EXACT RATE OF DECAY FOR SOLUTIONS TO DAMPED SECOND ORDER ODE'S WITH A DEGENERATE POTENTIAL

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Abstract. We prove exact rate of decay for solutions to a class of second order ordinary differential equations with degenerate potentials, in particular, for potential functions that grow as different powers in different directions in a neigborhood of zero. As a tool we derive some decay estimates for scalar second order equations with non-autonomous damping.

1. Introduction

In this paper we study rate of convergence to equilibrium of solutions to second order ordinary differential equations of the type

$$\ddot{u} + g(\dot{u})\dot{u} + \nabla E(u) = 0, \tag{DP}$$

which describe damped oscilations of a system. We assume that the potential energy $E : \mathbb{R}^n \to \mathbb{R}_+$ has its only local minimum in the origin and $g : \mathbb{R}^n \to \mathbb{R}$ is positive (except in the origin), so the term $g(\dot{u})\dot{u}$ has a damping effect.

The scalar case with $E(u) = a|u|^p$, $g(s) = b|s|^\alpha$ was studied by Haraux [9] and the vector valued case with $E(u) = \|A^{\frac{1}{2}}u\|^p$, $g(s) \in (c_1|s|^\alpha, c_2|s|^\alpha)$, A being a symmetric positive linear operator on a Hilbert space H was studied by Abdelli, Anguiano and Haraux [1]. For these cases exact decay rates were derived. Let us mention, that in both cases E satisfies $E(u) \sim \|u\|^p$, $\langle \nabla E(u), u\rangle \sim \|u\|^p$ on a neighborhood of zero (where $f \sim g$ means $cf \leq g \leq Cf$ for some positive constants c, C and $\langle \cdot, \cdot \rangle$ is the scalar product on H).

In [5] similar decay estimates as in [9], [1] were derived with the assumptions formulated in terms of the Łojasiewicz gradient inequality, namely for E satisfying

$$c\|\nabla E(u)\| \leq E(u)^{1-\theta} \leq C\|\nabla E(u)\| \tag{1}$$

and $\|\nabla^2 E(u)\| \leq \|\nabla E(u)\|^{\frac{1-2\theta}{1-\theta}}$ on a neighborhood of zero. The right inequality in (1) is called the Łojasiewicz gradient inequality. Let us mention that the

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potential functions E from [9], [1] satisfy (1) and also the condition on $\nabla^2 E$ with $\theta = \frac{1}{p}$.

The goal of this paper is to study degenerate cases, where the above assumptions do not hold, e.g. the behavior of E is not power-like or E does not satisfy the left inequality in (1) with the same θ as the right inequality[1]. A prototype of such E is

$$E(u) = \|u_1\|^{p_1} + \cdots + \|u_n\|^{p_n} \tag{2}$$

with $u = (u_1, \ldots, u_n)$ ($u_i \in \mathbb{R}^{n_i}$ are not neccessarily scalars) and $p_1 \geq p_2 \geq \cdots \geq p_n \geq 2$ are not all equal. We show that in such cases we obtain the same estimates (from above and from below) as for $E(u) = \|u\|^{p_1}$.

Further, we study the exact decay for the case where u_i in (2) are scalars. In the case studied in [9] and [1] the authors have shown that if $\alpha > 1 - \frac{2}{p}$ (i.e. the damping function is smaller than a treshold), then the solutions oscilate and all solutions converge to the origin with the same speed. On the other hand, if $\alpha < 1 - \frac{2}{p}$ (the damping function is larger than the treshold), then the solutions do not oscilate and there appear solutions with exactly two rates of convergence called fast solutions and slow solutions (see also [2] for existence of slow solutions). We show similar results for the degenerate case, in particular we show that for E given by (2) with u_i being scalars, at most $n + 1$ speeds of convergence occur (depending on p_i's).

While studying the exact decay for solutions to (DP) we look at the equations for single coordinates of u

$$\ddot{u}_i + g(\dot{u})\dot{u}_i + p\|u_i\|^{p-2}u_i = 0, \qquad i = 1, 2, \ldots, n. \tag{3}$$

Since we assume E to be in the special form (2) (a slightly more general case is considered below), these equations are coupled only by the term $g(\dot{u})u_i$. So, we consider these coordinate equations as non-autonomous problems

$$\ddot{u}_j + g_j(\dot{u}_j, t)\dot{u}_j + E(u_j) = 0, \tag{4}$$

where the dependence of g on other coordinates $\dot{u}_i$, $i \neq j$ is hidden in the dependence on t, in particular, g_j is defined by

$$g_j(s, t) = g((\dot{u}_1(t), \ldots, \dot{u}_{j-1}(t), s, \dot{u}_{j+1}(t), \ldots, \dot{u}_n(t))).$$

Therefore, we also give results on decay and oscilations for non-autonomous equations of the type (4) that may be of interest on their own. The results for $\alpha < 1 - \frac{2}{p}$ are again similar to those in [9], [1]. Decay estimates for another type of non-autonomous damping were derived in [3], [7], [10].

[1] Some decay estimates for even more general E satisfying only the Łojasiewicz inequality were obtained in [8], [6] and [4] but these estimates are in many cases not optimal and it is an open question, whether they are optimal at least for some problems.

