ROUGH MAXIMAL BILINEAR SINGULAR INTEGRALS

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ABSTRACT. We study the rough maximal bilinear singular integral

$$T^*_\Omega(f,g)(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}} f(x-y)g(x-z)dydz \right|,$$

where Ω is a function in $L^\infty(\mathbb{S}^{2n-1})$ with vanishing integral. We prove it is bounded from $L^p \times L^q \to L^r$, where $1 < p,q < \infty$ and $1/r = 1/p + 1/q$. We also discuss results for $\Omega \in L^s(\mathbb{S}^{2n-1})$, $1 < s < \infty$.

CONTENTS

1. INTRODUCTION

In this paper, we study the rough maximal bilinear singular integral. Singular integral theory was initiated in the seminal work of Calderón and Zygmund [1]. Bilinear singular operators were introduced by Coifman and Meyer in [4] and the theory was later developed by Grafakos and Torres in [11]. The boundedness of the smooth maximal multilinear singular integrals was obtained also by Grafakos and Torres in [10] via Cotlar type inequality. All these result were obtained for operators with smooth kernels, while the problem of boundedness of the bilinear rough singular integral remained open. Recently, in the paper of Grafakos, He and Honzík [9], the following was proved: For an operator T_Ω defined as

$$T_\Omega(f,g)(x) = \text{ p.v. } \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(y,z)|^{-2n} \Omega((y,z)/|(y,z)|) f(x-y)g(x-z)\, dydz$$

where Ω is a function in $L^q(\mathbb{S}^{2n-1})$ with vanishing integral it holds that for $q = \infty$ we obtain boundedness for T_Ω from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ where $p_1, p_2 \in$

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$(1,\infty)$ and $1/p = 1/p_1 + 1/p_2$. Also, for $q = 2$ it was proved that T_Ω is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and these two results were interpolated. These results were obtained using a bilinear technique based on tensor-type wavelet decomposition which we will briefly describe later on. In this paper we build on these results and our goal is to describe similar properties of the maximal version of the operator defined above.

2. Notation and results

To fix notation, we assume $q \in (1,\infty]$ and we let Ω in $L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$, where $\mathbb{S}^{2n-1}$ is the unit sphere in $\mathbb{R}^{2n}$. In this manuscript we will be working with the bilinear singular integral operator associated with Ω by

$$T^*_\Omega(f,g)(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} K(y,z) f(x-y) g(x-z) \, dy dz, \right| \tag{1}$$

where f, g are functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$,

$$K(y,z) = \Omega((y,z)')/|(y,z)|^{2n},$$

and $x' = x/|x|$ for $x \in \mathbb{R}^{2n}$.

For $q \in [1,\infty]$ we denote q' the conjugate index. We denote the set of positive integers by $\mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Finally, we adhere to the standard convention to denote by C a constant that depends only on inessential parameters of the problem.

Let us now state the main results of this paper.

Theorem 1. *For all $n \geq 1$, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, then, for T^*_Ω defined in* (1)*, we have*

$$\|T^*_\Omega\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C \|\Omega\|_\infty \tag{2}$$

whenever $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.

For more general Ω, we get the following:

Theorem 2. *For all $n \geq 1$, if $\Omega \in L^2(\mathbb{S}^{2n-1})$, then, for T^*_Ω defined in* (1)*, we have*

$$\|T^*_\Omega\|_{L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)} \leq C \|\Omega\|_2. \tag{3}$$

These two results give by interpolation:

Corollary 3. *For all $n \geq 1$, if $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $q \in (2,\infty)$, then, for T^*_Ω defined in* (1)*, we have*

$$\|T^*_\Omega\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \tag{4}$$

whenever $1/p = 1/p_1 + 1/p_2$ and the point $(1/p_1, 1/p_2)$ lies inside the quadrilateral with vertices $(1/q, 1/q)$, $(1/q, 1-1/q)$, $(1-1/q, 1-1/q)$ and $(1-1/q, 1/q)$.

Let us now give a brief introduction to wavelets which are the essential tool in the paper [9] and will be used in this manuscript also.

The wavelet system is a form of a complete orthonormal system for $L^2(\mathbb{R}^n)$. For our purposes we need product type smooth wavelets with compact supports, their existence is due to Daubechies [5] and can also be found in Meyer's book [13].

The construction of such objects can be found in Triebel [15] and the existence of a wavelet orthonormal base of $L^2\left(\mathbb{R}^{2n}\right)$ is described by the following statement.

Lemma 4. *For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, for $0 \le \alpha \le k$ we have $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$, and, if Ψ^G is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1)\cdots\psi_{G_{2n}}(x_{2n})$$

for $G = (G_1,\dots,G_{2n})$ in the set

$$\mathcal{I} := \Big\{(G_1,\dots,G_{2n}) : \ G_i \in \{F,M\}\Big\},$$

then the family of functions

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}} \left[\Big\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\Big\} \cup \bigcup_{\lambda=0}^{\infty} \Big\{2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}) : \ G \in \mathcal{I}\setminus\{(F,\dots,F)\}\Big\}\right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1,\dots,x_{2n})$.

Le us now describe the basic decomposition of the kernel. Now we fix a smooth function α in $\mathbb{R}^+$ such that $\alpha(t) = 1$ for $t \in (0,1]$, $\alpha(t) \in (0,1)$ for $t \in (1,2)$ and $\alpha(t) = 0$ for $t \in [2,\infty)$. For $(y,z) \in \mathbb{R}^{2n}$ and $j \in \mathbb{Z}$ we introduce the function

$$\beta_j(y,z) = \alpha(2^{-j}|(y,z)|) - \alpha(2^{-j+1}|(y,z)|).$$

We write $\beta = \beta_0$ and we note that this is a function supported in $[1/2,2]$. We denote Δ_j the Littlewood-Paley operator $\Delta_j f = \mathcal{F}^{-1}(\beta_j\widehat{f})$. Here and throughout this paper $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which is defined via $\mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i x\cdot\xi}d\xi = \widehat{g}(-x)$, where $\widehat{g}$ is the Fourier transform of g.

