0$ and $t_0 > 0$ such that

$$B(t) \leq A(ct) \quad \text{for } t \in (t_0, \infty) \quad [t \in [0, t_0)] \quad [t \in [0, \infty)].$$

The functions A and B are called equivalent near infinity [near zero] [globally] if they dominate each other near infinity [near zero] [globally]. Equivalence of the Young functions A and B will be denoted by $A \approx B$.

If $|\mathcal{E}| = \infty$, then

$$L^A(\mathcal{E}) \rightarrow L^B(\mathcal{E}) \quad \text{if and only if} \quad A \text{ dominates } B \text{ globally.}$$

If $|\mathcal{E}| < \infty$, then

$$L^A(\mathcal{E}) \rightarrow L^B(\mathcal{E}) \quad \text{if and only if} \quad A \text{ dominates } B \text{ near infinity.}$$

Given $1 \leq p, q \leq \infty$, we define the Lorentz functional $\|\cdot\|_{L^{p,q}(0,L)}$ by

$$\|f\|_{L^{p,q}(0,L)} = \left\| f^*(s) s^{\frac{1}{p} - \frac{1}{q}} \right\|_{L^q(0,L)}$$

for $f \in \mathcal{M}_+(0, L)$. This functional is a rearrangement-invariant function norm provided that $1 \leq q \leq p$, namely when $q \geq 1$ and the weight function $s^{\frac{1}{p} - \frac{1}{q}}$ is non-increasing. In general, it is known to be equivalent, up to multiplicative constants, to a rearrangement-invariant function norm if (and only if) either $p = q = 1$, or $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = \infty$. The rearrangement-invariant space on a measurable set $\mathcal{E} \subset \mathbb{R}^n$, built upon the latter rearrangement-invariant function norm, is the standard Lorentz space $L^{p,q}(\mathcal{E})$. Thus, $L^{p,q}(\mathcal{E})$ consists of all functions $u \in \mathcal{M}(\mathcal{E})$ such that the functional $\|u\|_{L^{p,q}(\mathcal{E})}$, defined as

$$(2.6) \quad \|u\|_{L^{p,q}(\mathcal{E})} = \|u^*\|_{L^{p,q}(0,|\mathcal{E}|)}$$

is finite.

The notion of Lorentz functional can be generalized on replacing the function $s^{\frac{1}{p} - \frac{1}{q}}$ by a more general weight $w(s) \in \mathcal{M}_+(0, L)$ in (2.6). The resulting functional is classically denoted by $\|\cdot\|_{\Lambda^q(w)(0,L)}$, and reads

$$\|f\|_{\Lambda^q(w)(0,L)} = \|f^*(t)w(t)\|_{L^q(0,L)}$$

for $f \in \mathcal{M}_+(0, L)$. A characterization of those exponents q and weights w for which the latter functional is equivalent to a rearrangement-invariant function norm is known. In particular, when $q \in (1, \infty)$, this is the case if and only if there exists a positive constant C such that

$$s^q \int_s^L \frac{w(r)^q}{r^q} dr \leq C \int_0^s w(r)^q dr \quad \text{for } 0 < s < L$$

[33, Theorem 2]. Moreover, $\|\cdot\|_{\Lambda^1(w)(0,L)}$ is equivalent to a rearrangement-invariant function norm if and only if there exists a constant C such that

$$\frac{1}{s} \int_0^s w(t) dt \leq \frac{C}{r} \int_0^r w(t) dt \quad \text{for } 0 < r \leq s < L$$

[10, Theorem 2.3]. Finally, $\|\cdot\|_{\Lambda^\infty(w)(0,L)}$ is equivalent to a rearrangement-invariant function norm if and only if there exists a constant C such that

$$\int_0^s \frac{dr}{w(r)} \leq \frac{Cs}{w(s)} \quad \text{for } 0 < s < L.$$

This follows from a result of [34, Theorem 3.1]. In any of these cases, we shall denote by $\Lambda^q(w)(\mathcal{E})$ the corresponding rearrangement-invariant space on a measurable set $\mathcal{E} \subset \mathbb{R}^n$.

A further extension of the notion of the Lorentz functional $\|\cdot\|_{\Lambda^q(w)(0,L)}$ is obtained when the role of the function norm $\|\cdot\|_{L^q(0,L)}$ is played by a more general Orlicz function norm $\|\cdot\|_{L^A(0,L)}$. The resulting functional will be denoted by $\|\cdot\|_{\Lambda^A(w)(0,L)}$, and defined as

$$\|f\|_{\Lambda^A(w)(0,L)} = \|f^*(s)w(s)\|_{L^A(0,L)}$$

for $f \in \mathcal{M}_+(0,L)$. A complete characterization of those Young functions A and weights w for which the functional $\|\cdot\|_{\Lambda^A(w)(0,L)}$ is equivalent to a rearrangement-invariant function norm seems not to be available in the literature. However, this functional is certainly equivalent to a rearrangement-invariant function norm whenever the weight w is equivalent, up to multiplicative constants, to a non-increasing function. This is the only case that will be needed in our applications. For such a choice of weights w , we shall denote by $\Lambda^A(w)(\mathcal{E})$ the corresponding rearrangement-invariant space on a measurable set $\mathcal{E} \subset \mathbb{R}^n$.

A comprehensive treatment of Lorentz type functionals can be found in [11] and [31].

3. MAIN RESULTS

Our first main result exhibits the optimal rearrangement-invariant space into which any assigned arbitrary-order Sobolev space is embedded. Given an open set Ω in \mathbb{R}^n and any rearrangement-invariant space $X(\Omega)$, the m -th order Sobolev type space $W^m X(\Omega)$ is defined as

$$W^m(X)(\Omega) = \{u \in X(\Omega) : u \text{ is } m\text{-times weakly differentiable, and } |\nabla^k u| \in X(\Omega) \text{ for } k = 1, \dots, m\}.$$

The space $W^m X(\Omega)$ is a Banach space endowed with the norm given by

$$(3.1) \quad \|u\|_{W^m X(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{X(\Omega)}$$

for $u \in \Omega$. In (3.1), and in what follows, the rearrangement-invariant norm of a vector is understood as the norm of its length.

Given a rearrangement-invariant space $X(\mathbb{R}^n)$, we say that $Y(\mathbb{R}^n)$ is the *optimal rearrangement-invariant target space* in the Sobolev embedding

$$(3.2) \quad W^m X(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n)$$

if it is the smallest one that renders (3.2) true, namely, if for every rearrangement-invariant space $U(\mathbb{R}^n)$ such that $W^m X(\mathbb{R}^n) \rightarrow U(\mathbb{R}^n)$, one has that $Y(\mathbb{R}^n) \rightarrow U(\mathbb{R}^n)$. The construction of the optimal space $Y(\mathbb{R}^n)$ in (3.2) requires a few steps. A first ingredient is the function norm $\|\cdot\|_{Z(0,1)}$ obeying

$$(3.3) \quad \|g\|_{Z'(0,1)} = \|s^{\frac{m}{n}} g^{**}(s)\|_{\tilde{X}'(0,1)} \quad \text{for } g \in \mathcal{M}(0,1),$$

where the localized rearrangement-invariant function norm $\|\cdot\|_{\tilde{X}(0,1)}$ is defined as in (2.3), and $\|\cdot\|_{\tilde{X}'(0,1)}$ stands for its associated function norm. The function norm $\|\cdot\|_{Z(0,1)}$ determines the optimal rearrangement-invariant space in the Sobolev embedding for the space $W^m \tilde{X}(\Omega)$ in any regular open

set in \mathbb{R}^n (with $|\Omega| = 1$), see [18, 24].

Next, we extend $\|\cdot\|_{Z(0,1)}$ to a function norm $\|\cdot\|_{X^m(0,\infty)}$ on $(0, \infty)$ by setting

$$(3.4) \quad \|f\|_{X^m(0,\infty)} = \|f^*\|_{Z(0,1)} \quad \text{for } f \in \mathcal{M}(0, \infty).$$

The optimal rearrangement-invariant target space $Y(\mathbb{R}^n)$ in embedding (3.2) is the space $X_{\text{opt}}^m(\mathbb{R}^n)$ defined as

$$(3.5) \quad X_{\text{opt}}^m(\mathbb{R}^n) = (X^m \cap X)(\mathbb{R}^n).$$

One has that

$$\{u \in X^m(\mathbb{R}^n) : |\text{supp } u| < \infty\} \subset X(\mathbb{R}^n).$$

This follows via Proposition 4.1, Section 4. Hence, if $|\text{supp } u| < \infty$, then $u \in X_{\text{opt}}^m(\mathbb{R}^n)$ if and only if $u \in X^m(\mathbb{R}^n)$. In this sense the optimal space $X_{\text{opt}}^m(\mathbb{R}^n)$ is determined by $X^m(\mathbb{R}^n)$ locally, and by $X(\mathbb{R}^n)$ near infinity.

Theorem 3.1 (Optimal Sobolev embedding). *Let $n \geq 2$, $m \in \mathbb{N}$, and let $X(\mathbb{R}^n)$ be a rearrangement-invariant space. Then*

$$(3.6) \quad W^m X(\mathbb{R}^n) \rightarrow X_{\text{opt}}^m(\mathbb{R}^n),$$

where $X_{\text{opt}}^m(\mathbb{R}^n)$ is the space defined by (3.5). Moreover, $X_{\text{opt}}^m(\mathbb{R}^n)$ is the optimal rearrangement-invariant target space in (3.6).

Theorem 3.1 has the following consequence.