The paper is organized as follows. In Section 2 we present basic definitions and assumptions valid throughout the rest of the paper. Section 3 is devoted to the scalar autonomous problems and Section 4 to scalar non-autonomous problems. The results in this section are based on comparison with the autonomous case. The degenerate vector-valued problem (DP) is studied in Section 5.

2. Basic definitions and preliminaries

In this paper we study three types of equations: the scalar autonomous problem

$$\ddot{u} + g(\dot{u})\dot{u} + E'(u) = 0, \tag{AP}$$

the scalar non-autonomous problem

$$\ddot{u} + g(\dot{u}, t)\dot{u} + E'(u) = 0, \tag{NP}$$

and the degenerate vector valued problem (DP). The assumption on $g \in C(\mathbb{R})$ for (AP), resp. $g \in C(\mathbb{R}^n)$ for (DP) is

$$c_g|s|^\alpha \leq g(s) \leq C_g|s|^\alpha, \tag{G}$$

in the non-autonomous case we assume only $g \in C(\mathbb{R} \times \mathbb{R}_+)$,

$$c_g|s|^\alpha \leq g(s, t) \tag{Gn}$$

for some $\alpha \in (0, 1)$, c_g, $C_g > 0$ and all s in any bounded set (with c_g, C_g depending on the set), and all $t \geq 0$ in case of (Gn). The potential function $E \in C^2(\mathbb{R})$ in (AP), (NP) is assumed to satisfy

$$c_E|s|^p \leq E(s) \leq C_E|s|^p, \quad c_E|s|^p \leq E'(s)s \leq C_E|s|^p \tag{E}$$

for some $p \geq 2$, c_E, $C_E > 0$ and all s in a bounded set. In case of (DP) we assume $E \in C^2(\mathbb{R}^n)$ is in the form

$$E(u) = E_1(u_1) + \cdots + E_n(u_n),$$

where $u = (u_1, u_2, \ldots, u_n)$, $E_i \in C^2(\mathbb{R})$ satisfy (E) with exponents p_i respectively, and $p_1 \geq p_2 \geq \cdots \geq p_n \geq 2$.

By a solution to (AP), (NP), (DP) we always mean a classical solution defined on $\mathbb{R}_+$. If u (resp. u_i) is a solution to one these equations, then v (resp. v_i) always denotes its velocity, i.e. $v = \dot{u}$ (resp. $v_i = \dot{u}_i$). We denote

$$\mathcal{E}(u, v) = \frac{1}{2}\|v\| + E(u).$$

This function is non-increasing along solutions since

$$\frac{d}{dt}\mathcal{E}(u(t), v(t)) = \langle v(t), \dot{v}(t)\rangle + \langle \nabla E(u(t)), v(t)\rangle = -g(v(t))\|v(t)\|^2 \leq 0,$$

whenever u is a solution to any of the studied equations. Sometimes, we write $\mathcal{E}(t)$ instead of $\mathcal{E}(u(t), v(t))$.

If $\alpha \geq 1 - \frac{2}{p}$ (p, α from (E), (Gn), (G)), we speak about *the oscilatory case*, otherwise we speak about *the non-oscilatory case*. In the non-oscilatory case, we say that the solution u is *a fast solution* if it converges to zero and $\lim_{t\to+\infty} \frac{\|v(t)\|}{E(u(t))} = +\infty$ (i.e. the kinetic energy is much bigger than the potential energy of u as t tends to infinity). On the other hand, u si called *a slow solution* if it converges to zero and $\lim_{t\to+\infty} \frac{\|v(t)\|}{E(u(t))} = 0$.

Let us now present two easy lemmas that show that the fast solutions converge to zero faster than slow solutions and how the speed of convergence depend on the trajectory in the uv plane, i.e. on the ratio of $\|u(t)\|$ and $\|v(t)\|$. Let $X(a,b) = \{u \in C^2((a,b)) : \ \dot{u} > 0 \text{ on } (a,b)\}$. By *trajectory* of u we mean the function $V_u : u(t) \mapsto v(t)$, i.e. $V_u(x) = v(u^{-1}(x))$, $x \in (u(a), u(b))$, where $v = \dot{u}$.

Lemma 1. *Let $a < x < y < b$ and let u_1, $u_2 \in X(a,b)$ with $V_{u_1} \geq V_{u_2}$ on $[x,y]$. Then u_2 needs more time than u_1 to get from x to y, i.e. if $u_1(t_1) = x = u_2(t_2)$ and $u_1(s_1) = y = u_2(s_2)$, then $s_1 - t_1 \leq s_2 - t_2$. Moreover, it holds that $u_2 \leq u_1$ on (t_1, s_1). If, moreover, $V_{u_1}(x) > V_{u_2}(x)$, then $s_1 - t_1 < s_2 - t_2$ and $u_2 < u_1$ on (t_1, s_1).*

Proof. We have for $i = 1, 2$

$$s_i - t_i = \int_{t_i}^{s_i} 1\, dt = \int_{t_i}^{s_i} \frac{\dot{u}_i(t)}{v_i(t)}\, dt = \int_{t_i}^{s_i} \frac{\dot{u}_i(t)}{V_{u_i}(u(t))}\, dt = \int_x^y \frac{1}{V_{u_i}(u)}\, du.$$

The assertion now follows easily from $V_{u_1} \geq V_{u_2}$ (resp. $V_{u_1} > V_{u_2}$ on a neighborhood of x). □

