Generalized Versions of Ilmanen Lemma: Insertion of $C^{1,\omega}$ or $C^{1,\omega}_{\mathrm{loc}}$ Functions

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Abstract. We prove that for a normed linear space X, if $f_1 : X \to \mathbb{R}$ is continuous and semiconvex with modulus ω, $f_2 : X \to \mathbb{R}$ is continuous and semiconcave with modulus ω and $f_1 \leq f_2$, then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \leq f \leq f_2$. Using this result we prove a generalization of Ilmanen lemma (which deals with the case $\omega(t) = t$) to the case of an arbitrary nontrivial modulus ω. This generalization (where a $C^{1,\omega}_{\mathrm{loc}}$ function is inserted) gives a positive answer to a problem formulated by A. Fathi and M. Zavidovique in 2010.

Keywords: Ilmanen lemma, semiconvex function with general modulus, $C^{1,\omega}$ function

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1 Introduction

Suppose $A \subset \mathbb{R}^n$ is a convex set. We say that $f : A \to \mathbb{R}$ is classically semiconvex if there exists $C > 0$ such that the function $x \mapsto f(x) + C|x|^2$, $x \in A$, is convex. We say that $f : A \to \mathbb{R}$ is classically semiconcave if $-f$ is classically semiconvex. T. Ilmanen proved the following result (so called Ilmanen lemma) [I, proof of 4F from 4G, p. 199].

Ilmanen lemma. *Let $G \subset \mathbb{R}^n$ be an open set and $f_1, f_2 : G \to \mathbb{R}$. Suppose that $f_1 \leq f_2$ and that for every $a \in G$ there exists $r > 0$ such that $U := U(a, r) \subset G$, $f_1 \restriction_U$ is classically semiconvex and $f_2 \restriction_U$ is classically semiconcave. Then there exists $f \in C^{1,1}_{\mathrm{loc}}(G)$ such that $f_1 \leq f \leq f_2$.*

Alternative proofs of Ilmanen lemma can be found in [B] and [FZ].

We will work with semiconvex, resp. semiconcave, functions with general modulus (see Definition 2.2 and cf. [CS, Definition 2.1.1]). Note that the classically semiconvex functions coincide with semiconvex functions with modulus $\omega(t) = Ct$ where $C > 0$.

A. Fathi and M. Zavidovique ([FZ, Problem 5.1]) asked if Ilmanen lemma can be generalized to the case of a general modulus ω.

More precisely, suppose that $G \subset \mathbb{R}^n$ is an open set, ω a modulus and $f_1, f_2 : G \to \mathbb{R}$ are continuous functions such that $f_1 \leq f_2$ and for every $a \in G$

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there exist $C, r > 0$ such that $f_1 \restriction_{U(a,r)}$ is semiconvex with modulus $C\omega$ and $f_2 \restriction_{U(a,r)}$ is semiconcave with modulus $C\omega$. Then the question is whether there exists $f \in C^{1,\omega}_{\mathrm{loc}}(G)$ with $f_1 \le f \le f_2$.

We prove (see Theorem 4.5) that the answer is positive if the modulus ω satisfies $\liminf_{t\to 0+} \omega(t)/t > 0$ (even if G is an open subset of a Hilbert space). Note (see implication (2) below) that if $\liminf_{t\to 0+} \omega(t)/t = 0$, then f_1, resp. f_2, is convex, resp. concave, on every convex $A \subset G$. In such a case it is well known that the answer is negative for many open G.

The proof of Theorem 4.5 is based on Corollary 3.2 which is a special case of Theorem 3.1 (which has a short and quite simple proof).

Corollary 3.2 can be equivalently reformulated (without using the symbol $SC^\omega(X)$) in the following way. Suppose that X is a normed linear space, ω a modulus and $f_1, f_2 : X \to \mathbb{R}$ are continuous functions such that f_1 is semiconvex with modulus ω, f_2 is semiconcave with modulus ω and $f_1 \le f_2$. Then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \le f \le f_2$.