Corollary 3.2 (Supercritical Sobolev embedding). *Let $n \geq 2$, $m \in \mathbb{N}$, and let $X(\mathbb{R}^n)$ be a rearrangement-invariant space such that*

$$(3.7) \quad \|\chi_{(0,1)}(s)s^{-1+\frac{m}{n}}\|_{X'(0,\infty)} < \infty.$$

(In particular, this is always the case when $m \geq n$, whatever is $X(\mathbb{R}^n)$.) Then $X_{\text{opt}}^m(\mathbb{R}^n) = (L^\infty \cap X)(\mathbb{R}^n)$, up to equivalent norms. Namely,

$$(3.8) \quad W^m X(\mathbb{R}^n) \rightarrow (L^\infty \cap X)(\mathbb{R}^n),$$

and $(L^\infty \cap X)(\mathbb{R}^n)$ is the optimal rearrangement-invariant target space in (3.8).

Our second main result is a reduction theorem for Sobolev embeddings on the entire \mathbb{R}^n .

Theorem 3.3 (Reduction principle). *Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant spaces. Then*

$$(3.9) \quad W^m X(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n)$$

if and only if there exists a constant C such that

$$(3.10) \quad \left\| \chi_{(0,1)}(s) \int_s^1 f(r)r^{-1+\frac{m}{n}} dr \right\|_{Y(0,\infty)} \leq C \|\chi_{(0,1)} f\|_{X(0,\infty)}$$

and

$$(3.11) \quad \|\chi_{(1,\infty)} f\|_{Y(0,\infty)} \leq C \|f\|_{X(0,\infty)}$$

for every non-increasing function $f : (0, \infty) \rightarrow [0, \infty)$.

Remark 3.4. In fact, inequality (3.10) turns out to hold for every non-increasing nonnegative function f if and only if it holds for every nonnegative function f – see [18, Corollary 9.8].

In the remaining part of this section we present applications of our general results to two important families of Sobolev type spaces: the Orlicz-Sobolev and the Lorentz-Sobolev spaces. In the light of Corollary 3.2, we shall mainly restrict our attention to the case when $m < n$.

Let us begin with embeddings for Orlicz-Sobolev spaces built upon a Young function A . The notation $W^{m,A}(\mathbb{R}^n)$ will also be adopted instead of $W^m L^A(\mathbb{R}^n)$. This kind of spaces is of use in applications to the theory of partial differential equations, whose nonlinearity is not necessarily of power type, which also arise in mathematical models for diverse physical phenomena – see e.g. [1, 3, 5, 6, 7, 8, 9, 22, 25, 26, 28, 36, 37, 41].

We first focus on embeddings of $W^m L^A(\mathbb{R}^n)$ into Orlicz spaces. We show that an optimal target space in this class always exists, and we exhibit it explicitly.

Define the function $H_{\frac{n}{m}} : [0, \infty) \rightarrow [0, \infty)$ as

$$(3.12) \quad H_{\frac{n}{m}}(\tau) = \left(\int_0^\tau \left(\frac{t}{A(t)} \right)^{\frac{m}{n-m}} dt \right)^{\frac{n-m}{n}} \quad \text{for } \tau \geq 0,$$

and the Young function $A_{\frac{n}{m}}$ as

$$(3.13) \quad A_{\frac{n}{m}}(t) = A(H_{\frac{n}{m}}^{-1}(t)) \quad \text{for } t \geq 0,$$

where $H_{\frac{n}{m}}^{-1}$ denotes the (generalized) left-continuous inverse of $H_{\frac{n}{m}}$ (see [13, 15]).

Theorem 3.5 (Optimal Orlicz target in Orlicz-Sobolev embeddings). *Let $n \geq 2$, $m \in \mathbb{N}$, and let A be a Young function. Assume that $m < n$, and let $A_{\frac{n}{m}}$ be the Young function defined by (3.13). Let A_{opt} be a Young function such that*

$$A_{\text{opt}}(t) \approx \begin{cases} A(t) & \text{near } 0 \\ A_{\frac{n}{m}}(t) & \text{near infinity.} \end{cases}$$

Then

$$(3.14) \quad W^{m,A}(\mathbb{R}^n) \rightarrow L^{A_{\text{opt}}}(\mathbb{R}^n),$$

and $L^{A_{\text{opt}}}(\mathbb{R}^n)$ is the optimal Orlicz target space in (3.14).

Remark 3.6. In equations (3.12) and (3.13), we may assume, without loss of generality, that

$$(3.15) \quad \int_0^\infty \left(\frac{t}{A(t)} \right)^{\frac{m}{n-m}} dt < \infty,$$

so that $H_{\frac{n}{m}}$ is well defined. Indeed, since only the behavior of $A_{\frac{n}{m}}$, and hence of $H_{\frac{n}{m}}$, near infinity is relevant in view of our result, A can be replaced in (3.12) and (3.13) (if necessary) by a Young function equivalent near infinity, which renders (3.15) true. Any such a replacement results into an equivalent function $A_{\frac{n}{m}}$ near infinity.

Remark 3.7. Notice that, if

$$(3.16) \quad \int_0^\infty \left(\frac{t}{A(t)} \right)^{\frac{m}{n-m}} dt < \infty,$$

then $H_{\frac{n}{m}}^{-1}(t) = \infty$ for large t , and equation (3.13) has accordingly to be interpreted as $A_{\frac{n}{m}}(t) = \infty$ for large t .

Remark 3.8. Condition (3.16) can be shown to agree with (3.7) when $X(\mathbb{R}^n) = L^A(\mathbb{R}^n)$, and embedding (3.14) recovers the conclusion of Corollary 3.2 in this case. Indeed, by Remark 3.7, under assumption (3.7) one has that

$$A_{\text{opt}}(t) \approx \begin{cases} A(t) & \text{if } t \in [0, 1] \\ \infty & \text{if } t \in (1, \infty). \end{cases}$$

Owing to Lemma 5.2 below, the resulting Orlicz target space $L^{A_{\text{opt}}}(\mathbb{R}^n)$ coincides with $(L^\infty \cap L^A)(\mathbb{R}^n)$. Hence, the target space $L^{A_{\text{opt}}}(\mathbb{R}^n)$ is in fact also optimal among all rearrangement-invariant spaces in (3.14).

We next determine the optimal target space for embeddings of $W^{m,A}(\mathbb{R}^n)$ among all rearrangement-invariant spaces. Such a space turns out to be a Lorentz-Orlicz space defined as follows.

Let A be a Young function. In the light of Remark 3.8, we may restrict our attention to the case when $m < n$, and

$$(3.17) \quad \int_0^\infty \left(\frac{t}{A(t)} \right)^{\frac{m}{n-m}} dt = \infty.$$

Define the Young function \widehat{A} by

$$(3.18) \quad \widehat{A}(t) = \int_0^t \widehat{a}(\tau) d\tau \quad \text{for } t \geq 0,$$

where

$$\widehat{a}^{-1}(s) = \left(\int_{a^{-1}(s)}^\infty \left(\int_0^t \left(\frac{1}{a(\tau)} \right)^{\frac{m}{n-m}} d\tau \right)^{-\frac{n}{m}} \frac{dt}{a(t)^{\frac{n}{n-m}}} \right)^{\frac{m}{m-n}} \quad \text{for } s \geq 0,$$

and a is the function appearing in (2.5). Notice that condition (3.15) is equivalent to requiring that

$$(3.19) \quad \int_0^\infty \left(\frac{1}{a(\tau)} \right)^{\frac{m}{n-m}} d\tau < \infty.$$

Thus, by a reason analogous to that explained in Remark 3.6, there is no loss of generality in assuming (3.19).

Theorem 3.9 (Optimal rearrangement-invariant target in Orlicz-Sobolev embeddings).

Let $n \geq 2$ and $m \in \mathbb{N}$, and let A be a Young function. Assume that $m < n$, and condition (3.17) is fulfilled, and let \widehat{A} be the Young function given by (3.18). Let E be a Young function such that

$$E(t) \approx \begin{cases} A(t) & \text{near } 0 \\ \widehat{A}(t) & \text{near infinity.} \end{cases}$$

and let $v : (0, \infty) \rightarrow (0, \infty)$ be the function defined as

$$(3.20) \quad v(s) = \begin{cases} s^{-\frac{m}{n}} & \text{if } s \in (0, 1) \\ 1 & \text{if } s \in [1, \infty). \end{cases}$$

Then

$$(3.21) \quad W^{m,A}(\mathbb{R}^n) \rightarrow \Lambda^E(v)(\mathbb{R}^n),$$

and the Orlicz-Lorentz space $\Lambda^E(v)(\mathbb{R}^n)$ is the optimal rearrangement-invariant target space in (3.21).