Lemma 2. *Let $u \in X(0, +\infty)$ with $\lim_{t\to+\infty}(u(t), v(t)) = 0$. If $V_u(x) \geq cx^a$ on $(-\varepsilon, 0)$ for some $a > 1$, $\varepsilon > 0$, then $u(t) \leq \tilde{c}t^{-\frac{1}{a-1}}$ for some $\tilde{c}$ and all t large enough. If $V_u(x) \leq cx^a$ on $(-\varepsilon, 0)$ for some $a > 1$, $\varepsilon > 0$, then $u(t) \geq \tilde{c}t^{-\frac{1}{a-1}}$ for some $\tilde{c}$ and all t large enough.*

Proof. $V_u(x) \geq c|x|^a$ means $v(t) = \dot{u}(t) \geq c|u(t)|^a$. Dividing by $|u(t)|^a$ and integrating from t_0 to t we get

$$\frac{1}{1-a}\left(|u(t_0)|^{1-a} - |u(t)|^{1-a}\right) \geq c(t - t_0),$$

i.e.

$$|u(t)| \leq \left((a-1)c(t-t_0) + |u(t_0)|^{1-a}\right)^{\frac{1}{1-a}} \leq \tilde{c}t^{-\frac{1}{a-1}}.$$

The opposite estimate follows similarly. □

Finally, $f(t) \sim h(t)$ means that there exist $T, c, C > 0$ such that $cf(t) \leq h(t) \leq Cg(t)$ for all $t \geq T$. By

$$|u(t)| \leq Cf(t),\ |u(t)| \geq Cf(t),\ |v(t)| \leq Cf(t),\ |v(t)| \geq Cf(t)$$

in the Theorems and Lemmas below we mean that there exist $C > 0$, $T > 0$ such that the inequality holds for all $t \geq T$.

3. Scalar autonomous problem

In this section we study the autonomous problem (AP). We assume that g satisfies (G) and E satisfies (E) with $\alpha < 1 - \frac{2}{p}$, i.e. the non-oscilatory case. We first formulate the main result, Theorem 3. In fact, it is a minor generalization of results proved by Haraux in [9]. However, important are the lemmas below leading to the proof of the Theorem, they are needed in the next section for investigation of the non-autonomous problem.

Theorem 3. *Let $\alpha < 1 - \frac{2}{p}$. Then all solutions converge to zero and do not oscilate (e.g. u, v change sign only finitely many times). Further, any solution to* (AP) *is either fast or slow. Moreover, every fast solution satisfies*

$$u(t) \sim t^{-\frac{1-\alpha}{\alpha}}, \qquad v(t) \sim t^{-\frac{1}{\alpha}} \tag{5}$$

and every slow solution satisfies

$$u(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}}, \qquad v(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}-1} = t^{-\frac{p-1}{p-2-\alpha}}. \tag{6}$$

We first show that some sets are positively invariant for solutions of (AP), namely sets $O_{\varepsilon,K}$, $N_{\varepsilon,K}$, $P_{\delta,\eta}$ defined below.

Lemma 4. *Denote $\kappa_0 = \frac{c_E}{C_g}$ and $\kappa = \frac{c_g}{C_E}$. Let $K \in (0, \kappa)$ and $\varepsilon > 0$ satisfy*

$$\varepsilon^{p-2-\alpha p} \leq \left(\frac{C_E(\alpha+1)}{p-1} \right)^{1+\alpha} (\kappa - K)^{1+\alpha} K^{1-\alpha}. \tag{7}$$

Then the sets

$$\begin{aligned} N_{\varepsilon,K} &= \left\{ (u,v) \in \mathbb{R}^2 : \ -\varepsilon \leq u \leq 0, \ \kappa_0^{\frac{1}{\alpha+1}} |u|^{\frac{p-1}{\alpha+1}} \leq v \leq K^{-\frac{1}{\alpha+1}} |u|^{\frac{p-1}{\alpha+1}} \right\} \\ O_{\varepsilon,K} &= \left\{ (u,v) \in \mathbb{R}^2 : \ -\varepsilon \leq u \leq 0, \ 0 \leq v \leq K^{-\frac{1}{\alpha+1}} |u|^{\frac{p-1}{\alpha+1}} \right\} \end{aligned} \tag{8}$$

are positively invariant for solutions $(u, \dot{u})$ of (AP)*. Moreover, any solution in $O_{\varepsilon,K}$ is a slow solution, it enters the set $N_{\varepsilon,K}$, and satisfies* (6)*.*

Proof. We show that the vectors $(\dot{u}, \dot{v})$ point into N resp. O (we omit the subscripts) if $(u, v) \in \partial N$ resp. ∂O. For $u = -\varepsilon$, $v \geq 0$ and $v = 0$, $u < 0$ it is obvious. For $v = \kappa_0^{\frac{1}{\alpha+1}} |u|^{\frac{p-1}{\alpha+1}}$, $u < 0$ it follows from

$$\dot{v} = -g(v)v - E'(u) \geq -C_g v^{\alpha+1} + c_E |u|^{p-1} = 0 > \frac{d}{du} \kappa_0^{-\frac{1}{\alpha+1}} |u|^{\frac{p-1}{\alpha+1}}.$$