So, Corollary 3.2 generalizes [B, Theorem 2].

2 Preliminaries

If X is a normed linear space, then we set $U(a,r) := \{x \in X : \|x - a\| < r\}$, $a \in X$, $r > 0$, and also $\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}$, $f : X \to \mathbb{R}$.

Notation 2.1. We denote by $\mathcal{M}$ the set of all $\omega : [0,\infty) \to [0,\infty)$ which are non-decreasing and satisfy $\lim_{t\to 0+} \omega(t) = 0$.

Definition 2.2. Let X be a normed linear space, $A \subset X$ a convex set and $\omega \in \mathcal{M}$.

- We say that $f : A \to \mathbb{R}$ is semiconvex with modulus ω if

  $$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda)\|x - y\|\omega(\|x - y\|)$$

  for every $x, y \in A$ and $\lambda \in [0,1]$.
- We say that $f : A \to \mathbb{R}$ is semiconcave with modulus ω if $-f$ is semiconvex with modulus ω.
- We denote by $SC^\omega(A)$ the set of all $f : A \to \mathbb{R}$ which are semiconvex with modulus $C\omega$ for some $C > 0$. We denote by $-SC^\omega(A)$ the set of all $f : A \to \mathbb{R}$ such that $-f \in SC^\omega(A)$.

If G is an open subset of a normed linear space and $\omega \in \mathcal{M}$, then we denote by $C^{1,\omega}(G)$ the set of all Fréchet differentiable $f : G \to \mathbb{R}$ such that f' is uniformly continuous with modulus $C\omega$ for some $C > 0$, and we denote by $C^{1,\omega}_{\mathrm{loc}}(G)$ the set of all $f : G \to \mathbb{R}$ which are locally $C^{1,\omega}$.

The following lemma is well known and follows directly from the definition (for (iv) cf. [CS, Proposition 2.1.5]).

Lemma 2.3. *Let X, A and ω be as in Definition 2.2. Then the following hold.*

(i) Let $f : A \to \mathbb{R}$. Then f is semiconvex with modulus ω if and only if f is semiconvex with modulus ω on every line, i.e. for every $x, h \in X$, $\|h\| = 1$, the function $t \mapsto f(x + th)$, $t \in \{t \in \mathbb{R} : x + th \in A\}$, is semiconvex with modulus ω.

(ii) Let $f : X \to \mathbb{R}$ be semiconvex with modulus ω and let $z \in X$. Then the function $x \mapsto f(x+z)$, $x \in X$, is semiconvex with modulus ω.

(iii) Let $f_1, f_2 : A \to \mathbb{R}$ be semiconvex with modulus ω, let $a_1, a_2 \in [0,\infty)$ and let $a_3 \in \mathbb{R}$. Then $a_1 f_1 + a_2 f_2 + a_3$ is semiconvex with modulus $(a_1+a_2)\omega$.

(iv) Let $\mathcal{S} \subset \mathbb{R}^A$ be such that every $s \in \mathcal{S}$ is semiconvex with modulus ω and $f(x) := \sup\{s(x) : s \in \mathcal{S}\} \in \mathbb{R}$, $x \in A$. Then the function f is semiconvex with modulus ω.

The notion of semiconvex functions is (up to a multiplicative constant) equivalent to the notion of strongly paraconvex functions (for definition see [R1]). More precisely, suppose that A is a convex subset of a normed linear space, $f : A \to \mathbb{R}$, $\omega \in \mathcal{M}$ and set $\alpha(t) := t\omega(t)$, $t \in [0,\infty)$, then (cf. [DZ1, Theorem 4.16])

$$f \in SC^{\omega}(A) \Leftrightarrow f \text{ is strongly } \alpha(\cdot)\text{-paraconvex.} \tag{1}$$

We also have

$$\left(f \in SC^{\omega}(A), \liminf_{t\to 0+} \omega(t)/t = 0 \right) \Rightarrow f \text{ is convex.} \tag{2}$$