Example 3.10. We focus here on Orlicz-Sobolev spaces built upon a special family of Orlicz spaces, called Zygmund spaces and denoted by $L^p(\log L)^\alpha(\mathbb{R}^n)$, where either $p = 1$ and $\alpha \geq 0$, or $p > 1$ and $\alpha \in \mathbb{R}$. They are associated with a Young function equivalent to $t^p(1 + \log_+ t)^\alpha$, where $\log_+(t) = \max\{\log t, 0\}$. Of course, $L^p(\log L)^0(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Owing to Theorem 3.5 and [15, Example 3.4], one has that

$$W^m L^p(\log L)^\alpha(\mathbb{R}^n) \rightarrow L^G(\mathbb{R}^n),$$

where

$$G(t) \approx \begin{cases} t^{\frac{np}{n-mp}} (\log t)^{\frac{n\alpha}{n-mp}} & \text{if } 1 \leq p < \frac{n}{m} \\ e^{t^{\frac{n}{n-m-\alpha m}}} & \text{if } p = \frac{n}{m}, \alpha < \frac{n}{m} - 1 \\ e^{e^{t^{\frac{n}{n-m}}}} & \text{if } p = \frac{n}{m}, \alpha = \frac{n}{m} - 1 \\ \infty & \text{otherwise} \end{cases} \quad \text{near infinity,} \quad G(t) \approx t^p \quad \text{near zero.}$$

Moreover, $L^G(\mathbb{R}^n)$ is optimal among all Orlicz spaces. Furthermore, by Theorem 3.9 and [15, Example 3.10],

$$W^m L^p(\log L)^\alpha(\mathbb{R}^n) \rightarrow \Lambda^E(v)(\mathbb{R}^n),$$

where v is given by (3.20), and

$$E(t) \approx \begin{cases} t^p (\log t)^\alpha & \text{if } 1 \leq p < \frac{n}{m} \\ t^{\frac{n}{m}} (\log t)^{\alpha - \frac{n}{m}} & \text{if } p = \frac{n}{m}, \alpha < \frac{n}{m} - 1 \\ t^{\frac{n}{m}} (\log t)^{-1} (\log(\log t))^{-\frac{n}{m}} & \text{if } p = \frac{n}{m}, \alpha = \frac{n}{m} - 1 \end{cases} \quad \text{near infinity,} \quad E(t) \approx t^p \quad \text{near zero.}$$

Moreover, the Orlicz-Lorentz space $\Lambda^E(v)$ is optimal among all rearrangement-invariant spaces. The optimal rearrangement-invariant target space when either $p > \frac{n}{m}$, or $p = \frac{n}{m}$ and $\alpha > \frac{n}{m} - 1$ agrees with the Orlicz space $L^G(\mathbb{R}^n)$ with G defined as above, and hence with $(L^\infty \cap L^p(\log L)^\alpha)(\mathbb{R}^n)$.

We conclude this section with an optimal embedding theorem for Lorentz-Sobolev spaces in \mathbb{R}^n , which relies upon Theorem 3.1.

Theorem 3.11 (Lorentz-Sobolev embeddings). (i) *Let $m < n$, and either $p = q = 1$, or $1 < p < \frac{n}{m}$ and $1 \leq q \leq \infty$. Then*

$$W^m L^{p,q}(\mathbb{R}^n) \rightarrow \Lambda^q(w)(\mathbb{R}^n),$$

where

$$w(s) = \begin{cases} s^{\frac{1}{p} - \frac{1}{q} - \frac{m}{n}} & \text{if } s \in (0, 1] \\ s^{\frac{1}{p} - \frac{1}{q}} & \text{if } s \in (1, \infty). \end{cases}$$

(ii) *Let $m < n$ and $1 < q \leq \infty$. Then*

$$W^m L^{\frac{n}{m},q}(\mathbb{R}^n) \rightarrow \Lambda^q(w)(\mathbb{R}^n),$$

where

$$w(s) = \begin{cases} s^{-\frac{1}{q}} (1 + \log \frac{1}{s})^{-1} & \text{if } s \in (0, 1] \\ s^{\frac{m}{n} - \frac{1}{q}} & \text{if } s \in (1, \infty). \end{cases}$$

(iii) *Let either $m > n$, or $m \leq n$ and $p > \frac{n}{m}$, or $m \leq n$, $p = \frac{n}{m}$ and $q = 1$. Then*

$$W^m L^{p,q}(\mathbb{R}^n) \rightarrow \Lambda^E(w)(\mathbb{R}^n),$$

where

$$w(s) = \begin{cases} 1 & \text{if } s \in (0, 1] \\ s^{\frac{1}{p} - \frac{1}{q}} & \text{if } s \in (1, \infty), \end{cases}$$

and

$$E(t) = \begin{cases} t^q & \text{if } t \in (0, 1] \\ \infty & \text{if } t \in (1, \infty). \end{cases}$$

Moreover, in each case, the target space is optimal among all rearrangement-invariant spaces.

4. PROOFS OF THE MAIN RESULTS

Let $\|\cdot\|_{X(0,\infty)}$ be a rearrangement-invariant function norm, and let $\|\cdot\|_{\tilde{X}(0,1)}$ be its localized rearrangement-invariant function norm on $(0,1)$ given by (2.3). The rearrangement-invariant function norm $\|\cdot\|_{Z(0,1)}$ defined as in (3.3) is the optimal one for which the Hardy type inequality

$$(4.1) \quad \left\| \int_s^1 f(r)r^{-1+\frac{m}{n}} dr \right\|_{Z(0,1)} \leq C\|f\|_{\tilde{X}(0,1)}$$

holds with some positive constant C independent of nonnegative functions $f \in \tilde{X}(0,1)$, see [24, Theorem A]. The following proposition tells us that such a norm is always at least as strong as that of $\tilde{X}(0,1)$.

Proposition 4.1. *Let $n \geq 2$ and let $m \in \mathbb{N}$. Assume that $\|\cdot\|_{X(0,\infty)}$ is a rearrangement-invariant function norm, and let $\|\cdot\|_{Z(0,1)}$ be rearrangement-invariant function norm defined as in (3.3). Then*

$$(4.2) \quad Z(0,1) \rightarrow \tilde{X}(0,1).$$

Proof. Embedding (4.2) is equivalent to

$$(4.3) \quad \tilde{X}'(0,1) \rightarrow Z'(0,1).$$

By the very definition of $Z(0,1)$, embedding (4.3) is in turn equivalent to the inequality

$$(4.4) \quad \|s^{\frac{m}{n}} f^{**}(s)\|_{\tilde{X}'(0,1)} \leq C\|f\|_{\tilde{X}'(0,1)}$$

for some constant C and for every $f \in \mathcal{M}_+(0,1)$. It is easily verified that the operator

$$\mathcal{M}_+(0,1) \ni f(s) \mapsto s^{\frac{m}{n}} f^{**}(s) \in \mathcal{M}_+(0,1)$$

is sublinear, and bounded in $L^1(0,1)$ and in $L^\infty(0,1)$. An interpolation theorem by Calderón [4, Theorem 2.12, Chapter 3] then tells us that it is bounded in any rearrangement-invariant space. Hence, inequality (4.4) follows. \square

The next auxiliary result will be critical in the proof of the sharpness of our embeddings. Throughout, we shall use the relation $\gtrsim [\lesssim]$ between two expressions to denote that the former bounds [is bounded by] the latter, up to a positive constant. The notation \simeq is adopted to denote that the relations \gtrsim and \lesssim hold simultaneously. Notice the different meanings of the relation \simeq and the relation \approx of equivalence between Young functions introduced in Section 2.

Proposition 4.2. *Let $\|\cdot\|_{X(0,\infty)}$ and $\|\cdot\|_{U(0,\infty)}$ be rearrangement-invariant function norms. Let $\alpha > 0$ and let $L > 0$. Assume that*

$$(4.5) \quad \left\| \chi_{(L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{U(0,\infty)} \leq C_1 \left(\|f\|_{X(0,\infty)} + \left\| \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} \right)$$

for some constant C_1 , and for every non-increasing function $f \in \mathcal{M}_+(0,\infty)$. Then there exists a constant C_2 such that

$$(4.6) \quad \|g^* \chi_{(2L,\infty)}\|_{U(0,\infty)} \leq C_2 \|g\|_{X(0,\infty)}$$

for every $g \in \mathcal{M}_+(0,\infty)$.

Proof. Let f be as in the statement. Assume, in addition, that f is constant on $(0,2L)$, and denote by $f_0 \in \mathbb{R}$ the constant value of f on $(0,2L)$. If $s \geq 2L$, then

$$\int_s^\infty f(r)r^{\alpha-1} dr \geq \int_s^{2s} f(r)r^{\alpha-1} dr \geq f(2s) \int_s^{2s} r^{\alpha-1} dr = (2^\alpha - 1) \frac{s^\alpha}{\alpha} f(2s) \geq (2^\alpha - 1) \frac{(2L)^\alpha}{\alpha} f(2s).$$

Furthermore, if $s \in (0, L)$, then

$$\int_s^\infty f(r)r^{\alpha-1} dr \geq \int_L^{2L} f(r)r^{\alpha-1} dr = \frac{1}{\alpha} [(2L)^\alpha - L^\alpha] f_0.$$

Thus,

$$\begin{aligned} (4.7) \quad \|f(s)\|_{X(0,\infty)} &\lesssim \|f(2s)\|_{X(0,\infty)} \leq \|f(2s)\chi_{(2L,\infty)}(s)\|_{X(0,\infty)} + \|f(2s)\chi_{(0,2L)}(s)\|_{X(0,\infty)} \\ &\lesssim \left\| \chi_{(2L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} + f_0 \|\chi_{(0,2L)}\|_{X(0,\infty)} \\ &\lesssim \left\| \chi_{(2L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} + f_0 \|\chi_{(0,L)}\|_{X(0,\infty)} \\ &\lesssim \left\| \chi_{(2L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} + \left\| \chi_{(0,L)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} \\ &\lesssim \left\| \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)}. \end{aligned}$$

Inequalities (4.5) and (4.7) imply that

$$(4.8) \quad \left\| \chi_{(L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{U(0,\infty)} \lesssim \left\| \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)}.$$

Consider functions f of the form

$$f = \sum_{i=1}^k a_i \chi_{(0, b_i)} \quad \text{with } a_i \geq 0 \quad \text{and} \quad b_i \geq 2L.$$