It remains to investigate the upper part of the boundary, i.e. $v(t) = K^{-\frac{1}{\alpha+1}}|u|^{\frac{p-1}{\alpha+1}}$, $u < 0$. Here we have

$$\frac{\dot v}{\dot u} \le \frac{-c_g|v|^{\alpha+1} + C_E|u|^{p-1}}{v} = C_E\left(K - \frac{c_g}{C_E}\right)v^\alpha = C_E\left(K - \frac{c_g}{C_E}\right)K^{-\frac{\alpha}{\alpha+1}}|u|^{\frac{\alpha(p-1)}{\alpha+1}}$$

and

$$\frac{d}{du}K^{-\frac{1}{\alpha+1}}|u|^{\frac{p-1}{\alpha+1}} = -K^{-\frac{1}{\alpha+1}}\frac{p-1}{\alpha+1}|u|^{\frac{p-2-\alpha}{\alpha+1}}.$$

Therefore, $\frac{\dot v}{\dot u} < \frac{d}{du}K^{-\frac{1}{\alpha+1}}|u|^{\frac{p-1}{\alpha+1}} < 0$ if and only if $K \in (0,\kappa)$ and

$$|u|^{\frac{p-2-\alpha p}{\alpha+1}} \le \frac{\alpha+1}{p-1}K^{\frac{1-\alpha}{1+\alpha}}C_E\left(\frac{c_g}{C_E} - K\right)$$

and the positive invariance is proved.

Since $\frac{p-1}{1+\alpha} > \frac{p}{2}$ we have $|u|^{\frac{p-1}{1+\alpha}} < C|u|^{\frac{p}{2}}$ (for all u in a bounded set), and therefore any solution in O is slow. Moreover, if a solution (u,v) belongs to $O \setminus N$, then functions u and v are increasing, and therefore the solution enters N. By Lemma 2, $u(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}}$ and due to $\kappa_0^{-\frac{1}{\alpha+1}}|u|^{\frac{p-1}{\alpha+1}} \le v \le K^{-\frac{1}{\alpha+1}}|u|^{\frac{p-1}{\alpha+1}}$ we have $v(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}\frac{p-1}{1+\alpha}} = t^{-\frac{p-1}{p-2-\alpha}}$. □

Lemma 5. *There exist $\delta, \eta > 0$ such that the set*

$$P_{\delta,\eta} = \left\{(u,v) \in \mathbb{R}^2 :\ -\delta \le u < 0,\ 0 \le v \le \eta|u|^{\frac{1}{1-\alpha}}\right\}$$

is positively invariant for (AP) *and any solution to* (AP) *with* $(u(t_0), v(t_0)) \in P_{\delta,\eta}$ *for some $t_0 > 0$ is a slow solution and satisfies* (6).

Proof. Let us define $K(u) = C^{-\frac{1}{1-\alpha}}|u|^{\frac{p-2-\alpha p}{1-\alpha}}$, where $C = \left(\frac{c_g(1+\alpha)}{2(p-1)}\right)^{1+\alpha}$. Let $\delta > 0$ be such that $K(u) \le \frac{c_g}{2C_E}$ for all $u \in [-\delta, 0]$. Then for any $u \in [-\delta, 0]$, inequality (7) holds with $(\varepsilon, K) = (u, K(u))$ and therefore (by Lemma 4) the set $O_{-u,K(u)}$ is positively invariant. We have

$$K(u)^{-\frac{1}{1+\alpha}}|u|^{\frac{p-1}{\alpha+1}} = C^{\frac{1}{1-\alpha^2}}|u|^{-\frac{p-2-\alpha p}{1-\alpha^2}}|u|^{\frac{p-1}{\alpha+1}} = C^{\frac{1}{1-\alpha^2}}|u|^{\frac{1}{1-\alpha}}.$$

Set $\eta = C^{\frac{1}{1-\alpha^2}}$. If $0 \le v(t_0) \le \eta|u(t_0)|^{\frac{1}{1-\alpha}}$, then $0 \le v(t_0) \le K(u(t_0))^{-\frac{1}{1+\alpha}}|u(t_0)|^{\frac{p-1}{\alpha+1}}$, i.e. $(u(t_0), v(t_0)) \in O_{-u(t_0),K(u(t_0))}$. Then $(u(t), v(t)) \in O_{-u(t_0),K(u(t_0))}$ for all $t \ge t_0$ and by Lemma 4 it is a slow solution and satisfies (6). □

Lemma 6. *Let us consider two sequences (u_n), (v_n) satisfying $\lim u_n = 0$ and $u_n < 0, 0 < v_n < M|u_n|^{\frac{p}{2}}$ for all n. Then, for all n large enough, $(u_n, v_n) \in P_{\delta,\eta}$, with δ, η from Lemma 5.*

Proof. Let δ, η be the constants form Lemma 5, then obviously $u(t_n) \geq -\delta$ for all n large enough and

$$v(t_n) \leq M|u(t_n)|^{\frac{p}{2}} = \eta|u(t_n)|^{\frac{1}{1-\alpha}} \frac{M}{\eta}|u(t_n)|^{\frac{p-2-\alpha p}{2(1-\alpha)}} \leq \eta|u(t_n)|^{\frac{1}{1-\alpha}}$$

for large n since $\frac{p-2-\alpha p}{2(1-\alpha)} > 0$. □

Proposition 7. *Let u be a solution to* (AP) *satisfying $u < 0, v = \dot{u} > 0$ on* $(T_0, +\infty)$ *and* $(u(t), v(t)) \to (0, 0)$. *Then u is either fast solution or slow solution. In the latter case, u satisfies* (6).