For this implication see [R1, Proposition 7] (the proof is not quite rigorous but one can easily correct it) or [DZ1, Corollary 3.6]. Hence we may (and sometimes will) consider only the case $\liminf_{t\to 0+} \omega(t)/t > 0$. Note that for $\omega \in \mathcal{M}$ we have

$$\liminf_{t\to 0+} \omega(t)/t > 0 \Leftrightarrow (\forall d \in [0,\infty))(\exists M \in (0,\infty))(\forall t \in [0,d])\ t \le M\omega(t). \tag{3}$$

We will need the following two propositions. The first one was proved in [DZ2, Proposition 2.8].

Proposition 2.4. *Let $I \subset \mathbb{R}$ be an open interval, $\omega \in \mathcal{M}$ and let $f : I \to \mathbb{R}$ be continuous. Then the following hold.*

(i) If f semiconvex with modulus ω, then $f'_+(x) \in \mathbb{R}$ for every $x \in I$ and

$$f'_+(x_1) - f'_+(x_2) \le 2\omega(x_2 - x_1), \quad x_1, x_2 \in I, x_1 \le x_2.$$

(ii) If $f'_+(x) \in \mathbb{R}$ for every $x \in I$ and

$$f'_+(x_1) - f'_+(x_2) \le \omega(x_2 - x_1), \quad x_1, x_2 \in I, x_1 \le x_2,$$

then f is semiconvex with modulus ω.

Proposition 2.5. *Let X be a normed linear space, $A \subset X$ an open convex set and $f \in \bigcup_{\omega\in\mathcal{M}} SC^{\omega}(A)$. Then the following conditions are equivalent.*

(i) f is locally Lipschitz.

(ii) f is continuous.

(iii) f is locally bounded.

Proof. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If (iii) holds, then (i) holds by (1) and [R1, Proposition 5]. □

We will need the following theorem whose part (i) is well known. Part (ii) is essentially known at least in its local version (see [CS, Theorem 3.3.7, p. 60], [FF, Theorem A.19] and [JTZ, Theorem 6.1]) but the present version is probably new.

Theorem 2.6. *Let X be a normed linear space, $A \subset X$ an open convex set and $\omega \in \mathcal{M}$. Then the following hold (where $C(A)$ denotes the set of all continuous $f : A \to \mathbb{R}$).*

(i) $C^{1,\omega}(A) \subset C(A) \cap SC^{\omega}(A) \cap (-SC^{\omega}(A))$.

(ii) If $A = X$ or A is bounded, then

$$C^{1,\omega}(A) = C(A) \cap SC^{\omega}(A) \cap (-SC^{\omega}(A)). \tag{4}$$

Proof. (i) It follows easily from Lemma 2.3 (i) and [CS, Proposition 2.1.2]. It can be also deduced from Lemma 2.3 (i) and Proposition 2.4 (ii).

(ii) Let $f \in C(A) \cap SC^{\omega}(A) \cap (-SC^{\omega}(A))$. By Proposition 2.5, f is locally Lipschitz. Hence f and $-f$ have nonempty Clarke subdifferential at every point of A (cf. [CLSW, Proposition 1.5, p. 73]). Thus, by (1) and [R2, Theorem 3], there exists $C > 0$ such that for every $x \in A$ we can find $\phi_x, \psi_x \in X^*$ with

$$\begin{aligned} f(x+h) - f(x) - \phi_x(h) &\geq -C\|h\|\omega(\|h\|), \quad h \in A - x, \\ -f(x+h) + f(x) - \psi_x(h) &\geq -C\|h\|\omega(\|h\|), \quad h \in A - x. \end{aligned}$$

Adding these two inequalities together and using the standard argument we obtain that $\psi_x = -\phi_x$, $x \in A$. Hence for every $x \in A$

$$|f(x+h) - f(x) - \phi_x(h)| \leq C\|h\|\omega(\|h\|), \quad h \in A - x,$$

and $f'(x) = \phi_x$. Thus $f \in C^{1,\omega}(A)$ by [HJ, Corollary 126, p. 58]. □