We decompose the kernel K as follows: we denote $K^i = \beta_i K$ and we set $K^i_j = \Delta_{j-i}K^i$ for $i,j \in \mathbb{Z}$. Then we write

$$K = \sum_{j=-\infty}^{\infty} K_j,$$

where

$$K_j = \sum_{i=-\infty}^{\infty} K^i_j.$$

We also denote $m_j = \widehat{K_j}$.

Then the operator can be written as

$$T^*(f,g)(x) = \sup_{\varepsilon>0}\sum_j \int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} K_j(y,z)f(x-y)g(x-z)\,dydz.$$

We have the following lemma whose proof is known (see for instance [6]) and is omitted.

Lemma 5. *Given $q \in (1,\infty]$, $\Omega \in L^q(\mathbb{S}^{2n-1})$, $\delta \in (0,1/q')$ and $\vec{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^{2n}$ we have*

$$|\widehat{K^0}(\vec{\xi}\,)| \le C\|\Omega\|_{L^q}\min(|\vec{\xi}\,|,|\vec{\xi}\,|^{-\delta})$$

and for all multiindices α in $\mathbb{Z}^{2n}$ with $\alpha \neq 0$ we have

$$|\partial^{\alpha}\widehat{K^0}(\vec{\xi}\,)| \le C_{\alpha}\|\Omega\|_{L^q}\min(1,|\vec{\xi}\,|^{-\delta}).$$

We recall that M is the Hardy-Littlewood maximal function defined for locally integrable functions f as

$$Mf(x) = \sup_{r>0}\frac{1}{|B(0,r)|}\int_{B(0,r)}|f(x-y)|\,dy,$$

where $x \in \mathbb{R}^n$.

3. From T^* to $T^{\#}$

In this section we will show that to obtain the main results of this paper we can equivalently instead of the operator T^* use the operator $T^{\#}$ defined as follows. For functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ let us define

$$T^{\#}(f,g)(x) = \sup_{j\in\mathbb{Z}}\left|\sum_{i>j}\int_{\mathbb{R}^{2n}}K^i(y,z)\,f(x-y)\,g(x-z)\,dydz\right|.$$

As already mentioned the above defined operator will be the essential tool that we will be working with in this text.

The following two propositions is will give us immediately the statement of Theorem 2.

Proposition 6. *Let $\Omega \in L^2(\mathbb{S}^{2n-1})$, and for $j \in \mathbb{Z}$ consider the bilinear operator*

$$T_j^{\#}(f,g)(x) = \sup_{k\in\mathbb{Z}}\left|\sum_{i>k}\int_{\mathbb{R}^{2n}}K_j^i(y,z)f(x-y)g(x-z)dydz\right|.$$

If $j \ge 0$, then $T_j^{\#}$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^2}2^{-\delta j}$, where δ is a fixed positive constant.

Furthermore, we can define an operator

$$\widetilde{T}^{\#}(f,g)(x) = \sup_{j\in\mathbb{Z}}\left|\sum_{i>j}\int_{\mathbb{R}^{2n}}\sum_{\gamma<0}K_{\gamma}^i(y,z)\,f(x-y)\,g(x-z)\,dydz\right|.$$

Clearly

$$T^{\#}(f,g)(x) \le \widetilde{T}^{\#}(f,g)(x) + \sum_{j\ge 0}T_j^{\#}(f,g)(x).$$

We have the following

Proposition 7. *For $\Omega \in L^2(\mathbb{S}^{2n-1})$, and for $1 < p_1,\, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$ the operator $\widetilde{T}^{\#}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. Let us consider the kernel

$$\widetilde{K} = \sum_{i\in\mathbb{Z}}\sum_{\gamma<0}K_{\gamma}^i.$$

It is easy to check that this is a smooth Calderón-Zygmund convolution kernel and therefore we immediately get from the Cotlar inequality [10] the boundedness of the maximal operator $T^*_{\widetilde{K}}$. Next, we write

$$\begin{aligned}\widetilde{T}^{\#}(f,g)(x) &= \sup_{j\in\mathbb{Z}}\left|\sum_{i>j}\int_{\mathbb{R}^{2n}}\sum_{\gamma<0}K^i_\gamma(y,z)\,f(x-y)\,g(x-z)\,dydz\right| \\ &\leq T^*_{\widetilde{K}}(f,g)(x)+ \\ &\sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^{2n}}\left(\chi_{\mathbb{R}^{2n}\setminus B(0,2^j)}\widetilde{K}-\sum_{i>j}\sum_{\gamma<0}K^i_\gamma\right)(y,z)\,f(x-y)\,g(x-z)\,dydz\right|.\end{aligned}$$

Standard calculation shows that the error term

$$\left|\chi_{B(0,2^j)}\widetilde{K}-\sum_{i>j}\sum_{\gamma<0}K^i_\gamma\right|$$

is dominated by $C_N\frac{2^{-2nj}}{1+|2^{-j}x|^N}$ for every $N>0$ and therefore

$$\widetilde{T}^{\#}(f,g)(x)\leq T^*_{\widetilde{K}}(f,g)(x)+CMf(x)Mg(x)$$

and the proof is finished. $\square$

Next lemmas are important tools for the forgoing considerations. They allow us to work with the operator $T^{\#}$ instead of T^*. For $\Omega\in L^\infty\left(\mathbb{S}^{2n-1}\right)$ the following estimate holds.