Proof. If u is not a fast solution, then there exists $M > 0$ such that $\frac{v(t_n)}{|u(t_n)|^{\frac{p}{2}}} \leq M$ for a sequence $t_n \nearrow +\infty$. By Lemma 6 we have $(u(t_n), v(t_n)) \in P_{\delta,\eta}$ for large n. Hence u is a slow solution by Lemma 5 and (6) holds. □

Lemma 8. *Any fast solution u of* (AP) *with $u < 0, v > 0$ on* $(T, +\infty)$ *satisfies* (5).

Proof. By Lemma 5, any fast solution satisfies $v(t) > \eta|u(t)|^{\frac{1}{1-\alpha}}$ for all t sufficiently large. By Lemma 2, $u(t) \leq ct^{-\frac{1-\alpha}{\alpha}}$. It follows that

$$\dot{v} \leq -c_g v^{\alpha+1} + C_E|u|^{p-1} \leq -\kappa v^{\alpha+1} + C|v|^{(p-1)(1-\alpha)} \leq (-\kappa + \varepsilon)|v|^{1+\alpha}$$

since $(p-1)(1-\alpha) = p - \alpha p - 1 + \alpha = p - 2 - \alpha p + (1+\alpha) > 1 + \alpha$. By Lemma 2 we have $v(t) \leq Ct^{-\frac{1}{\alpha}}$. Since u is a fast solution, we have $\mathcal{E}(t) \sim v(t)^2$ and due to $\mathcal{E}(t) \geq ct^{-\frac{2}{\alpha}}$ we have $v(t) \geq ct^{-\frac{1}{\alpha}}$. Now $v(t) \sim t^{-\frac{1}{\alpha}}$ and by integration we have $u(t) \sim t^{-\frac{1-\alpha}{\alpha}}$. □

Proof of Theorem 3. Convergence to zero follows from Theorem 9 below and absence of oscilations follows from Proposition 11 below. Then any solution satisfies $u < 0, v > 0$ on $(T, +\infty)$ or symmetrically $u > 0, v < 0$. By Proposition 7, any solution is slow or fast and slow solutions satisfy (6). By Lemma 8, fast solutions satisfy (5). □

4. Nonautonomous damping

In this section we study the non-autonomous problem (NP). We keep the assumption (E) and assume that g satisfies (Gn). We show that for g bounded all solutions converge to zero and that they do not oscilate if $\alpha < 1 - \frac{2}{p}$. Then we study decay of the non-oscilatory solutions.

Theorem 9. *Let g satisfy* (Gn) *for some $\alpha \geq 0$ and $g(s,t) \leq M$ for all s from a bounded set and all $t \geq 0$. Then any solution to* (NP) *converges to zero as $t \to +\infty$.*

Proof. Let u be a solution to (NP). Since $\frac{d}{dt}\mathcal{E}(u(t), v(t)) = -g(v(t), t)v^2(t) \leq 0$, it follows that $(u(t), v(t))$ is bounded and the omega-limit set

$$\omega(u, v) = \left\{(\varphi, \psi) \in \mathbb{R}^2 : \exists t_n \nearrow +\infty,\ u(t_n) \to \varphi,\ v(t_n) \to \psi\right\}$$

is nonempty. Let $(\varphi,\psi)\in\omega(u,v)$. If $\psi\neq 0$, then $\frac{d}{dt}\mathcal{E}(u(t),v(t)) < -c_g|v(t)|^{\alpha+2}\leq -\varepsilon<0$ for all t such that $(u(t),v(t))$ belongs to a small neighborhood N of (φ,ψ). Due to boundedness of $\dot u$, $\dot v$ the solution (u,v) spends infinite time in N, which is a contradiction with boundedness of $\mathcal{E}(u(t),v(t))$ from below. So, $\psi=0$. Since ω is connected, it is an interval $[a,b]\times\{0\}$. However, $\mathcal{E}$ is constant on ω, hence ω is a singleton, i.e. $\lim u(t)=\varphi$.

Since $g(v(t),t)$ is bounded we have for $t\to+\infty$ $g(v(t),t)v(t)\to 0$. Since $E'(u(t))\to E'(\varphi)$ we get from (NP) $\dot v(t)\to -E'(\varphi)$. Therefore, $E'(\varphi)=0$ (otherwise, we have a contradiction with $v(t)\to 0$). It follow by (E) that $\varphi=0$. □

Remark 1. *It can be seen from the proof of Theorem 9 that if we omit the assumtion on boundedness of g, then we would still have $(u(t),v(t))\to(\varphi,0)$. However, φ is not neccessarilly zero. In fact, we show that $u(t)=1+t^{-1}$ solves (for t large enough)* (NP) *with*

$$g(s,t)=|s|^\alpha+\max\left\{0,2t^{-1}+t^2E(1+t^{-1})-t^{-2\alpha}\right\}.$$

Since $v(t)=-t^{-2}$ we have for large t (such that $2t^{-1}+t^2E(1+t^{-1})-t^{-2\alpha}>0$)

$$g(v(t),t)v(t)=\left(t^{-2\alpha}+2t^{-1}+t^2E(1+t^{-1})-t^{-2\alpha}\right)(-t^{-2})=-2t^{-3}-E(1+t^{-1}),$$

which is exactly $-\dot v(t)-E(u(t))$.