For this choice of f , one has that

$$\int_s^\infty f(r)r^{\alpha-1} dr = \sum_{i=1}^k \chi_{(0, b_i)}(s) \frac{a_i}{\alpha} (b_i^\alpha - s^\alpha) \quad \text{for } s > 0,$$

whence

$$\left\| \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{X(0,\infty)} \lesssim \left\| \sum_{i=1}^k \chi_{(0, b_i)}(s) a_i b_i^\alpha \right\|_{X(0,\infty)}.$$

On the other hand,

$$\begin{aligned} (4.9) \quad &\left\| \chi_{(L,\infty)}(s) \int_s^\infty f(r)r^{\alpha-1} dr \right\|_{U(0,\infty)} \gtrsim \left\| \chi_{(L,\infty)}(s) \sum_{i=1}^k \chi_{(0, \frac{b_i}{2})}(s) a_i (b_i^\alpha - s^\alpha) \right\|_{U(0,\infty)} \\ &\gtrsim \left\| \chi_{(L,\infty)}(s) \sum_{i=1}^k \chi_{(0, \frac{b_i}{2})}(s) a_i (b_i^\alpha - (b_i/2)^\alpha) \right\|_{U(0,\infty)} \gtrsim \left\| \chi_{(L,\infty)}(s) \sum_{i=1}^k \chi_{(0, \frac{b_i}{2})}(s) a_i b_i^\alpha \right\|_{U(0,\infty)} \\ &\simeq \left\| \chi_{(L,\infty)}(s/2) \sum_{i=1}^k \chi_{(0, \frac{b_i}{2})}(s/2) a_i b_i^\alpha \right\|_{U(0,\infty)} \simeq \left\| \chi_{(2L,\infty)}(s) \sum_{i=1}^k \chi_{(0, b_i)}(s) a_i b_i^\alpha \right\|_{U(0,\infty)}. \end{aligned}$$

Inequalities (4.8) and (4.9) imply, via an approximation argument, that

$$(4.10) \quad \|\chi_{(2L,\infty)} g\|_{U(0,\infty)} \lesssim \|g\|_{X(0,\infty)}$$

for every non-increasing function $g \in \mathcal{M}_+(0, \infty)$ that is constant on $(0, 2L)$. Observe that property (P3) of rearrangement-invariant function norms plays a role in the approximation in question.

Assume now that g is any function in $\mathcal{M}_+(0, \infty)$. An application of (4.10) with g replaced by $g^*(2L)\chi_{(0,2L)} + g^*\chi_{(2L,\infty)}$ yields

$$(4.11) \quad \begin{aligned} \|\chi_{(2L,\infty)} g^*\|_{U(0,\infty)} &= \|\chi_{(2L,\infty)} g^*(2L)\chi_{(0,2L)} + g^*\chi_{(2L,\infty)}\|_{U(0,\infty)} \\ &\lesssim \|g^*(2L)\chi_{(0,2L)} + g^*\chi_{(2L,\infty)}\|_{X(0,\infty)} \leq \|g^*\|_{X(0,\infty)} = \|g\|_{X(0,\infty)}. \end{aligned}$$

Hence, inequality (4.6) follows. \square

A key step towards the general embedding theorem of Theorem 3.1 is a first-order embedding, for somewhat more general non-homogeneous Sobolev type spaces defined as follows. Given two rearrangement-invariant function norms $\|\cdot\|_{X(0,\infty)}$ and $\|\cdot\|_{Y(0,\infty)}$, we define the space $W^1(X, Y)(\mathbb{R}^n)$ as

$$W^1(X, Y)(\mathbb{R}^n) = \{u \in X(\mathbb{R}^n) : u \text{ is weakly differentiable in } \mathbb{R}^n, \text{ and } |\nabla u| \in Y(\mathbb{R}^n)\}$$

endowed with the norm

$$\|u\|_{W^1(X, Y)(\mathbb{R}^n)} = \|u\|_{X(\mathbb{R}^n)} + \|\nabla u\|_{Y(\mathbb{R}^n)}.$$

Theorem 4.3 (Non-homogeneous first-order Sobolev embeddings). *Let $\|\cdot\|_{X(0,\infty)}$ and $\|\cdot\|_{Y(0,\infty)}$ be rearrangement-invariant function norms. Let $\|\cdot\|_{Y^1(0,\infty)}$ be defined as in (3.4), with $m = 1$ and $X(\mathbb{R}^n)$ replaced by $Y(\mathbb{R}^n)$. Then*

$$(4.12) \quad W^1(X, Y)(\mathbb{R}^n) \rightarrow (Y^1 \cap X)(\mathbb{R}^n).$$

Proof. We begin by showing that there exists a constant C such that

$$(4.13) \quad \|u\|_{Y^1(\mathbb{R}^n)} \leq C \|\nabla u\|_{Y(\mathbb{R}^n)}$$

for every function u such that $|\text{supp } u| \leq 1$ and $|\nabla u| \in Y(\mathbb{R}^n)$. Indeed, a general form of the Pólya-Szegő principle on the decrease of gradient norms under spherically symmetric symmetrization ensures that, for any such function u , the decreasing rearrangement u^* is locally absolutely continuous in $(0, \infty)$, and

$$(4.14) \quad \|\nabla u\|_{Y(\mathbb{R}^n)} \geq n\omega_n^{\frac{1}{n}} \left\| s^{\frac{1}{n'}} u^{*'}(s) \right\|_{Y(0,\infty)},$$

where ω_n denotes the Lebesgue measure of the unit ball in \mathbb{R}^n – see e.g. [16, Lemma 4.1]. Moreover, there exists a constant C such that

$$(4.15) \quad \left\| s^{\frac{1}{n'}} u^{*'}(s) \right\|_{Y(0,\infty)} = \left\| s^{\frac{1}{n'}} u^{*'}(s) \right\|_{\tilde{Y}(0,1)} \geq C \left\| \int_s^1 u^{*'}(r) dr \right\|_{Z(0,1)} = C \|u^*(s)\|_{Z(0,1)} = \|u\|_{Y^1(\mathbb{R}^n)},$$

where the first equality holds since u^* vanishes in $(1, \infty)$, the inequality is a consequence of the Hardy type inequality (4.1), the second equality holds since $u^*(1) = 0$, and the last inequality by the very definition of the norm in $Y^1(\mathbb{R}^n)$. Inequality (4.13) is a consequence of (4.14) and (4.15).

Now, let $u \in W^1(X, Y)(\mathbb{R}^n)$ such that

$$(4.16) \quad \|u\|_{W^1(X, Y)(\mathbb{R}^n)} \leq 1.$$

Then,

$$(4.17) \quad 1 \geq \|u\|_{X(\mathbb{R}^n)} \geq t \|\chi_{\{|u|>t\}}\|_{X(\mathbb{R}^n)} = t \varphi_X(\{ |u| > t \}) \quad \text{for } t > 0,$$

where φ_X is the fundamental function of $\|\cdot\|_{X(0,\infty)}$ defined as in (2.2). Let φ_X^{-1} denote its generalized left-continuous inverse. One has that

$$\lim_{t \rightarrow 0^+} \varphi_X^{-1}(t) = 0,$$

and

$$\varphi_X^{-1}(\varphi_X(t)) \geq t \quad \text{for } t > 0.$$

Therefore, by (4.17),

$$|\{|u| > t\}| \leq \varphi_X^{-1}(\varphi_X(|\{|u| > t\}|)) \leq \varphi_X^{-1}(1/t) \quad \text{for } t > 0.$$

Hence,

$$|\{|u| > t\}| \leq \varphi_X^{-1}(1/t) \quad \text{for } t > 0.$$

Choose t_0 such that $\varphi_X^{-1}(1/t_0) \leq 1$, whence

$$|\{|u| > t_0\}| \leq 1.$$

Let us decompose the function u as $u = u_1 + u_2$, where $u_1 = \text{sign}(u) \min\{|u|, t_0\}$ and $u_2 = u - u_1$. By standard properties of truncations of Sobolev functions, we have that $u_1, u_2 \in W^1(X, Y)$. Moreover, $\|u_1\|_{L^\infty(\mathbb{R}^n)} \leq t_0$, and $|\{|u_2| > 0\}| = |\{|u| > t_0\}| \leq 1$. We claim that there exist positive constants c_1 and c_2 such that

$$(4.18) \quad \|u_1\|_{Y^1(\mathbb{R}^n)} \leq c_1,$$

and

$$(4.19) \quad \|u_2\|_{Y^1(\mathbb{R}^n)} \leq c_2.$$

Inequality (4.18) is a consequence of the fact that

$$\|u_1\|_{Y^1(\mathbb{R}^n)} \leq t_0 \|1\|_{Y^1(\mathbb{R}^n)} < \infty,$$

where the last inequality holds thanks to the definition of the norm $\|\cdot\|_{Y^1(\mathbb{R}^n)}$. Inequality (4.19) follows from inequalities (4.13) and (4.16). From (4.18), (4.19) and (4.16) we infer that

$$\|u\|_{X(\mathbb{R}^n)} + \|u\|_{Y^1(\mathbb{R}^n)} \leq \|u\|_{X(\mathbb{R}^n)} + \|u_1\|_{Y^1(\mathbb{R}^n)} + \|u_2\|_{Y^1(\mathbb{R}^n)} \leq \|u\|_{X(\mathbb{R}^n)} + c_1 + c_2 \leq 1 + c_1 + c_2$$

for every function u fulfilling (4.16). This establishes embedding (4.12). \square

We are now in a position to accomplish the proof of Theorem 3.1.

Proof of Theorem 3.1. We shall prove by induction on m that

$$(4.20) \quad \|u\|_{X^m(\mathbb{R}^n)} \leq C \|u\|_{W^m X(\mathbb{R}^n)}$$

for some constant C , and for every $u \in W^m X(\mathbb{R}^n)$. This inequality, combined with the trivial embedding $W^m X(\mathbb{R}^n) \rightarrow X(\mathbb{R}^n)$, yields embedding (3.6).