Lemma 8. *Let $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$, where $p,q\in(1,\infty)$ and $r\in[1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Then*

$$\|T^*(f,g)\|_r\leq\left\|T^{\#}(f,g)\right\|_r+C\,\|Mf\|_p\,\|Mg\|_q\,.$$

The simple proof is omitted, instead we will show that the same results holds for more general Ω. We first state and prove the estimate for a directional bilinear maximal function and then use the method of rotations. We start with a definition of the directional maximal functions.

$$M_\alpha(f)(x)=\sup_{t>0}\frac{1}{t}\int_0^t|f(x-\alpha y)|\,dy,$$

$$M_{\alpha,\beta}(f,g)(x)=\sup_{t>0}\frac{1}{t}\int_0^t|f(x-\alpha y)\,g(x-\beta y)|\,dy,$$

where $\alpha,\beta\in\mathbb{S}^{n-1}$, $x\in\mathbb{R}^n$ and f,g are locally integrable functions on $\mathbb{R}^n$.

The following lemma describes boundedness of the directional bilinear maximal function.

Lemma 9. *Let $p,q\in(1,\infty)$ and $r\in(1,\infty)$ and $\alpha,\beta\in\mathbb{S}^{n-1}$. Then*

$$M_{\alpha,\beta}:L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^r(\mathbb{R}^n)\,.$$

Proof:

$$\left\|M_{\alpha,\beta}(f,g)\right\|_r=\left(\int_{\mathbb{R}^n}\left[\sup_{t>0}\int_0^t|f(x-\alpha y)\,g(x-\beta y)|\,\frac{dy}{t}\right]^r dx\right)^{\frac{1}{r}}$$

$$\leq\left(\int_{\mathbb{R}^n}\left[\sup_{t>0}\left(\frac{1}{t}\int_0^t|f(x-\alpha y)|^s\,dy\right)^{\frac{1}{s}}\sup_{t>0}\left(\frac{1}{t}\int_0^t|g(x-\beta y)|^z\,dy\right)^{\frac{1}{z}}\right]^r dx\right)^{\frac{1}{r}},$$

where $s=\frac{p}{r}, z=\frac{q}{r}$ and the inequality follows from Hölder inequality since $\frac{r}{p}+\frac{r}{q}=1$. The last expression can be written as

$$\left(\int_{\mathbb{R}^n}\left[M_\alpha\left(|f|^s\right)(x)\right]^{\frac{r}{s}}\left[M_\beta\left(|g|^z\right)(x)\right]^{\frac{r}{z}}dx\right)^{\frac{1}{r}}$$

and $f\in L^p$ implies $f^s\in L^{\frac{p}{s}}$ which implies $Mf^s\in L^{\frac{p}{s}}$, therefore $(Mf^s)^{\frac{1}{s}}\in L^p$ and $(Mf^s)^{\frac{r}{s}}\in L^{\frac{p}{r}}$. Then again from Hölder inequality and also Hardy-Littlewood maximal theorem we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[M_\alpha\left(|f|^s\right)(x)\right]^{\frac{r}{s}}\left[M_\beta\left(|g|^z\right)(x)\right]^{\frac{r}{z}}dx\right)^{\frac{1}{r}}&\leq\left(\left\|M_\alpha\left(|f|^s\right)^{\frac{r}{s}}\right\|_{\frac{p}{r}}\left\|M_\beta\left(|g|^z\right)^{\frac{r}{z}}\right\|_{\frac{q}{r}}\right)^{\frac{1}{r}}\\&=\left(\left\|M_\alpha|f|^s\right\|_{\frac{p}{s}}\right)^{\frac{1}{s}}\left(\left\|M_\beta|g|^z\right\|_{\frac{q}{z}}\right)^{\frac{1}{z}}\\&\leq C\left(\|f^s\|_{\frac{p}{s}}\right)^{\frac{1}{s}}\left(\|g^z\|_{\frac{q}{z}}\right)^{\frac{1}{z}}\\&=C\,\|f\|_p\,\|g\|_q.\end{aligned}$$

□

In the following we show boundedness of another type of maximal function.

Lemma 10. *Let p,q,r be like in the previous lemma and define*

$$M_\Omega(f,g)(x)=\sup_{R>0}\left|\int_{\mathbb{R}^{2n}}H_\Omega^R(y,z)\,f(x-y)\,g(x-z)\,dydz\right|$$

where $\Omega\in L^1\left(\mathbb{S}^{2n-1}\right),\Omega\geq 0$ and

$$H_\Omega^R=\left|\Omega\left(\frac{(y,z)}{|(y,z)|}\right)\right|R^{-2n}\chi_{B(0,R)}(y,z),$$

where $x,y,z\in\mathbb{R}^n$. Then $\|M_\Omega(f,g)\|_r\leq\|\Omega\|_1\,\|f\|_p\,\|g\|_q$.

Proof: We can express and estimate the term $M_\Omega(f,g)$ as follows

$$M_\Omega(f,g)(x)=\sup_{R>0}\left|\int_{\mathbb{S}^{2n-1}}\Omega\left(u'\right)R^{-2n}\int_0^R t^{2n-1}f(x-tu_1)\,g(x-tu_2)\,dtdu\right|,$$

where $u=(u_1,u_2)$ and we recall $u'=\frac{u}{|u|}$ for $u\in\mathbb{R}^{2n}$. Since $\frac{t^{2n-1}}{R^{2n-1}}\leq 1$ we get

$$M_\Omega(f,g)(x)\leq\sup_{R>0}\left|\frac{1}{R}\int_{\mathbb{S}^{2n-1}}\Omega\left(u'\right)\int_0^R f(x-tu_1)\,g(x-tu_2)\,dtdu\right|$$

$$\leq \int_{\mathbb{S}^{2n-1}} \Omega\left(u'\right) M_u\left(f,g\right)(x)\, du$$

where $M_u\left(f,g\right)(x) = \sup_{R>0} \frac{1}{R}\int_0^R \left|f\left(x-tu_1\right)g\left(x-tu_2\right)\right| dt$. According to the previous lemma $M_u : L^p \times L^q \to L^r$ therefore $\left\|M_\Omega\left(f,g\right)\right\|_r \leq \left\|\Omega\right\|_1 \left\|f\right\|_p \left\|g\right\|_q$. $\square$

The previous lemma gives boundedness of the opertor M_Ω, which is a necessary tool for the transition from T^* to $T^{\#}$, but only when $r>1$. In order to prove the Theorem 2, we need to do this precisely when $r=1$. We use interpolation to extend the results on M_Ω to the region where $r\leq 1$. While in the rest of the paper we mostly use $r=1$, we include the whole result for general interest.