Let us prove the following comparison Lemma.

Lemma 10. *Let $0\leq g_1(s,t)<g_2(s,\tilde t)$ for any $s\in\mathbb{R}$, $t,\tilde t\geq 0$ and let $E'(s)s>0$ for all $s\neq 0$. Let u_i, $i=1,2$ be, respectively, solutions to*

$$\ddot u_i+g_i(\dot u_i,t)\dot u_i+E'(u_i)=0,\qquad i=1,2 \tag{9}$$

with $u_1(t_0)=u_2(t_0)<0$, $v_1(t_0)=v_2(t_0)>0$ for some $t_0\geq 0$. Let $t_1>t_0$ be such that $u_1<0$, $v_1>0$ on (t_0,t_1). Then $\dot u_2>0$ and $u_2<u_1$ on (t_0,t_1) and the trajectories $V_i(x)=\dot u_i(u_i^{-1}(x))$ satisfy $V_2(x)<V_1(x)$ on $(u_1(t_0),u_1(t_1))$.

Proof. Obviously, the solution (u_2,v_2) cannot cross the halfline $\{u<0,v=0\}$ since $\dot v_2(t)=-E(u_2(t))>0$ on this halfline. So, $v_2=\dot u_2$ stays positive as long as $u_2<0$. Let $t_2=\sup\{t\in[t_0,t_1]:u_2<0\text{ on }[t_0,t]\}$. Then either $t_2=t_1$ or $u(t_2)=0$. Then trajectories V_1, resp. V_2 are well defined on $(u_1(t_0),u_1(t_1))$, resp. $(u_2(t_0),u_2(t_2))$. For $t\in[t_0,t_i]$ it holds that

$$V_i'(u_i(t))=\frac{\dot v_i(t)}{\dot u_i(t)}=\frac{-g_i(v_i(t),t)v_i(t)-E'(u_i(t))}{v_i(t)}. \tag{10}$$

So, if for any $s,t\in(t_0,t_2)$ we have $u_1(t)=u_2(s)$, $v_1(t)=v_2(s)$, then $V_2'(u_2(s))<V_1'(u_2(s))$ (since $g_1<g_2$ and other terms in (10) are equal for $i=1$ and $i=2$). This leads to contradiction (take infimum of such s), and therefore $V_2<V_1$ on $(u_1(t_0),\min\{u_1(t_1),u_2(t_2)\})$. It follows from Lemma 1 that $u_2<u_1$ on (t_0,t_2) and $t_2=t_1$. □

Proposition 11. *Let g satisfy* (Gn) *for some $\alpha < 1 - \frac{2}{p}$ and let u be a solution to* (NP) *such that* $\lim_{t\to+\infty} = 0$*. Then u does not oscilate, i.e. u, $\dot{u}$ do not change sign on $(t_0, +\infty)$ for some $t_0 \geq 0$.*

Proof. Let us assume for contradiction that a solution u to (NP) oscilates, i.e. there exists a sequence $t_n \nearrow +\infty$ such that $u(t_n) = 0$ or $v(t_n) = 0$. We show that for every $\varepsilon > 0$ there exists T_ε such that $v(T_\varepsilon) = 0$ and $|u(T_\varepsilon)| \leq \varepsilon$. In fact, if $v(t) \neq 0$ on some $(T, +\infty)$, then u would be monotone on $(T, +\infty)$ and it would be a contradiction with existence of t_n. So, there exists a sequence $s_n \nearrow +\infty$ with $v(s_n) = 0$ and since any solution converges to zero, for large n we have $|u(s_n)| \leq \varepsilon$.

Let us without loss of generality assume that $u(s_n) < 0$. Then $\dot{v}(s_n) = -E'(u(s_n)) > 0$, so the solution enters the set $P_{\delta,\eta}$ defined in Lemma 5. We show that $P_{\delta,\eta}$ is positively invariant for solutions of (NP). Obviously, for $u = -\delta$, $v > 0$ we have $\dot{u} > 0$, for $v = 0$, $u < 0$ we have $\dot{v} = -E(u) > 0$ and for the remaining part of the boundary $v = \eta|u|^{\frac{1}{1-\alpha}}$ we have $\dot{u} = \dot{u}_1$, $\dot{v} \leq \dot{v}_1$, where u_1 is the solution to (AP) with $g(s) = c_g|s|^\alpha$ going through the same point of the boundary. □

In the following we consider only solutions satisfying $u < 0$, $v = \dot{u} > 0$ on $(T, +\infty)$. We now formulate and prove two main theorems of this section. Theorem 12 is applied in the next section. In fact, it says that any fast solution converges faster than any slow solution, even for solutions to different problems with the same α (and possibly different p's). Theorem 13 says that if the non-autonomous part of the damping is smaller than the natural damping given by the velocity of slow solutions to the corresponding autonomous problem, then the non-autonomous part does not influence the decay.