Remark 2.7. [HJ, Corollary 126, p. 58] and the proof of Theorem 2.6 show that (4) holds also for A such that there exists $a \in X$, $r > 0$ and a sequence $(u_n)_{n=1}^{\infty}$ in X such that $\|u_n\| = n$ and $\overline{U(a + u_n, rn)} \subset A$ for every $n \in \mathbb{N}$. But (4) doesn't hold for an arbitrary open convex set A ([K, Example 2.10, Remark 2.11]). However, if $\omega(t) = t$, $t \in [0, \infty)$, then (4) holds for any open convex A ([K, Theorem 2.9 (iv)]).

3 Insertion of a $C^{1,\omega}$ function on the whole space

Here we prove the principal observation of this article. The main idea is based on the choice of the function s in the proof of Theorem 3.1.

Theorem 3.1. *Let X be a normed linear space, $f_1, f_2 : X \to \mathbb{R}$ and $\omega_1, \omega_2 \in \mathcal{M}$. Suppose that f_1 is semiconvex with modulus ω_1, f_2 is semiconcave with modulus ω_2 and $f_1 \leq f_2$. Denote by $\mathcal{S}$ the set of all $s : X \to \mathbb{R}$ which are semiconvex with modulus ω_1 and satisfy $s \leq f_2$. Then the function*

$$f(x) := \sup\{s(x) : s \in \mathcal{S}\}, \quad x \in X,$$

is semiconvex with modulus ω_1, semiconcave with modulus ω_2 and satisfies $f_1 \leq f \leq f_2$.

Proof. It is clear that $f_1 \le f \le f_2$. By Lemma 2.3 (iv), f is semiconvex with modulus ω_1. Now we will prove that f is semiconcave with modulus ω_2.

Let $u, v \in X$ and $\lambda \in [0,1]$. Set $w := \lambda u + (1-\lambda)v$ and define a function s by

$$\begin{aligned} s(x) &= \lambda f(x-w+u) + (1-\lambda) f(x-w+v) \\ &\quad - \lambda(1-\lambda)\|u-v\|\omega_2(\|u-v\|), \quad x \in X. \end{aligned}$$

By Lemma 2.3 (ii), (iii), s is semiconvex with modulus $\lambda\omega_1 + (1-\lambda)\omega_1 = \omega_1$. Since f_2 is semiconcave with modulus ω_2, we have

$$\begin{aligned} s(x) &\le \lambda f_2(x-w+u) + (1-\lambda) f_2(x-w+v) - \lambda(1-\lambda)\|u-v\|\omega_2(\|u-v\|) \\ &\le f_2(\lambda(x-w+u) + (1-\lambda)(x-w+v)) = f_2(x), \quad x \in X. \end{aligned}$$

Hence $s \in \mathcal{S}$ and consequently $s \le f$. So

$$f(\lambda u + (1-\lambda)v) \ge s(w) = \lambda f(u) + (1-\lambda) f(v) - \lambda(1-\lambda)\|u-v\|\omega_2(\|u-v\|).$$

□

Corollary 3.2. *Let X be a normed linear space, $\omega \in \mathcal{M}$, $f_1 \in SC^\omega(X)$ and $f_2 \in -SC^\omega(X)$. Suppose that f_1, f_2 are continuous and $f_1 \le f_2$. Then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \le f \le f_2$.*

Proof. By Theorem 3.1 there exists $f \in SC^\omega(X) \cap (-SC^\omega(X))$ such that $f_1 \le f \le f_2$. Since f_1, f_2 are continuous, f is locally bounded. Hence, by Proposition 2.5, f is continuous and thus, by Theorem 2.6, $f \in C^{1,\omega}(X)$. □

4 Insertion of a $C^{1,\omega}_{\mathrm{loc}}$ function

In this section we will use Corollary 3.2 and partitions of unity to obtain a version (Theorem 4.5) of Ilmanen lemma which works with locally semiconvex and locally semiconcave functions defined on an open subset of a Hilbert space. Recall that Theorem 4.5 gives a positive answer to a problem formulated by A. Fathi and M. Zavidovique ([FZ, Problem 5.1]).