Lemma 11. *Let $r \in (1/2,\infty)$ and $p,q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right), \Omega \geq 0$ where $s\in(1,\infty)$ is such that $\frac{s}{2s-1}<q$. Then $M_\Omega : L^p\left(\mathbb{R}^n\right)\times L^q\left(\mathbb{R}^n\right) \to L^r\left(\mathbb{R}^n\right)$.*

Proof. Without loss of generality we can assume $\left\|\Omega\right\|_s = 1$. We decompose Ω as $\Omega = \sum_{i\geq 0}\Omega_i$ where $\Omega_i\left(x'\right) = \Omega\left(x'\right)\chi_{E_i}(x)$ and $E_i = \left\{x\in\mathbb{R}^{2n} : \Omega\left(x'\right) \in \left(2^i, 2^{i+1}\right]\right\}$ for $i>0$ and $E_0 = \left\{x \in \mathbb{R}^{2n} : \Omega\left(x'\right)\in[0,2]\right\}$. Then $\Omega_i \in L^1\left(\mathbb{S}^{2n-1}\right)$ and $\left\|\Omega_i\right\|_1 \leq 2^{-i\frac{s}{s'}}$ since from Hölder inequality we have

$$\left\|\Omega_i\right\|_1 = \int_{\mathbb{S}^{2n-1}} \Omega\left(x'\right)\chi_{E_i}(x)\,dx \leq \left(\int_{E_i} 1dx\right)^{\frac{1}{s'}} \left(\int_{\mathbb{S}^{2n-1}} \Omega^s\left(x'\right)dx\right)^{\frac{1}{s}}$$

$$= \left|\mathrm{supp}\Omega_{\mathrm{i}}\right|^{\frac{1}{s'}} K \leq K2^{-i\frac{s}{s'}}$$

where the last inequality follows from the fact that $\int_{\mathrm{supp}\Omega_{\mathrm{i}}}\Omega^s \leq 1$ and $\int_{\mathrm{supp}\Omega_{\mathrm{i}}}\Omega^s \cong 2^{is}\left|\mathrm{supp}\Omega_{\mathrm{i}}\right|$.

Now let $\varepsilon>0$. We will use the multilinear interpolation (Theorem 7.2.2 in [8]). Let

- $(p_{11},p_{12},q_1) = \left((1+\varepsilon)^2, \frac{(1+\varepsilon)^2}{\varepsilon}, \cdot\right)$
- $(p_{21},p_{22},q_2) = \left(\frac{(1+\varepsilon)^2}{\varepsilon}, (1+\varepsilon)^2, \cdot\right)$
- $(p_{31},p_{32},q_3) = \left((1+\varepsilon)^2, (1+\varepsilon)^2, \cdot\right)$,

where q_i is such that $\frac{1}{p_{i1}}+\frac{1}{p_{i2}}=\frac{1}{q_i}$ and therefore $q_1=q_2=\frac{1}{1+\varepsilon}$ and $q_3=\frac{(1+\varepsilon)^2}{2}$.

Then

- $\left\|M_{\Omega_i}\left(f,g\right)\right\|_{L^{q_1}} \leq 2^{-i\frac{s}{s'}}\left\|f\right\|_{p_{11}}\left\|g\right\|_{p_{12}}$
- $\left\|M_{\Omega_i}\left(f,g\right)\right\|_{L^{q_2}} \leq 2^{-i\frac{s}{s'}}\left\|f\right\|_{p_{21}}\left\|g\right\|_{p_{22}}$
- $\left\|M_{\Omega_i}\left(f,g\right)\right\|_{L^{q_3}} \leq 2^{i}\left\|f\right\|_{p_{31}}\left\|g\right\|_{p_{32}}$

Define

$$\left(\frac{1}{p_1},\frac{1}{p_2}\right) = \left(\frac{\mu_1}{p_{11}}+\frac{\mu_2}{p_{21}}+\frac{\mu_3}{p_{31}}, \frac{\mu_1}{p_{21}}+\frac{\mu_2}{p_{22}}+\frac{\mu_3}{p_{32}}\right)$$

$$= \left(\frac{\mu_1+\mu_3+\mu_2\varepsilon}{(1+\varepsilon)^2}, \frac{\mu_2+\mu_3+\mu_1\varepsilon}{(1+\varepsilon)^2}\right)$$

where $\mu_i \in (0,1)$ and $\mu_1+\mu_2+\mu_3=1$ and also

$$\frac{1}{q}=\frac{\mu_1}{q_1}+\frac{\mu_2}{q_2}+\frac{\mu_3}{q_3}$$
$$\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}.$$

We note that q from above is dependent on ε and μ_i. Then $\|M_{\Omega_i}(f,g)\|_{L^{q,r_1}} \leq 2^{-i\left(\frac{s}{s'}+\mu_3\left(-\frac{s}{s'}-1\right)\right)}$ and we need the exponent to be greater than 0, therefore $s(1-\mu_3) > 1$ which implies $\mu_3 < \frac{1}{s'}$. Then

$$\frac{1}{q} < \frac{(1+\varepsilon)^3+\frac{1}{s'}\left(2-(1+\varepsilon)^3\right)}{(1+\varepsilon)^2}.$$

Therefore if we choose (p_1,p_2,q) such that they satisfy the assumptions and $\frac{1}{q} < \frac{2s-1}{s}$ then we can find ε small enough so that the above inequality holds. So we get

$$\|M_\Omega(f,g)\|_q \leq \sum_i \|M_{\Omega_i}(f,g)\|_q \leq C\|f\|_{p_1}\|g\|_{p_2}.$$

□

Now we can formulate the version of Lemma 8 for $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right)$ where s is like in the previous Lemma. Again, this lemma is most important in the situation where $p=q=2$ and $r=1$, but we state it in full generality.