Theorem 12. *Let g satisfy* (Gn) *with $\alpha < 1 - \frac{2}{p}$. Then any solution to* (NP) *which converges to zero is either fast or slow. Further, slow solutions satisfy $|u(t)| \geq ct^{-\frac{\alpha+1}{p-2-\alpha}}$, $v(t) \leq C|u(t)|^{\frac{p-1}{1+\alpha}}$ and fast solutions satisfy $|u(t)| \leq ct^{-\frac{1-\alpha}{\alpha}}$, $c|u(t)|^{\frac{1}{1-\alpha}} \leq v(t) \leq Ct^{-\frac{1}{\alpha}}$.*

Proof. Let u be a solution that is not fast. Then there exists $M > 0$ such that $\frac{v(t_n)}{|u(t_n)|^{\frac{p}{2}}} \leq M$ for a sequence $t_n \nearrow +\infty$. By Lemma 6, there exists $n \in \mathbb{N}$ such that $(u(t_n), v(t_n)) \in P_{\delta,\eta}$. Let us consider the solution u_1 of the autonomous problem (AP) with $u_1(t_n) = u(t_n)$, $v_1(t_n) = v(t_n)$. By Lemma 5, u_1 is a slow solution to (AP) and it satisfies (6) by Theorem 3. By the comparison Lemma 10 and (Gn), the trajectories satisfy $V(x) < V_1(x)$ and $u(x) \geq u_1(x) \sim t^{-\frac{\alpha+1}{p-2-\alpha}}$ (and $(u(t), v(t))$ belongs to $O_{\varepsilon,K}$ for some ε, K, what we use in the next Theorem).

Let u be a fast solution. Then $v(t) > \eta|u(t)|^{\frac{1}{1-\alpha}}$ on some interval $(T, +\infty)$ (otherwise, we would proceed as in the first paragraph of this proof and obtain that u is a slow solution). By Lemma 2 we have $u(t) \leq Ct^{-\frac{1-\alpha}{\alpha}}$. Further, we have

$$\dot{v} = -g(v,t)v + E'(u) \leq -c_g v^{1+\alpha} + C_E \eta^{1-p} v^{(p-1)(1-\alpha)} \leq (-c+\varepsilon)v^{1+\alpha}$$

since $1+\alpha > (p-1)(1-\alpha)$. Therefore, (again by Lemma (2)) we obtain $v(t) \leq Ct^{-\frac{1}{\alpha}}$. □

Theorem 13. *Let g satisfy* (Gn) *with $\alpha < 1 - \frac{2}{p}$ and*

$$g(s,t) \leq D_g(|s| + t^{-\frac{p-1}{p-2-\alpha}})^{\alpha}. \tag{11}$$

Then any slow solution satisfies (6).

Proof. In the proof of Theorem 12 we have shown that any slow solution $(u(t), v(t))$ belongs to $O_{\varepsilon,K}$ for some ε, K and all $t \geq t_n$. Let us set $T = t_n$ and take $\varepsilon > 0$ such that $v(T) > \varepsilon T^{-\frac{p-1}{p-2-\alpha}}$. We show that that $v(t) > \varepsilon t^{-\frac{p-1}{p-2-\alpha}}$ for all $t > T$. In fact, let $t_0 = \inf\{t > T : \ v(t) \leq \varepsilon t^{-\frac{p-1}{p-2-\alpha}}\}$. Then $v(t_0) = \varepsilon t_0^{-\frac{p-1}{p-2-\alpha}}$ and by Theorem 12 we have $|u(t_0)| \geq ct_0^{-\frac{\alpha+1}{p-2-\alpha}}$, and therefore

$$\begin{aligned}\dot{v}(t_0) = -g(v,t_0)v + E'(u) &\geq -D_g(1+\varepsilon)^{\alpha} t_0^{-\frac{p-1}{p-2-\alpha}\alpha} \varepsilon t_0^{-\frac{p-1}{p-2-\alpha}} + c_E c^{p-1} t_0^{-\frac{(\alpha+1)(p-1)}{p-2-\alpha}} \\ &= (c^{p-1}c_E - D_g(1+\varepsilon)^{\alpha}\varepsilon) t_0^{-\frac{\alpha+1}{p-2-\alpha}(p-1)} > 0\end{aligned}$$

if ε is small enough. It follows that for $t \in (t_0 - \delta, t_0)$ we have

$$v(t) < v(t_0) = \varepsilon t_0^{-\frac{p-1}{p-2-\alpha}} < \varepsilon t^{-\frac{p-1}{p-2-\alpha}},$$

contradiction with definition of t_0. Hence, $v(t) > \varepsilon t^{-\frac{p-1}{p-2-\alpha}}$ holds on $(T, +\infty)$, and therefore

$$g(v,t)v < C(\varepsilon)v^{\alpha+1} \qquad \text{on } (T, +\infty).$$

Now, if we compare the solution u with the solution u_2 of $\ddot{u} + C(\varepsilon)\dot{u}^{\alpha+1} - p|u|^{p-1} = 0$, $u_2(T) = u(T)$, $v_2(T) = v(T)$, the comparison Lemma 10 yields $|u(t)| \leq |u_2(t)| \sim Ct^{-\frac{\alpha+1}{p-2-\alpha}}$. Now we have $u(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}}$ and due to $v \leq cu^{\frac{p-1}{\alpha+1}}$ (since $(u(t), v(t)) \in O_{\varepsilon,K}$) we have $v(t) \sim t^{-\frac{\alpha+1}{p-2-\alpha}-1}$. □