We will need the following obvious fact.

Fact 4.1. *Let X, Y be normed linear spaces, $A \subset X$ and $f : A \to Y$. If A is bounded and f is uniformly continuous with some modulus $\omega \in \mathcal{M}$, then f is bounded.*

Lemma 4.2. *Let X be a normed linear space, $A \subset X$ a bounded open convex set, $\omega \in \mathcal{M}$, $g_1 \in C^{1,\omega}(A)$ and $g_2 \in SC^\omega(A)$. Suppose that $g_1 \ge 0$, g_2 is Lipschitz and $\liminf_{t\to 0+} \omega(t)/t > 0$. Then $g_1 \cdot g_2 \in SC^\omega(A)$.*

Proof. By Fact 4.1, g_1' is bounded and thus, by [HJ, Proposition 71, p. 29], g_1 is Lipschitz. By the assumptions and Fact 4.1 we can find $C > 0$ big enough such that $0 \le g_1 \le C$, $|g_2| \le C$, g_1' is uniformly continuous with modulus $C\omega$, g_2 is semiconvex with modulus $C\omega$ and g_1, g_2 are C-Lipschitz. By (3) there exists $M > 0$ such that $t \le M\omega(t)$, $t \in [0, \mathrm{diam}(A)]$. We will show that $g_1 \cdot g_2$ is semiconvex with modulus $(2M+3)C^2\omega$.

Let $x, h \in X$, $\|h\| = 1$. Set $I := \{t \in \mathbb{R} : x + th \in A\}$ and for $i = 1, 2$ define a function $f_i(t) := g_i(x + th)$, $t \in I$. By Lemma 2.3 (i), it is sufficient to show that $f_1 \cdot f_2$ is semiconvex with modulus $(2M + 3)C^2\omega$. Since g_1' is uniformly continuous with modulus $C\omega$, we easily obtain that $f_1'(t) \in \mathbb{R}$ for every $t \in I$ and

$$|f_1'(t_1) - f_1'(t_2)| \leq C\omega(t_2 - t_1), \quad t_1, t_2 \in I, t_1 \leq t_2.$$

By Lemma 2.3 (i), f_2 is semiconvex with modulus $C\omega$ and thus, by Proposition 2.4 (i), $(f_2)'_+(t) \in \mathbb{R}$ for every $t \in I$ and

$$(f_2)'_+(t_1) - (f_2)'_+(t_2) \leq 2C\omega(t_2 - t_1), \quad t_1, t_2 \in I, t_1 \leq t_2.$$

Clearly f_1, f_2 are C-Lipschitz and hence also $|f_1'| \leq C$ and $|(f_2)'_+| \leq C$. Thus $(f_1 \cdot f_2)'_+(t) \in \mathbb{R}$ for every $t \in I$ and

$$\begin{aligned}
&(f_1 f_2)'_+(t_1) - (f_1 f_2)'_+(t_2) \\
&= f_1'(t_1) f_2(t_1) + f_1(t_1)(f_2)'_+(t_1) - f_1'(t_2) f_2(t_2) - f_1(t_2)(f_2)'_+(t_2) \\
&= f_1'(t_1)(f_2(t_1) - f_2(t_2)) + f_2(t_2)(f_1'(t_1) - f_1'(t_2)) \\
&+ (f_2)'_+(t_1)(f_1(t_1) - f_1(t_2)) + f_1(t_2)((f_2)'_+(t_1) - (f_2)'_+(t_2)) \\
&\leq C^2(t_2 - t_1) + C^2\omega(t_2 - t_1) + C^2(t_2 - t_1) + 2C^2\omega(t_2 - t_1) \\
&\leq (2M + 3)C^2\omega(t_2 - t_1)
\end{aligned}$$

for every $t_1, t_2 \in I$, $t_1 \leq t_2$. Hence $f_1 \cdot f_2$ is semiconvex with modulus $(2M + 3)C^2\omega$ by Proposition 2.4 (ii). $\square$