Lemma 12. *Let $r \in (1/2,\infty)$ and $p,q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right)$ where $s \in (1,\infty)$ is such that $\frac{s}{2s-1} < q$. Then*

$$\|T^*(f,g)\|_r \leq \left\|T^{\#}(f,g)\right\|_r + C\left\|M_{|\Omega|}(f,g)\right\|_r.$$

Proof. Let us define functions $K_\varepsilon(x,y) = K(x,y)\chi_{[\varepsilon,\infty)}(|(x,y)|)$ and $\tilde{K}_\varepsilon(x,y) = K(x,y)\left(1-\alpha\left(\frac{1}{\varepsilon}|(x,y)|\right)\right)$. Then

$$\begin{aligned} T^*(f,g)(x) \leq {} & \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^{2n}} K_\varepsilon(x-y,x-z)f(y)g(z) - \tilde{K}_{\varepsilon_j}(x-y,x-z)f(y)g(z)\,dydz\right| \\ & + \sup_{i\in\mathbb{Z}}\left|\int \tilde{K}_i(x-y,x-z)f(y)g(z)\,dydz\right|, \end{aligned}$$

where in the first part we take j such that $\varepsilon \approx \varepsilon_j = 2^j$. The second part equals $T^{\#}(f,g)(x)$ since it can be written as

$$\begin{aligned} & \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^{2n}} K(x-y,x-z)\left(1-\alpha\left(2^{-j}|(x-y,x-z)|\right)\right)dydz\right| \\ = {} & \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^{2n}} \sum_{i>j} K^i(x-y,x-z)\left(1-\alpha\left(2^{-j}|(x-y,x-z)|\right)\right)dydz\right|. \end{aligned}$$

Then we can further estimate the first part with

$$\sup_{\varepsilon>0}\left|\int_{\mathbb{R}^{2n}} H^\varepsilon_{|\Omega|}(y,z)f(x-y)g(x-z)\,dydz\right|.$$

□

4. BOUNDEDNESS: A SPECIAL CASE

Now we prove the Theorem 2. In view of Proposition 6, Theorem 2 will be a consequence of the following proposition.

Proposition 13. *Given $q \in [2,\infty]$ and $\delta \in (0, 1/8q')$, then for any $j \in \mathbb{N}_0$, the operator $T_j^{\#}$ maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^q} 2^{-\delta j}$.*

To obtain this estimate, we follow the proof from the paper [9] and first decompose the symbol into dyadic pieces, estimate them separately, and then use orthogonality arguments to put them back together which is the point where the arguments will differ from the paper [9]. Let us remind certain properties of the symbol $\widehat{K_j^0}$ which we denote $m_{j,0}$. The classical estimates show that

$$\|m_{j,0}\|_{L^\infty} = \|\widehat{K_j^0}\|_{L^\infty} \le C\|\Omega\|_{L^q} 2^{-\delta j}, \qquad \delta \in \left(0, 1/q'\right), \tag{5}$$

while for $q \in [2,\infty]$ it holds

$$\|m_{j,0}\|_{L^2} = \|\beta_j(\widehat{\beta_0 K})\|_{L^2} \le C\|\widehat{\beta_0 K}\|_{L^2} \le C\|\Omega\|_{L^2} \le C\|\Omega\|_{L^q}. \tag{6}$$

We observe that for the case $i \neq 0$ we have the identity $m_{j,i} = \widehat{K_j^i} = m_{j,0}(2^i \cdot)$ from the homogeneity of the symbol, and thus $m_{j,i}$ also lies in $L^2\left(\mathbb{R}^{2n}\right)$.

In this manuscript we will be using the wavelet transform of $m_{j,0}$ taking product wavelets described above in Lemma 4 with compact supports and M vanishing moments, where M is a large number to be determined later. We choose generating functions with support diameter approximately 1. The wavelets with the same dilation factor 2^λ have some bounded overlap N independent of λ. With

$$\Psi_{\vec{\mu}}^{\lambda,G}(\vec{x}) = 2^{\lambda n}\Psi^G\left(2^\lambda \vec{x} - \vec{\mu}\right),$$

where $\vec{x} \in \mathbb{R}^{2n}$, we have the next lemma which is another tool to enable us to work with the wavelet technique, the proof can be found in [9].

Lemma 14. *Using the preceding notation, for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{N}_0$ we have*

$$|\langle \Psi_{\vec{\mu}}^{\lambda,G}, m_{j,0}\rangle| \le C\|\Omega\|_{L^q} 2^{-\delta j} 2^{-(M+1+n)\lambda}, \tag{7}$$

where M is the number of vanishing moments of ψ_M and δ is as in (5).

Again as in the paper [9] the wavelets sharing the same generation index λ may be organized into $C_{n,M,N}$ groups so that members of the same group have disjoint supports and are of the same product type, i.e., they have the same index $G \in \mathcal{I}$. For $1 \le \kappa \le C_{n,M,N}$ we denote by $D_{\lambda,\kappa}$ one of these groups consisting of wavelets whose supports have diameters about $2^{-\lambda}$. We now have that the wavelet expansion

$$m_{j,0} = \sum_{\substack{\lambda \ge 0 \\ 1 \le \kappa \le C_{n,M,N}}} \sum_{\omega \in D_{\lambda,\kappa}} a_\omega \omega$$

and ω all have disjoint supports within the group $D_{\lambda,\kappa}$. We recall the following estimates: For the sequence $a=\{a_\omega\}$ we get $\|a\|_{\ell^2}\le C$, if we set $b_\omega=\|a_\omega\omega\|_{L^\infty}$, we have

$$\|\{b_\omega\}_{\omega\in D_{\lambda,\kappa}}\|_{\ell^2}\le C\|\Omega\|_{L^2}2^{n\lambda}.$$