If $m = 1$, then inequality (4.20) is a straightforward consequence of Theorem 4.3 applied in the special case when $Y(\mathbb{R}^n) = X(\mathbb{R}^n)$. Now, assume that (4.20) is fulfilled for some $m \in \mathbb{N}$. Let $u \in W^{m+1} X(\mathbb{R}^n)$. By induction assumption applied to the function u_{x_i} for $i = 1, \dots, n$,

$$\|u_{x_i}\|_{X^m(\mathbb{R}^n)} \leq C \|u_{x_i}\|_{W^m X(\mathbb{R}^n)} \leq C' \|u\|_{W^{m+1} X(\mathbb{R}^n)}$$

for some constants C and C' . Consequently,

$$(4.21) \quad \|\nabla u\|_{X^m(\mathbb{R}^n)} \leq C \|u\|_{W^{m+1} X(\mathbb{R}^n)}$$

for some constant C . From Theorem 4.3 with $Y(\mathbb{R}^n) = X^m(\mathbb{R}^n)$, and inequality (4.21), we obtain

$$(4.22) \quad \|u\|_{(X^m)^1(\mathbb{R}^n)} \leq C (\|u\|_{X(\mathbb{R}^n)} + \|\nabla u\|_{X^m(\mathbb{R}^n)}) \leq C' \|u\|_{W^{m+1} X(\mathbb{R}^n)}$$

for some constants C and C' . From [18, Corollary 9.6] one can deduce that

$$(4.23) \quad \|u\|_{(X^m)^1(\mathbb{R}^n)} \simeq \|u\|_{X^{m+1}(\mathbb{R}^n)}$$

for every $u \in W^{m+1}X(\mathbb{R}^n)$. Inequality (4.20), with m replaced by $m + 1$, follows from (4.22) and (4.23).

It remains to prove the optimality of the space $(X^m \cap X)(\mathbb{R}^n)$. Assume that $U(\mathbb{R}^n)$ is another rearrangement-invariant space such that

$$W^m X(\mathbb{R}^n) \rightarrow U(\mathbb{R}^n).$$

Then there exists a constant C such that

$$(4.24) \quad \|u\|_{U(\mathbb{R}^n)} \leq C \|u\|_{W^m X(\mathbb{R}^n)}$$

for every $u \in W^m X(\mathbb{R}^n)$. We have to show that

$$(X \cap X^m)(\mathbb{R}^n) \rightarrow U(\mathbb{R}^n),$$

or, equivalently, that

$$(4.25) \quad \|f\|_{U(0,\infty)} \leq C (\|f\|_{X(0,\infty)} + \|f\|_{X^m(0,\infty)})$$

for some constant C , and every $f \in \mathcal{M}_+(0, \infty)$. Inequality (4.25) will follow if we show that

$$(4.26) \quad \|f^* \chi_{(0,1)}\|_{U(0,\infty)} \leq C \|f\|_{X^m(0,\infty)}$$

and

$$(4.27) \quad \|f^* \chi_{(1,\infty)}\|_{U(0,\infty)} \leq C \|f\|_{X(0,\infty)}$$

for some constant C , and for every $f \in \mathcal{M}_+(0, \infty)$.

Let \mathcal{B} be the ball in \mathbb{R}^n , centered at 0, such that $|\mathcal{B}| = 1$. We claim that inequality (4.24) implies that

$$(4.28) \quad \|v\|_{U(\mathcal{B})} \leq C \|v\|_{W^m X(\mathcal{B})}$$

for some constant C , and for every $v \in W^m X(\mathcal{B})$. Here, $X(\mathcal{B})$ and $U(\mathcal{B})$ denote the rearrangement-invariant spaces built upon the function norms $\|\cdot\|_{\tilde{X}(0,1)}$ and $\|\cdot\|_{\tilde{U}(0,1)}$ defined as in (2.3). To verify this claim, one can make use of the fact that there exists a bounded extension operator $T : W^m X(\mathcal{B}) \rightarrow W^m X(\mathbb{R}^n)$ [19, Theorem 4.1]. Thus,

$$(4.29) \quad \|Tv\|_{W^m X(\mathbb{R}^n)} \leq C \|v\|_{W^m X(\mathcal{B})}$$

for some constant C and for every $v \in W^m X(\mathcal{B})$. Coupling (4.24) with (4.29) we deduce that

$$\|v\|_{U(\mathcal{B})} = \|Tv\|_{U(\mathcal{B})} \leq \|Tv\|_{U(\mathbb{R}^n)} \leq C \|Tv\|_{W^m X(\mathbb{R}^n)} \leq C' \|v\|_{W^m X(\mathcal{B})}$$

for some constants C and C' , and for every $v \in W^m X(\mathcal{B})$. Hence, inequality (4.28) follows.

Inequality (4.28) in turn implies that

$$(4.30) \quad \left\| \int_s^1 g(r) r^{-1+\frac{m}{n}} dr \right\|_{\tilde{U}(0,1)} \leq C \|g\|_{\tilde{X}(0,1)}$$

for some positive constant C , and every $g \in \mathcal{M}_+(0, 1)$. This implication can be found in the proof of [24, Theorem A]. For completeness, we provide a proof hereafter, that also fixes some details in that of [24].

Let us preliminarily note that we can restrict our attention to the case when $m < n$. Indeed, if $m \geq n$, inequality (4.30) holds with $\|\cdot\|_{\tilde{U}(0,1)} = \|\cdot\|_{L^\infty(0,1)}$, and hence for every rearrangement-invariant norm $\|\cdot\|_{U(0,\infty)}$.

Given any bounded function $f \in \mathcal{M}_+(0, 1)$, define

$$(4.31) \quad u(x) = \int_{\omega_n |x|^n}^1 \int_{s_1}^1 \int_{s_2}^1 \cdots \int_{s_{m-1}}^1 f(s_m) s_m^{-m+\frac{m}{n}} ds_m \cdots ds_1 \quad \text{for } x \in \mathcal{B}.$$

Set $M = \omega_n^{-\frac{1}{n}}$. We need to derive a pointwise estimate for $|D^m u|$. As a preliminary step, consider any function $v : \mathcal{B} \rightarrow [0, \infty)$ given by

$$v(x) = g(|x|) \quad \text{for } x \in \mathcal{B},$$

where $g : (0, M) \rightarrow [0, \infty)$ is an m -times weakly differentiable function. One can show that every ℓ -th order derivative of v , with $1 \leq \ell \leq m$, is a linear combination of terms of the form

$$\frac{x_{\alpha_1} \dots x_{\alpha_i} g^{(j)}(|x|)}{|x|^k} \quad \text{for a.e. } x \in \mathcal{B},$$

where $\alpha_1, \dots, \alpha_i \in \{1, \dots, n\}$, and

$$1 \leq j \leq \ell, \quad 0 \leq i \leq \ell, \quad k - i = \ell - j.$$

Here, $g^{(j)}$ denotes the j -th order derivative of g . As a consequence,

$$(4.32) \quad \sum_{\ell=1}^m |\nabla^\ell v(x)| \leq C \sum_{\ell=1}^m \sum_{k=1}^{\ell} \frac{|g^{(k)}(|x|)|}{|x|^{\ell-k}} \quad \text{for a.e. } x \in \mathcal{B}.$$

Next, consider functions g defined by

$$g(s) = \int_{\omega_n s^n}^1 \int_{s_1}^1 \dots \int_{s_{m-1}}^1 f(s_m) s_m^{-m+\frac{m}{n}} ds_m \dots ds_1 \quad \text{for } s \in (0, M),$$

where f is as in (4.31). It can be verified that, for each $1 \leq k \leq m-1$, the function $g^{(k)}(s)$ is a linear combination of functions of the form

$$s^{jn-k} \int_{\omega_n s^n}^1 \int_{s_{j+1}}^1 \dots \int_{s_{m-1}}^1 f(s_m) s_m^{-m+\frac{m}{n}} ds_m \dots ds_{j+1} \quad \text{for } s \in (0, M),$$

where $j \in \{1, 2, \dots, k\}$, whereas $g^{(m)}(s)$ is a linear combination of functions of the form

$$(4.33) \quad s^{jn-m} \int_{\omega_n s^n}^1 \int_{s_{j+1}}^1 \dots \int_{s_{m-1}}^1 f(s_m) s_m^{-m+\frac{m}{n}} ds_m \dots ds_{j+1} \quad \text{for } s \in (0, M),$$

where $j \in \{1, 2, \dots, m-1\}$, and of the function $f(\omega_n s^n)$. Note that, if $j = m-1$, then the expression in (4.33) has to be understood as

$$s^{(m-1)n-m} \int_{\omega_n s^n}^1 f(s_m) s_m^{-m+\frac{m}{n}} ds_m \quad \text{for } s \in (0, M).$$

As a consequence of these formulas, we can infer that, if $1 \leq k \leq m-1$, then

$$(4.34) \quad |g^{(k)}(s)| \lesssim \sum_{j=1}^k s^{jn-k} \int_{\omega_n s^n}^1 f(r) r^{-j+\frac{m}{n}-1} dr \quad \text{for } s \in (0, M),$$

and

$$(4.35) \quad |g^{(m)}(s)| \lesssim \sum_{j=1}^{m-1} s^{jn-m} \int_{\omega_n s^n}^1 f(r) r^{-j+\frac{m}{n}-1} dr + f(\omega_n s^n) \quad \text{for a.e. } s \in (0, M).$$