Lemma 4.3. *Let X be a normed linear space, $f : X \to \mathbb{R}$ and $\omega \in \mathcal{M}$. Suppose that there exists an open convex set $U \subset X$ such that* $\operatorname{supp} f \subset U$ *and $f \restriction_U$ is semiconvex with modulus ω. Then f is semiconvex with modulus 2ω.*

Proof. By Lemma 2.3 (i) we may suppose that $X = \mathbb{R}$. Then f is continuous on U by [CS, Theorem 2.1.7]. Since $\operatorname{supp} f \subset U$, it follows that f is continuous and $f'(x) = 0$ for every $x \in \mathbb{R} \setminus U$. By Proposition 2.4 (i), $f'_+(x) \in \mathbb{R}$ for every $x \in U$ and

$$f'_+(x_1) - f'_+(x_2) \leq 2\omega(x_2 - x_1) \tag{5}$$

for every $x_1, x_2 \in U$, $x_1 \leq x_2$. Let $x_1, x_2 \in \mathbb{R}$, $x_1 \leq x_2$. By Proposition 2.4 (ii) it is enough to show that (5) holds. This is clear if $x_1, x_2 \in U$ or $x_1, x_2 \in \mathbb{R} \setminus U$. Suppose that $x_1 \in \mathbb{R} \setminus U$ and $x_2 \in U$. Then $f'(x_1) = 0$ and there exists $c \in U$ such that $x_1 < c \leq x_2$ and $f'(c) = 0$. Hence

$$f'_+(x_1) - f'_+(x_2) = f'_+(c) - f'_+(x_2) \leq 2\omega(x_2 - c) \leq 2\omega(x_2 - x_1).$$

The case $x_1 \in U$, $x_2 \in \mathbb{R} \setminus U$ is analogous. $\square$

Lemma 4.4. *Let X be a Hilbert space, $a \in X$, $r > 0$ and $\omega \in \mathcal{M}$. Suppose that* $\liminf_{t \to 0+} \omega(t)/t > 0$. *Then there exists $b \in C^{1,\omega}(X)$ such that $0 \leq b \leq 1$,* $\operatorname{supp} b \subset U(a, 2r)$ *and $b = 1$ on $U(a, r)$.*

Proof. Set $g(x) := \|x - a\|^2$, $x \in X$, and $\varphi(t) := t$, $t \in [0, \infty)$. It is well known that $g \in C^{1,\varphi}(X)$, g is Lipschitz on $U := U(a, 2r)$ and that we can find $f \in C^{1,\varphi}(\mathbb{R})$ such that $0 \leq f \leq 1$, $\operatorname{supp} f \subset (-1, 4r^2)$ and $f = 1$ on $[0, r^2]$.

Set $b = f \circ g$. Then clearly $0 \leq b \leq 1$, $\operatorname{supp} b \subset U$ and $b = 1$ on $U(a, r)$. By Fact 4.1 and [HJ, Proposition 128, p. 59] we have $b \restriction_U \in C^{1,\varphi}(U)$. Hence $b \restriction_U \in C^{1,\omega}(U)$ by (3). Since $\operatorname{supp} b \subset U$, we easily obtain that $b \in C^{1,\omega}(X)$. $\square$

Theorem 4.5. *Let X be a Hilbert space, $G \subset X$ an open set, $f_1, f_2 : G \to \mathbb{R}$ and $\omega \in \mathcal{M}$. Suppose that f_1, f_2 are continuous, $f_1 \leq f_2$, $\liminf_{t\to 0+} \omega(t)/t > 0$ and the following condition holds.*

- *For every $a \in G$ there exist $r, C > 0$ such that $U := U(a, r) \subset G$, $f_1 \restriction_U$ is semiconvex with modulus $C\omega$ and $f_2 \restriction_U$ is semiconcave with modulus $C\omega$.*

Then there exists $f \in C^{1,\omega}_{\mathrm{loc}}(G)$ such that $f_1 \leq f \leq f_2$.