Then we also have

$$\|\{b_\omega\}_{\omega\in D_{\lambda,\kappa}}\|_{\ell^\infty}\le C\|\Omega\|_{L^q}2^{-\delta j-(M+1)\lambda}. \tag{8}$$

Now, we split the group $D_{\lambda,\kappa}$ into three parts. Recall the fixed integer j in the statement of Proposition 13. Let us also assume that $j\ge 100\sqrt{n}$ since for $j<100\sqrt{n}$, Proposition 13 is an easy consequence of Proposition 6. We define sets

$$D^1_{\lambda,\kappa}=\Big\{\omega\in D_{\lambda,\kappa}:a_\omega\neq 0,\ \operatorname{supp}\omega\subset\{(\xi_1,\xi_2)\in\mathbb{R}^n\times\mathbb{R}^n:\ 2^{-j}|\xi_1|\le|\xi_2|\le 2^j|\xi_1|\}\Big\},$$

$$D^2_{\lambda,\kappa}=\Big\{\omega\in D_{\lambda,\kappa}:a_\omega\neq 0,\ \operatorname{supp}\omega\cap\{(\xi_1,\xi_2)\in\mathbb{R}^n\times\mathbb{R}^n:\ 2^{-j}|\xi_1|\ge|\xi_2|\}\neq\emptyset\Big\},$$

and

$$D^3_{\lambda,\kappa}=\Big\{\omega\in D_{\lambda,\kappa}:a_\omega\neq 0,\ \operatorname{supp}\omega\cap\{(\xi_1,\xi_2)\in\mathbb{R}^n\times\mathbb{R}^n:\ 2^{-j}|\xi_2|\ge|\xi_1|\}\neq\emptyset\Big\}.$$

These groups are disjoint for large j. Notice that $D^1_{\lambda,\kappa}\cap D^2_{\lambda,\kappa}=\emptyset$ is obvious. For $D^2_{\lambda,\kappa}$ and $D^3_{\lambda,\kappa}$ the worst case is $\lambda=0$ when we have balls of radius 1 centered at integers, and $D^2_{\lambda,\kappa}\cap D^3_{\lambda,\kappa}=\emptyset$ if j is sufficiently large. We choose $j\ge 100\sqrt{n}$ which works, since if $a_\omega\neq 0$, then ω is supported in an annulus centered at the origin of size about 2^j which is large enough.

We denote, for $\iota=1,2,3$,

$$m^\iota_{j,0}=\sum_{\lambda,\kappa}\sum_{\omega\in D^\iota_{\lambda,\kappa}}a_\omega\omega,$$

and define

$$m^\iota_j=\sum_{k=-\infty}^{\infty}m^\iota_{j,k}$$

with $m^\iota_{j,k}(\vec{\xi}\,)=m^\iota_{j,0}(2^k\vec{\xi}\,)$. We prove boundedness for each piece m^1_j,m^2_j,m^3_j. We call m^1_j the diagonal part of m_j and m^2_j,m^3_j the off-diagonal parts of $m_j=\widehat{K_j}$.

5. The diagonal part

We first deal with the first group $D^1_{\lambda,\kappa}$. Using the same arguments like in the paper [9] we will obtain (omitting details) again the similar estimate of $T_{m^1_{j,k}}$, where

$$T_{m^1_{j,k}}(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}m^1_{j,k}(y,z)\widehat{f}(y)\widehat{g}(z)e^{2\pi i(y+z)\cdot x}dydz.$$

We denote by f_k the function whose Fourier transform is $\widehat{f}(2^{-k}\xi_1)$ and $E_{j,k}=\{\xi_1\in\mathbb{R}^n: c_1 2^k\le|\xi_1|\le c_2 2^{j+k}\}$, where c_1,c_2 are suitable constants such that $\|T_{m^1_{j,0}}(f,g)\|_{L^1}=\|T_{m^1_{j,0}}(\mathcal{F}^{-1}\left(\widehat{f}\chi_{\{c_1 2^j\le|\xi_1|\le c_2 2^{j+1}\}}\right),\mathcal{F}^{-1}\left(\widehat{g}\chi_{\{c_1 2^j\le|\xi_1|\le c_2 2^{j+1}\}}\right))\|_{L^1}$. Then

$$\|T_{m^1_{j,k}}(f,g)\|_{L^1}=\ 2^{-2kn}\|T_{m^1_{j,0}}(f_k,g_k)(2^{-k}\cdot)\|_{L^1}$$

$$\begin{aligned}
&= 2^{-kn}\|T_{m^1_{j,0}}(f_k,g_k)\|_{L^1}\\
&\leq C\|\Omega\|_{L^q}2^{-kn}2^{-\delta j/5}\|\widehat{f}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2}\|\widehat{g}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2}\\
&= C\|\Omega\|_{L^q}2^{-\delta j/5}\|\widehat{f}\|_{L^2(E_{j,k})}\|\widehat{g}\|_{L^2(E_{j,k})},
\end{aligned}$$

where the inequality is described in detail in [9].

Using this estimate, applying the Cauchy-Schwarz inequality and modifying the first part to get the estimate for our operator we obtain for the diagonal part

$$\begin{aligned}
\|\sup_{\gamma\in\mathbb{Z}}\sum_{k>\gamma}T_{m^1_{j,k}}(f,g)\|_{L^1} &\leq \sum_{k=-\infty}^{\infty}\|T_{m^1_{j,k}}(f,g)\|_{L^1}\\
&\leq C\|\Omega\|_{L^q}j2^{-\delta j/5}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

where again the last inequality was explained in detail in [9]. This completes the estimate of the first piece m^1_j.