From equations (4.32), (4.34) and (4.35) one can deduce that

$$(4.36) \quad |D^m u(x)| \lesssim f(\omega_n |x|^n) + \int_{\omega_n |x|^n}^1 f(s) s^{-1+\frac{m}{n}} ds + \sum_{j=1}^{m-1} |x|^{jn-m} \int_{\omega_n |x|^n}^1 f(s) s^{-j+\frac{m}{n}-1} ds$$

for a.e. $x \in \mathcal{B}$. On the other hand, by Fubini's theorem,

$$(4.37) \quad u(x) = \int_{\omega_n |x|^n}^1 f(s) s^{-m+\frac{m}{n}} \frac{(s - \omega_n |x|^n)^{m-1}}{(m-1)!} ds \gtrsim \chi_{(0,1)}(2\omega_n |x|^n) \int_{2\omega_n |x|^n}^1 f(s) s^{\frac{m}{n}-1} ds$$

for $x \in \mathcal{B}$. The following chain holds:

$$\begin{aligned}
(4.38) \quad & \left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{\tilde{U}(0,1)} \lesssim \left\| \chi_{(0,\frac{1}{2})}(t) \int_{2s}^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{\tilde{U}(0,1)} \lesssim \|u\|_{U(\mathcal{B})} \lesssim \|D^m u\|_{X(\mathcal{B})} \\
& \lesssim \|f\|_{\tilde{X}(0,1)} + \left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{\tilde{X}(0,1)} + \sum_{j=1}^{m-1} \left\| s^{j-\frac{m}{n}} \int_s^1 f(r) r^{-j+\frac{m}{n}-1} dr \right\|_{\tilde{X}(0,1)} \\
& \lesssim \|f\|_{\tilde{X}(0,1)}.
\end{aligned}$$

Here, we have made use of (4.37), (4.24), (4.36), and of the boundedness of the operators

$$f(s) \mapsto s^{j-\frac{m}{n}} \int_s^1 f(r) r^{-j+\frac{m}{n}-1} dr \quad \text{and} \quad f(s) \mapsto \int_s^1 f(r) r^{\frac{m}{n}-1} dr$$

on $\tilde{X}(0,1)$ for every j as above. The boundedness of these operators follows from Calderón interpolation theorem, owing to their boundedness in $L^1(0,1)$ and $L^\infty(0,1)$. Inequality (4.30) follows from (4.38). Inequality (4.30) implies, via the optimality of the norm $\|\cdot\|_{Z(0,1)}$ in (4.1), that

$$\|g\|_{\tilde{U}(0,1)} \lesssim \|g\|_{Z(0,1)}$$

for every $g \in \mathcal{M}_+(0,1)$. Given any function $f \in \mathcal{M}_+(0,\infty)$, we can apply the latter inequality to $f^* \chi_{(0,1)}$, and obtain

$$\|f^* \chi_{(0,1)}\|_{\tilde{U}(0,1)} \leq C \|f^* \chi_{(0,1)}\|_{Z(0,1)}.$$

By (2.3) and (3.4), this entails (4.26).

Let us next focus on (4.27). Fix $L > 0$ and consider trial functions in (4.24) of the form

$$u(x) = \varphi(x') \psi(x_1) \int_{x_1}^\infty \int_{s_1}^\infty \cdots \int_{s_{m-1}}^\infty f(s_m) ds_m \cdots ds_1 \quad \text{for } x \in \mathbb{R}^n,$$

where $x = (x_1, x')$, $x' \in \mathbb{R}^{n-1}$, $x_1 \in \mathbb{R}$. Here, $f \in \mathcal{M}_+(\mathbb{R})$ and has bounded support; $\psi \in \mathcal{C}^\infty(\mathbb{R})$, with $\psi = 0$ in $(-\infty, L]$, $\psi \equiv 1$ in $[2L, +\infty)$, $0 \leq \psi \leq 1$ in \mathbb{R} ; $\varphi \in \mathcal{C}_0^\infty(\mathcal{B}_2^{n-1})$, with $\varphi = 1$ in \mathcal{B}_1^{n-1} , where \mathcal{B}_ρ^{n-1} denotes the ball in \mathbb{R}^{n-1} , centered at 0, with radius ρ . An application of Fubini's theorem yields

$$u(x) = \varphi(x') \psi(x_1) \int_{x_1}^\infty f(s) \frac{(s-x_1)^{m-1}}{(m-1)!} ds \quad \text{for } x \in \mathbb{R}^n.$$

Thus,

$$\begin{aligned}
(4.39) \quad u(x) & \geq \chi_{\mathcal{B}_1^{n-1}}(x') \frac{\psi(x_1)}{2^{m-1}(m-1)!} \int_{2x_1}^\infty f(s) s^{m-1} ds \\
& \geq \frac{\chi_{\mathcal{B}_1^{n-1}}(x') \chi_{(2L,\infty)}(x_1)}{2^{m-1}(m-1)!} \int_{2x_1}^\infty f(s) s^{m-1} ds \quad \text{for } x \in \mathbb{R}^n,
\end{aligned}$$

and

$$(4.40) \quad u(x) \leq \frac{\chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1)}{(m-1)!} \int_{x_1}^\infty f(s) s^{m-1} ds \quad \text{for } x \in \mathbb{R}^n.$$

Moreover, if $1 \leq k \leq m-1$, then

$$(4.41) \quad |\nabla^k u(x)| \lesssim \chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1) \sum_{j=1}^{k+1} \int_{x_1}^\infty \int_{s_j}^\infty \int_{s_{j+1}}^\infty \cdots \int_{s_{m-1}}^\infty f(s_m) ds_m \cdots dt_j$$

$$\begin{aligned}
&= \chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1) \sum_{j=1}^{k+1} \int_{x_1}^{\infty} f(s_m) \frac{(s_m - x_1)^{m-j}}{(m-j)!} ds_m \\
&\lesssim \chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1) \sum_{j=1}^{k+1} \int_{x_1}^{\infty} f(s_m) s_m^{m-j} ds_m \\
&\lesssim \chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1) \int_{x_1}^{\infty} f(s) s^{m-1} ds \quad \text{for a.e. } x \in \mathbb{R}^n
\end{aligned}$$

and, similarly,

$$(4.42) \quad |\nabla^m u(x)| \lesssim \chi_{\mathcal{B}_2^{n-1}}(x') \chi_{(L,\infty)}(x_1) \left(\int_{x_1}^{\infty} f(s) s^{m-1} ds + f(x_1) \right) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Now, observe that, if a function $w \in \mathcal{M}_+(\mathbb{R}^n)$ has the form

$$w(x) = g(x_1) \chi_{\mathcal{B}_N^{n-1}}(x') \quad \text{for a.e. } x \in \mathbb{R}^n,$$

for some $g \in \mathcal{M}_+(\mathbb{R})$ and $N > 0$, then

$$|\{x \in \mathbb{R}^n : w(x) > t\}| = \omega_{n-1} N^{n-1} |\{x_1 \in \mathbb{R} : g(x_1) > t\}| \quad \text{for } t > 0,$$

whence

$$(4.43) \quad w^*(s) = g^* \left(\frac{s}{\omega_{n-1} N^{n-1}} \right) \quad \text{for } s \geq 0.$$

From (4.39)–(4.43) we thus deduce that

$$(4.44) \quad u^*(s) \gtrsim \left(\chi_{(2L,\infty)}(\cdot) \int_{2(\cdot)}^{\infty} f(\tau) \tau^{m-1} d\tau \right)^* \left(\frac{s}{c} \right) \quad \text{for } s > 0$$

and

$$(4.45) \quad |\nabla^m u|^*(s) \lesssim \int_{cs}^{\infty} f(r) r^{m-1} dr + f^*(cs) \quad \text{for } s > 0,$$

for some constant $c > 0$. Note that here we have made use of (2.1). An application of inequality (4.24) yields, via (4.44), (4.45) and the boundedness of the dilation operator in rearrangement-invariant spaces,

$$(4.46) \quad \left\| \chi_{(4L,\infty)}(s) \int_s^{\infty} f(r) r^{m-1} dr \right\|_{U(0,\infty)} \lesssim \left\| \int_s^{\infty} f(r) r^{m-1} dr \right\|_{X(0,\infty)} + \|f\|_{X(0,\infty)}$$

for every $f \in \mathcal{M}_+(0, \infty)$ with bounded support. Now, if f is any function from $\mathcal{M}_+(0, \infty)$, one can apply inequality (4.46) with f replaced by the function $f \chi_{(0,k)}$, for $k \in \mathbb{N}$, and pass to the limit as $k \rightarrow \infty$ to deduce (4.46). Note that property (P3) of function norms plays a role in this argument. Finally, choosing $L = \frac{1}{8}$ in (4.46) and applying Proposition 4.2 tell us that

$$\|f^* \chi_{(1,\infty)}\|_{U(0,\infty)} \lesssim \|f\|_{X(0,\infty)}$$

for every $f \in \mathcal{M}_+(0, \infty)$, namely, (4.27). \square

Proof of Corollary 3.2. Under assumption (3.7), the function norm $\|\cdot\|_{Z(0,1)}$, defined by (3.3), is equivalent to $\|\cdot\|_{L^\infty(0,1)}$, up to multiplicative constants. Indeed,

$$\|g\|_{Z'(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^s g^*(r) dr \right\|_{\tilde{X}'(0,1)} \leq \left(\int_0^1 g^*(r) dr \right) \left\| \chi_{(0,1)}(s) s^{-1+\frac{m}{n}} \right\|_{X'(0,\infty)} \leq C \|g\|_{L^1(0,1)},$$

for some constant C , and for every $g \in \mathcal{M}(0,1)$. This chain establishes the embedding $L^1(0,1) \rightarrow Z'(0,1)$. The converse embedding follows from (P5). Thus $Z'(0,1) = L^1(0,1)$, whence $Z(0,1) = L^\infty(0,1)$. The coincidence of the space $X_{\text{opt}}^m(\mathbb{R}^n)$ with $(L^\infty \cap X)(\mathbb{R}^n)$ then follows by the very definition of the former. \square

The last proof of this section concerns Theorem 3.3.