Proof. We claim that for every $a \in G$ there exists $r_a > 0$ and $F_a \in C^{1,\omega}(X)$ such that $U(a, r_a) \subset G$ and

$$f_1(x) \leq F_a(x) \leq f_2(x), \quad x \in U(a, r_a). \tag{6}$$

To prove this, choose $a \in G$. By the assumptions and Proposition 2.5 there exists $r_a > 0$ such that $U := U(a, 2r_a) \subset G$, f_1, f_2 are Lipschitz on U, $f_1 \restriction_U \in SC^\omega(U)$ and $f_2 \restriction_U \in -SC^\omega(U)$. By Lemma 4.4 there exists $b \in C^{1,\omega}(X)$ such that $b \geq 0$, $\operatorname{supp} b \subset U$ and $b = 1$ on $U(a, r_a)$. For $i = 1, 2$ we define a function

$$b_i(x) := \begin{cases} b(x)f_i(x), & x \in U, \\ 0, & x \in X \setminus U. \end{cases}$$

Then $b_1 \leq b_2$, $\operatorname{supp} b_1 \subset U$, $\operatorname{supp} b_2 \subset U$ and b_1, b_2 are continuous. By Lemma 4.2 we have $b_1 \restriction_U \in SC^\omega(U)$ and $-b_2 \restriction_U \in SC^\omega(U)$. Thus $b_1 \in SC^\omega(X)$ and $-b_2 \in SC^\omega(X)$ by Lemma 4.3. Hence, by Corollary 3.2, there exists $F_a \in C^{1,\omega}(X)$ such that $b_1 \leq F_a \leq b_2$. Then (6) holds and we are done.

Since $\{U(a, r_a) : a \in G\}$ forms an open cover of G, we can, by [T, Theorem 3] and [KK, Lemma 2.5], find a locally finite C^∞-partition of unity $\mathcal{Q}$ on G subordinated to $\{U(a, r_a) : a \in G\}$. So, for every $q \in \mathcal{Q}$ there exists $a_q \in G$ such that $\operatorname{supp} q \subset U(a_q, r_{a_q})$. Set

$$f(x) := \sum_{q\in\mathcal{Q}} q(x)F_{a_q}(x), \quad x \in G.$$

It follows from [HJ, Proposition 71, p. 29] that q, q' and F_{a_q} are locally Lipschitz whenever $q \in \mathcal{Q}$. Hence $q \cdot F_{a_q} \in C^{1,\omega}_{\mathrm{loc}}(X)$, $q \in \mathcal{Q}$, by (3) and [HJ, Proposition 129, p. 59]. Since $\mathcal{Q}$ is locally finite, it follows that f is well defined and $f \in C^{1,\omega}_{\mathrm{loc}}(G)$. Finally, for every $x \in G$ we have $\sum_{q\in\mathcal{Q}} q(x)f_i(x) = f_i(x)$, $i = 1, 2$, and $q(x)f_1(x) \leq q(x)F_{a_q}(x) \leq q(x)f_2(x)$, $q \in \mathcal{Q}$. Thus $f_1 \leq f \leq f_2$. □

Theorem 4.5 holds also for some non-Hilbertian Banach spaces as noted in the following remark.

Remark 4.6. If, in the Theorem 4.5, X is a Banach space and G admits locally finite $C^{1,\omega}$-partitions of unity, then the proof works essentially the same. Moreover, it can be proved that if a Banach space X admits an equivalent norm with modulus of smoothness of power type 2 (e.g. $X = \ell^p$ for $p \geq 2$) and $\omega \in \mathcal{M}$ is such that $\liminf_{t\to 0+} \omega(t)/t > 0$, then every open $G \subset X$ admits locally finite $C^{1,\omega}$-partitions of unity. The proof of this fact is quite technical and thus we restricted ourself to the case of a Hilbert space.

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