6. The off-diagonal parts

We now estimate the off-diagonal parts of the operator, namely

$$\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma}T_{m^2_{j,k}}(f,g)\right|.$$

At first we need to have an approximate size of the Fourier support of

$$T_{m^2_{j,k}}(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}m^2_{j,k}(\alpha,\beta)\hat{f}(\alpha)\hat{g}(\beta)e^{2\pi i(\alpha+\beta)\cdot x}d\alpha d\beta.$$

We will estimate the support of $m^2_{j,k}$ since knowing that we can get the required support of $\widehat{T_{m^2_{j,k}}}(f,g)$ from the following lemma.

Lemma 15. *Let L^∞ function m defined on $\mathbb{R}^n\times\mathbb{R}^n$ be a symbol of a multiplier operator T_m. Suppose $I,J\subset\mathbb{R}^n$ are measurable sets and let* supp $m\subset I\times J$. *Then for $f,g\in\mathcal{S}(\mathbb{R}^n)$ it holds that* supp $T_m(f,g)\subset I+J$.

Proof. Using change of variables we obtain

$$T_m(f,g)(x)=\int_{\mathbb{R}^n}G(\alpha)e^{2\pi i x\cdot\alpha}d\alpha=\check{G}(x),$$

where

$$G(\alpha)=\int_{\mathbb{R}^n}m(\alpha-\beta,\beta)\hat{f}(\alpha-\beta)\hat{g}(\beta)d\beta.$$

Therefore $\widehat{T_m}(f,g)(x)=G(x)$ and we need to find the support of G which is:

$G(\alpha)\neq 0$ if and only if there exists $\beta\in\mathbb{R}^n$ such that $m(\alpha-\beta,\beta)\neq 0$, therefore if supp $m\in I\times J$ then supp $G\in I+J$.

□

Now we want to find the approximate size of the support of $m_{j,k}^2$. We first deal with $m_{j,0}^2$ since the general case will follow from the fact that $m_{j,k}^2(x) = m_{j,0}^2\left(2^k x\right)$ and simple modification of the calculation.

It holds that a point $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ is in the support of $m_{j,0}^2$ if there exists $\omega \in D_{\lambda,\kappa}^2$ such that $(x,y) \in \omega$ and

$$\operatorname{supp}\,\omega \cap \left\{(u,v) : 2^{-j}|u| \geq |v|\right\} \cap \left\{(u,v) : 2^{j-1} \leq |(u,v)| \leq 2^{j+1}\right\} \neq \emptyset.$$

The size of the wavelet ω (again in the case $m_{j,0}^2$) can be estimated by 1, therefore by simple calculation we get

$$\operatorname{supp}\, m_{j,0}^2 \subseteq A \times B,$$

where

$$A = \left\{\xi \in \mathbb{R}^n : 2^{j-2} - 1 \leq |\xi| \leq 2^{j+1} + 1\right\}$$

and

$$B = \left\{\xi \in \mathbb{R}^n : -3 - 2^{-j} \leq |\xi| \leq 3 + 2^{-j}\right\}.$$

Therefore

$$\operatorname{supp}\, \widehat{T_{m_{j,0}^2}}(f,g) \subseteq \left\{\xi \in \mathbb{R}^n : 2^{j-3} \leq |\xi| \leq 2^{j+2}\right\}.$$

If we now consider $m_{j,k}^2$ we obtain

$$\operatorname{supp}\, \widehat{T_{m_{j,k}^2}}(f,g) \subseteq \left\{\xi \in \mathbb{R}^n : 2^{j-3-k} \leq |\xi| \leq 2^{j+3-k}\right\}.$$

Now we can proceed to estimating $\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma} T_{m_{j,k}^2}(f,g)\right|$. From the paper [9] we know that for every $f,g \in L^2(\mathbb{R}^n)$ the following estimate holds

$$\left\|\left(\sum_{k\in 5\mathbb{Z}}\left|T_{m_{j,k}^2}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1} \leq C\,\|\Omega\|_{L^q} 2^{-j\delta}\,\|f\|_{L^2}\,\|g\|_{L^2}. \tag{9}$$

For $\mu = 0,\dots,10$ we have

$$T_{m_j^2}^{\#}(f,g) = \sup_{\gamma\in\mathbb{Z}} \sum_{\mu} \left|\sum_{\substack{k>\gamma \\ k\in 10\mathbb{Z}+\mu}} T_{m_{j,k}^2}(f,g)\right|,$$

so we fix μ. Then we have

$$\sup_{\gamma\in\mathbb{Z}}\left|\sum_{\substack{k>\gamma \\ k\in 10\mathbb{Z}+\mu}} T_{m_{j,k}^2}(f,g)\right| \leq \sup_{\beta>0}\left|\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g) - \psi_\beta * \left(\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right)\right|,$$

where ψ_β is a smooth function such that it is equal to 1 on $B\left(0,2^{j-\beta+3}\right)$ and vanishes outside of $B\left(0,2^{j-\beta+10}\right)$, more precisely we have a smooth function ψ such that $\psi \equiv 1$ on $B\left(0,2^{j+3}\right)$, $\psi \equiv 0$ on $B\left(0,2^{j+10}\right)$ and $\psi \in (0,1)$ otherwise. Then we define $\psi_\beta = \psi\left(2^\beta \cdot\right)$.