Proof of Theorem 3.3. Suppose that conditions (3.10) and (3.11) are in force. By Theorem 3.1,

$$W^m X(\mathbb{R}^n) \rightarrow (X^m \cap X)(\mathbb{R}^n).$$

Thus, in order to prove (3.9), it suffices to show that

$$(4.47) \quad (X^m \cap X)(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n).$$

Observe that inequality (3.10) can be written in the form

$$\left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{\tilde{Y}(0,1)} \lesssim \|f\|_{\tilde{X}(0,1)}$$

for every non-increasing function $f : (0,1) \rightarrow [0, \infty)$, where the function norms $\|\cdot\|_{\tilde{X}(0,1)}$ and $\|\cdot\|_{\tilde{Y}(0,1)}$ are defined as in (2.3). By the optimality of the space $Z(0,1)$ in inequality (4.1), one has that

$$Z(0,1) \rightarrow \tilde{Y}(0,1),$$

whence

$$(4.48) \quad \|g^* \chi_{(0,1)}\|_{Y(0,\infty)} \lesssim \|g^*\|_{Z(0,1)}$$

for any $g \in \mathcal{M}_+(0, \infty)$. Owing to (4.48) and (3.11),

$$\begin{aligned} \|u\|_{(X^m \cap X)(\mathbb{R}^n)} &= \|u\|_{X(\mathbb{R}^n)} + \|u\|_{X^m(\mathbb{R}^n)} = \|u^*\|_{X(0,\infty)} + \|u^*\|_{X^m(0,\infty)} = \|u^*\|_{X(0,\infty)} + \|u^*\|_{Z(0,1)} \\ &\gtrsim \|u^* \chi_{(1,\infty)}\|_{X(0,\infty)} + \|u^* \chi_{(0,1)}\|_{Y(0,\infty)} \geq \|u^*\|_{Y(0,\infty)} = \|u\|_{Y(\mathbb{R}^n)} \end{aligned}$$

for any $u \in (X^m \cap X)(\mathbb{R}^n)$. Inequality (4.47) is thus established.

Conversely, assume that embedding (3.9) holds. Owing to the optimality of the rearrangement-invariant target norm $X_{\text{opt}}^m(\mathbb{R}^n)$ in (3.1),

$$(X^m \cap X)(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n).$$

By (3.4), the latter embedding implies that

$$(4.49) \quad \|f\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)} + \|f^*\|_{Z(0,1)}$$

for any $f \in \mathcal{M}_+(0, \infty)$. In particular, applying this inequality to functions f of the form $g^* \chi_{(0,1)}$, and making use of Proposition 4.1 tell us that

$$\|g^* \chi_{(0,1)}\|_{Y(0,\infty)} \lesssim \|g^* \chi_{(0,1)}\|_{X(0,\infty)} + \|g^*\|_{Z(0,1)} = \|g^*\|_{\tilde{X}(0,1)} + \|g^*\|_{Z(0,1)} \lesssim \|g^*\|_{Z(0,1)}.$$

In particular,

$$\left\| \chi_{(0,1)}(s) \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{Y(0,\infty)} \lesssim \left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{Z(0,1)}$$

for every $f \in \mathcal{M}_+(0, \infty)$. Since, by (4.1),

$$\left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{Z(0,1)} \lesssim \|f\|_{\tilde{X}(0,1)} = \|\chi_{(0,1)} f\|_{X(0,\infty)}$$

for any non-increasing function $f : (0, \infty) \rightarrow [0, \infty)$, inequality (3.10) follows. On the other hand, on applying (4.49) to functions of the form $f^* \chi_{(1,\infty)}$, we obtain

$$\begin{aligned} \|f^* \chi_{(1,\infty)}\|_{Y(0,\infty)} &\lesssim \|f^* \chi_{(1,\infty)}\|_{X(0,\infty)} + \|f^*(1^-)\|_{1\|_{Z(0,1)}} \\ &\lesssim \|f^* \chi_{(1,\infty)}\|_{X(0,\infty)} + \|f^*(1^-)\|_{\chi_{(0,1)}\|_{X(0,\infty)}} \lesssim \|f\|_{X(0,\infty)}, \end{aligned}$$

namely (3.11). □

5. PROOFS OF THEOREMS 3.5, 3.9 AND 3.11

The proofs of our results about Orlicz-Sobolev embeddings require a couple of preliminary lemmas.

Lemma 5.1. *Let F and G be Young functions. Assume that there exist constants $t_0 > 0$ and $c > 0$ such that*

$$F(t) \leq G(ct) \quad \text{if } 0 \leq t \leq t_0.$$

Let $L \geq 0$. Then

$$(5.1) \quad \|f^*\|_{L^F(L, \infty)} \leq \max \left\{ f^*(L)/t_0, c \|f^*\|_{L^G(L, \infty)} \right\}$$

for every $f \in \mathcal{M}(0, \infty)$. In particular,

$$(5.2) \quad (L^G \cap L^\infty)(0, \infty) \rightarrow L^F(0, \infty).$$

Proof. If $f^*(L) = 0$, then $f^* = 0$ in (L, ∞) , and (5.1) holds trivially. Since the case when $f^*(L) = \infty$ is trivial as well, we can in fact assume that $f^*(L) \in (0, \infty)$. On replacing f with $\frac{f}{f^*(L)}$, we may suppose that $f^*(L) = 1$. Let

$$\lambda = \max \left\{ 1/t_0, c \|f^*\|_{L^G(L, \infty)} \right\}.$$

Then,

$$\int_L^\infty F\left(\frac{f^*(t)}{\lambda}\right) dt \leq \int_L^\infty G\left(\frac{c f^*(t)}{\lambda}\right) dt \leq \int_L^\infty G\left(\frac{f^*(t)}{\|f^*\|_{L^G(L, \infty)}}\right) dt \leq 1,$$

and hence

$$\|f^*\|_{L^F(L, \infty)} \leq \max \left\{ 1/t_0, c \|f^*\|_{L^G(L, \infty)} \right\}.$$

Inequality (5.2) is a consequence of (5.1), applied with $L = 0$, since $\|f\|_{L^\infty(0, \infty)} = f^*(0)$, and

$$\|f\|_{(L^G \cap L^\infty)(0, \infty)} \simeq \max \left\{ \|f^*\|_{L^G(0, \infty)}, \|f\|_{L^\infty(0, \infty)} \right\}.$$

□

Lemma 5.2. *Let F and G be Young functions such that*

$$F \text{ dominates } G \text{ near infinity.}$$

Assume that the function H , defined as

$$H(t) = \begin{cases} G(t) & \text{if } t \in [0, 1] \\ F(t) & \text{if } t \in (1, \infty), \end{cases}$$

is a Young function. Then

$$(5.3) \quad \|f^*\|_{L^F(0, 1)} + \|f^*\|_{L^G(0, \infty)} \simeq \|f^*\|_{L^H(0, \infty)}$$

for every $f \in \mathcal{M}(0, \infty)$.

Proof. Define the rearrangement-invariant function norm $\|\cdot\|_{X(0, \infty)}$ as

$$\|f\|_{X(0, \infty)} = \|f^*\|_{L^F(0, 1)} + \|f^*\|_{L^G(0, \infty)}$$

for $f \in \mathcal{M}_+(0, \infty)$. Thanks to [4, Theorem 1.8, Chapter 1], equation (5.3) will follow if we show that

$$X(0, \infty) = L^H(0, \infty)$$

as a set equality. Assume first that $f \in X(0, \infty)$. We have that

$$(5.4) \quad \|f^*\|_{L^H(0, \infty)} \leq \|f^*\|_{L^H(0, 1)} + \|f^*\|_{L^H(1, \infty)} \lesssim \|f^*\|_{L^F(0, 1)} + \|f^*\|_{L^H(1, \infty)},$$

where the second inequality holds since H is equivalent to F near infinity. By Lemma 5.1, $\|f^*\|_{L^H(1,\infty)} < \infty$ if $f \in L^G(0, \infty)$. Thus, equation (5.4) implies that $f \in L^H(0, \infty)$. Suppose next that $f \in L^H(0, \infty)$. Then

$$(5.5) \quad \begin{aligned} \|f^*\|_{L^F(0,1)} + \|f^*\|_{L^G(0,\infty)} &\leq \|f^*\|_{L^F(0,1)} + \|f^*\|_{L^G(0,1)} + \|f^*\|_{L^G(1,\infty)} \\ &\lesssim \|f^*\|_{L^F(0,1)} + \|f^*\|_{L^G(1,\infty)} \\ &\lesssim \|f^*\|_{L^H(0,1)} + \|f^*\|_{L^G(1,\infty)}, \end{aligned}$$

where the second inequality holds since F dominates G near infinity, and the third one since F and H agree near infinity. Since $f \in L^H(0, \infty)$, then $f \in L^G(1, \infty)$ by Lemma 5.1. Thus, the right-hand side of (5.5) is finite, whence $f \in X(0, \infty)$. \square

The main ingredients for a proof of Theorem 3.9 are now at our disposal.