Then we can further estimate the previous expression with

$$\left|\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right| + \sup_{\beta>0}\left|\psi_\beta * \left(\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right)\right|.$$

The second part can be estimated with maximal function defined as $M_\psi f(x) = \sup_{\beta>0}\left|\psi_\beta * f\right|$, therefore

$$\sup_{\beta>0}\left|\psi_\beta * \left(\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right)(x)\right| \le CM_\psi\left(\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right)(x).$$

If we consider now the L^1 norm of the expression $\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\beta} T_{m_{j,k}^2}(f,g)\right|$, we get

$$\left\|\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma} T_{m_{j,k}^2}(f,g)\right|\right\|_{L^1} \le \left\|\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right\|_{L^1} + C\left\|M_\psi\left(\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right)\right\|_{L^1}$$

The first expression can be estimated as follows. Since the Hörmander condition for multilinear multipliers holds for $m_{j,k}^2$ and $m_{j,k}^2$ is smooth, then $\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)$ is in $L^1(\mathbb{R}^n)$. Also since we have the square function estimate in (9), there exists polynomial Q_j^μ such that

$$\begin{aligned}\left\|\sum_{k\in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g) - Q_j^\mu\right\|_{L^1} &\le \left\|\left(\sum_{k\in 10\mathbb{Z}}\left|T_{m_{j,k}^2}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1}\\ &\le C\|\Omega\|_{L^q} 2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

Therefore $Q_j^\mu = 0$ and

$$\left\|\sum_{10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g)\right\|_{L^1} \le C\|\Omega\|_{L^q} 2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2}.$$

In fact (9) also implies that $\sum_{k\in\mathbb{Z}} T_{m_{j,k}^2}(f,g)$ is in H^1, therefore we can estimate the second expression as

$$\left\|M_\psi\left(\sum_{k\in\mathbb{Z}} T_{m_{j,k}^2}(f,g)\right)\right\|_{L^1} \le \left\|\sum_{k\in\mathbb{Z}} T_{m_{j,k}^2}(f,g)\right\|_{H^1}.$$

This finishes the proof of Proposition 13 and therefore also the proof of the Theorem 2

7. Interpolation

Finally, we want to prove the Theorem 1 and the Corollary 3. We recall Lemma 10 from the article [9], which states that the kernel K_j is a Calderón-Zygmund kernel with ε-Lipschitz constant

$$A_\varepsilon \le C_\varepsilon\|\Omega\|_\infty 2^{|j|\varepsilon},$$

for $\varepsilon\in(0,1)$. We are only using $j\ge 0$ in what follows.

Using the Cotlar inequality from [10], we can therefore get for any combination $1 < p, q < \infty$, $1/r = 1/p + 1/q$ and any $\varepsilon \in (0,1)$ the bound

$$\|T_j^*\|_{L^p \times L^q \to L^r} \le C_{p,q,\varepsilon} \|\Omega\|_\infty 2^{j\varepsilon}.$$

By T_j^* we denote the maximal singular bilinear operator with kernel K_j. Next, we need to observe that

$$\|T_j^\#\|_{L^p \times L^q \to L^r} \le \|T_j^*\|_{L^p \times L^q \to L^r} + C\|\Omega\|_\infty \|M\|_{L^p \times L^q \to L^r}.$$

This is rather trivial, the reasoning is very similar to the proof of the Proposition 7 and so we do not give full detail here. The proof of the Theorem 1 is now simple. If we have a fixed point (p,q), $1 < p,q < \infty$, we find a pair of points (p_1,q_1), $1 < p_1,q_1 < \infty$, and (p_2,q_2), $1 < p_2,q_2 < \infty$ such that $(1/p,1/q)$ lies inside the triangle $(1/2,1/2)$, $(1/p_1,1/q_1)$ and $(1/p_2,1/q_2)$. According to the Proposition 13 the operators $T_j^\#$ have norm at the point $(2,2)$ at most $C\|\Omega\|_{L^q} 2^{-\delta j}$, where $\delta > 0$ is fixed, while at the remaining two points the norm is $C_{p_i,q_i,\varepsilon}\|\Omega\|_\infty 2^{j\varepsilon}$, where $i = 1,2$ and we may choose $\varepsilon > 0$ arbitrarily small. Therefore, for a suitable choice of $\varepsilon > 0$, we get from the interpolation (Theorem 7.2.2 in [8]) that the series of norms of $T_j^\#$ is convergent at (p,q). This finishes the proof of Theorem 2.

To prove the Corollary 3, we apply interpolation argument similar to the proof of Lemma 11. We split Ω is similar way, but we need to make sure that the integral over sphere vanishes. Let us assume that $\|\Omega\|_q \le 1$. We decompose Ω as $\Omega = \sum_{i\ge 0} \widetilde{\Omega}_i$ where we first denote $\Omega_i(x') = \Omega(x')\chi_{E_i}(x)$ and

$$E_i = \left\{x \in \mathbb{R}^{2n} : |\Omega|\left(x'\right) \in \left(2^i, 2^{i+1}\right]\right\}$$

for $i > 0$ and $E_0 = \left\{x \in \mathbb{R}^{2n} : |\Omega|(x') \in [0,2]\right\}$, and then we set

$$\widetilde{\Omega}_i = \Omega_i - \int_{\mathbb{S}^{2n-1}} \Omega_i(x)dx.$$

We have $\|\Omega_i\|_1 \le C2^{-i\frac{q}{q'}}$ and therefore the sum converges back to Ω. We see that $\|\widetilde{\Omega}_i\|_2 \le C2^{\frac{-i(q-2)}{2}}$, while $\|\widetilde{\Omega}_i\|_\infty \le C2^i$. Now, we estimate

$$T_\Omega^*(f,g)(x) \le \sum_{i\ge 0} T_{\widetilde{\Omega}_i}^*(f,g)(x).$$

We interpolate the norm of each of the operators $T_{\widetilde{\Omega}_i}^*$ in a triangle which contains $(1/p_1,1/p_2)$, and has one vertex in the point $(1/2,1/2)$, where the norm $\|T_{\widetilde{\Omega}_i}^*\|_{L^2\times L^2\to L^1} \le C2^{\frac{-i(q-2)}{2}}$ according to the Theorem 2 and remaining two vertices close to points $(0,0)$, $(0,1)$, $(1,1)$, or $(1,0)$ where the operator has norm less than $C2^i$. The interpolated norms form convergent series precisely when the point $(1/p_1,1/p_2)$ lies inside the quadrilateral described in the Corollary 3 and the proof is finished.

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