Proof of Theorem 3.9. It suffices to show that

$$(5.6) \quad \|f\|_{(L^A)_{\text{opt}}^m(0,\infty)} \simeq \|v f^*\|_{L^E(0,\infty)}$$

for $f \in \mathcal{M}_+(0, \infty)$. The sharp Orlicz-Sobolev embedding theorem on domains with finite measure asserts that the optimal rearrangement-invariant function norm $\|\cdot\|_{Z(0,1)}$ in inequality (4.1), defined as in (3.3) with $X(0, \infty) = L^A(0, \infty)$, obeys

$$(5.7) \quad \|f\|_{Z(0,1)} = \|s^{-\frac{m}{n}} f^*(s)\|_{L^{\widehat{A}}(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$ [15, Theorem 3.7] (see also [14] for the case when $m = 1$). Hence, owing to Theorem 3.1,

$$\|f\|_{(L^A)_{\text{opt}}^m(0,\infty)} = \|s^{-\frac{m}{n}} f^*(s)\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^A(0,\infty)}$$

for $f \in \mathcal{M}_+(0, \infty)$. Thus, equation (5.6) will follow if we show that

$$(5.8) \quad \|s^{-\frac{m}{n}} f^*(s)\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^A(0,\infty)} \simeq \|v(s) f^*(s)\|_{L^E(0,\infty)}$$

for $f \in \mathcal{M}_+(0, \infty)$. Since the two sides of (5.8) define rearrangement-invariant function norms, by [4, Theorem 1.8, Chapter 1] it suffices to prove that the left-hand side of (5.8) is finite if and only if the right-hand side is finite. Assume that the left-hand side of (5.8) is finite for some $f \in \mathcal{M}_+(0, \infty)$. We have that

$$\begin{aligned} \|v(s) f^*(s)\|_{L^E(0,\infty)} &\leq \|v(s) f^*(s)\|_{L^E(0,1)} + \|v(s) f^*(s)\|_{L^E(1,\infty)} \\ &= \|s^{-\frac{m}{n}} f^*(s)\|_{L^E(0,1)} + \|f^*\|_{L^E(1,\infty)} \\ &\lesssim \|s^{-\frac{m}{n}} f^*(s)\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^E(1,\infty)}, \end{aligned}$$

where the second inequality holds inasmuch as E and \widehat{A} are equivalent near infinity. By Lemma 5.1, $\|f^*\|_{L^E(1,\infty)} < \infty$, since we are assuming that $\|f^*\|_{L^A(1,\infty)} < \infty$. Therefore, $\|v(s) f^*(s)\|_{L^E(0,\infty)} < \infty$. Suppose next that the right-hand side of (5.8) is finite. Then

$$\begin{aligned} \left\| s^{-\frac{m}{n}} f^*(s) \right\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^A(0,\infty)} &\leq \left\| s^{-\frac{m}{n}} f^*(s) \right\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^A(0,1)} + \|f^*\|_{L^A(1,\infty)} \\ &\lesssim \left\| s^{-\frac{m}{n}} f^*(s) \right\|_{L^{\widehat{A}}(0,1)} + \left\| s^{-\frac{m}{n}} f^*(s) \right\|_{L^{\widehat{A}}(0,1)} + \|v(s) f^*(s)\|_{L^A(1,\infty)} \\ &\lesssim \|v(s) f^*(s)\|_{L^E(0,1)} + \|v(s) f^*(s)\|_{L^A(1,\infty)}, \end{aligned}$$

where the second inequality holds by equation (5.7) and Proposition 4.1, and the last one since E and \widehat{A} are equivalent near infinity. Inasmuch as $v f^* \in L^E(1, \infty)$, one has that $v f^* \in L^A(1, \infty)$, by Lemma 5.1. Thus, $\|s^{-\frac{m}{n}} f^*(s)\|_{L^{\widehat{A}}(0,1)} + \|f^*\|_{L^A(0,\infty)} < \infty$. \square

Theorem 3.5 relies upon a specialization of the reduction principle of Theorem 3.3 to Orlicz spaces.

Proposition 5.3 (Reduction principle for Orlicz spaces). *Let A and B be Young functions. Then*

$$W^{m,A}(\mathbb{R}^n) \rightarrow L^B(\mathbb{R}^n)$$

if and only if there exists a constant C such that

$$(5.9) \quad \left\| \int_s^1 f(r) r^{-1+\frac{m}{n}} dr \right\|_{L^B(0,1)} \leq C \|f\|_{L^A(0,1)}$$

for every function $f \in \mathcal{M}(0,1)$, and

$$(5.10) \quad A \text{ dominates } B \text{ near } 0.$$

Proof. By Theorem 3.3 and Remark 3.4, it suffices to show that condition (5.10) is equivalent to the inequality

$$(5.11) \quad \|f\|_{L^B(1,\infty)} \leq C \|f\|_{L^A(0,\infty)}$$

for some constant C , and for every non-increasing function $f : (0, \infty) \rightarrow [0, \infty)$. Assume that (5.10) holds. Given any function f of this kind, we have that

$$\|f\|_{L^A(0,1)} \geq f(1)/A^{-1}(1).$$

Hence, owing to assumption (5.10), Lemma 5.1 implies that

$$\|f\|_{L^B(1,\infty)} \lesssim \max\{f(1), \|f\|_{L^A(0,\infty)}\} \lesssim \|f\|_{L^A(0,\infty)}.$$

Inequality (5.11) is thus established.

Conversely, assume that inequality (5.11) is in force. Fix $r > 1$. An application of this inequality with $f = \chi_{(0,r)}$ yields

$$\|1\|_{L^B(1,r)} \leq C \|1\|_{L^A(0,r)} \quad \text{for } r > 1,$$

namely

$$\frac{1}{B^{-1}(\frac{1}{r-1})} \leq \frac{C}{A^{-1}(\frac{1}{r})} \quad \text{for } r > 1.$$

Hence, (5.10) follows. \square

Proof of Theorem 3.5. Since A_{opt} is equivalent to $A_{\frac{n}{m}}$ near infinity, by [13, Inequality (2.7)] inequality (5.9) holds with $B = A_{\text{opt}}$. Moreover, since A_{opt} is equivalent to A near zero, condition (5.10) is satisfied with $B = A_{\text{opt}}$. Proposition 5.3 thus tells us that embedding (3.14) holds.

On the other hand, if B is any Young function such that $W^{m,A}(\mathbb{R}^n) \rightarrow L^B(\mathbb{R}^n)$, then, by Proposition 5.3, conditions (5.9) and (5.10) are fulfilled. The former ensures that $A_{\frac{n}{m}}$ dominates B near infinity – see [15, Proof of Theorem 3.1]. The latter tells us that A dominates B near 0. Altogether, A_{opt} dominates B globally, whence $L^{A_{\text{opt}}}(\mathbb{R}^n) \rightarrow L^B(\mathbb{R}^n)$. \square

Proof of Theorem 3.11. One has that $(L^{p,q}(\mathbb{R}^n))' = L^{p',q'}(\mathbb{R}^n)$, up to equivalent norms – see e.g. [4, Chapter 4, Theorem 4.7]. Since $(\tilde{f})^* = f^*$ for every $f \in \mathcal{M}_+(0,1)$,

$$\|f\|_{(\widetilde{L^{p,q}})'(0,1)} \simeq \|f\|_{L^{p',q'}(0,1)}.$$

Thus, on denoting by $\|\cdot\|_{Z(0,1)}$ the rearrangement-invariant function norm associated with $\|\cdot\|_{L^{p,q}(0,\infty)}$ as in (3.3), one has that

$$\|f\|_{Z'(0,1)} \simeq \|s^{\frac{m}{n}} f^{**}(s)\|_{L^{p',q'}(0,1)}$$

for $f \in \mathcal{M}_+(0, 1)$. Note that here there is some slight abuse of notation, since the functionals $\|\cdot\|_{\widetilde{L}^{p,q}(0,1)}$ and $\|\cdot\|_{\widetilde{L}^{p,q}'(0,1)}$ are, in general, only equivalent to rearrangement-invariant function norms. As shown in [17, Proof of Theorem 5.1],

$$\|f\|_{Z(0,1)} \simeq \begin{cases} \|s^{\frac{1}{p}-\frac{m}{n}-\frac{1}{q}} f^*(s)\|_{L^q(0,1)} & \text{if } p = q = 1, \text{ or } 1 < p < \frac{n}{m} \text{ and } 1 \leq q \leq \infty, \\ \|s^{-\frac{1}{q}} (\log \frac{2}{s})^{-1} f^*(s)\|_{L^q(0,1)} & \text{if } p = \frac{n}{m} \text{ and } 1 < q \leq \infty, \\ \|f^*\|_{L^\infty(0,1)} & \text{if } p > \frac{n}{m}, \text{ or } p = \frac{n}{m} \text{ and } q = 1, \text{ or } m > n \end{cases}$$

for every $f \in \mathcal{M}_+(0, 1)$. By (3.5),

$$\begin{aligned} \|u\|_{(L^{p,q})_{\text{opt}}^m(\mathbb{R}^n)} &\approx \|u\|_{(L^{p,q})^m(\mathbb{R}^n)} + \|u\|_{L^{p,q}(\mathbb{R}^n)} = \|u^*\|_{Z(0,1)} + \|u^*\|_{L^{p,q}(0,\infty)} \\ &= \|u^*\|_{Z(0,1)} + \|s^{\frac{1}{p}-\frac{1}{q}} u^*(s)\|_{L^q(0,\infty)}. \end{aligned}$$

Hence, the conclusion follows, via an analogous argument as in the proof of equation (5.8). \square

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