5. Degenerate potential

In this section we investigate the problem (DP). We assume that g satisfies (G) and $E \in C^2(\mathbb{R}^n)$ is in the form

$$E(u) = E_1(u_1) + \cdots + E_n(u_n),$$

where $u = (u_1, u_2, \dots, u_n)$, $E_i \in C^2(\mathbb{R})$ satisfy (E) with exponents p_i respectively, and $p_1 \geq p_2 \geq \cdots \geq p_n \geq 2$. Then (DP) can be written as the following system of equations for $u = (u_1, \dots, u_n)$

$$\ddot{u}_i + g(\dot{u})\dot{u}_i + E_i'(u_i) = 0, \qquad i = 1, 2, \dots, n. \tag{12}$$

The equations are coupled only by the term $g(\dot{u})$. Let us start with the decay estimates for solutions of (DP).

Theorem 14. *If $\alpha \geq 1 - \frac{2}{p_1}$, then*

$$\mathcal{E}(t) \sim C_2 t^{-\frac{2}{\alpha}}. \tag{13}$$

If $\alpha < 1 - \frac{2}{p_1}$, then for any solution u of (DP) *it holds that*

$$C_1 t^{-\frac{(1+\alpha)p_1}{p_1-2-\alpha}} \geq \mathcal{E}(t) \geq C_2 t^{-\frac{2}{\alpha}}. \tag{14}$$

Remark 2. *Let us remark that Theorem 14 remains valid (with the same proof) if u_i are vector valued functions with values in $\mathbb{R}^{n_i}$, $g : \mathbb{R}^{\sum n_i} \to \mathbb{R}$ and $E_i : \mathbb{R}^{n_i} \to \mathbb{R}$. We can also assume that E_i satisfy* (1) *and $\|\nabla^2 E(u)\| \leq \|\nabla E(u)\|^{\frac{1-2\theta}{1-\theta}}$ with $\theta = \frac{1}{p_i}$ instead of* (E). *Then Theorem* (14) *remains valid with a similar proof where we define $H_j(t) = \mathcal{E}_j(t) + \varepsilon \|\nabla E_j(u_j(t))\|^{\beta_j} \langle \nabla E_j(u_j), v_j \rangle$ with appropriate β_j's, cf.* [5].

Proof of Theorem 14. Let $u = (u_1, \dots, u_n)$ be a solution to (DP). Let us define

$$\mathcal{E}_j(t) = \frac{1}{2n}\|v(t)\|^2 + E_j(u_j(t))$$

and

$$H_j(t) = \mathcal{E}_j(t) + \varepsilon |u_j(t)|^{\beta_j} u_j v_j$$

with $\beta_j = \frac{\alpha p_j}{2}$ if $\alpha \geq 1 - \frac{2}{p_j}$ and $\beta_j = \frac{p_j - 2 - \alpha}{\alpha + 1}$ otherwise.

The last term in the definition of H_j is estimated by (we write u_j instead of $u_j(t)$)

$$\varepsilon\left(|u_j|^{2(\beta_j+1)} + |v_j|^2\right) \leq C\varepsilon\left(E_j(u_j)^{\frac{2}{p_j}(\beta_j+1)} + \|v\|^2\right) \leq C\varepsilon\left(E_j(u_j) + \|v\|^2\right),$$

where we applied the Young inequality, then $E_j(u) \sim u^{p_j}$ and finally $2(\beta_j+1) \geq p_j$ and boundedness of $E_j(u_j(t))$. It follows that $H_j(t) \sim \mathcal{E}_j(t)$. Further, we have (we write v, u_j instead of $v(t)$, $u_j(t)$) for every $t \geq 0$

$$\begin{aligned} H_j'(t) = &-\langle g(v)v_j, v_j\rangle - \varepsilon |u_j|^{\beta_j} u_j E_j'(u_j) \\ &+ \varepsilon \beta_j |u_j|^{\beta_j} v_j^2 \\ &+ \varepsilon |u_j|^{\beta_j} v_j^2 \\ &- \varepsilon |u_j|^{\beta_j} u_j g(v) v_j. \end{aligned} \tag{15}$$

Here the first line satisfies (due to (G), (E))

$$-\langle g(v)v_j, v_j\rangle - \varepsilon|u_j|^{\beta_j}u_jE_j'(u_j) \sim -|v_j|^{\alpha+2} - |v_j|^2\sum_{k\neq j}|v_k|^\alpha - \varepsilon|u_j|^{\beta_j+p_j}.$$

The second and third lines in (15) are by the Young inequality estimated by

$$C\varepsilon|v_j|^{\alpha+2} + \frac{\varepsilon}{4}|u_j|^q \leq C\varepsilon|v_j|^{\alpha+2} + \frac{\varepsilon}{4}|u_j|^{\beta_j+p_j}$$

since $q = \frac{\alpha+2}{\alpha}\beta_j \geq \beta_j + p_j$. The last line in (15) is estimated by

$$\varepsilon|u_j|^{\beta_j+1}|v_j|\sum_{k=1}^n|v_k|^\alpha \leq \frac{\varepsilon}{4}|u_j|^{\beta_j+p_j} + C\varepsilon|v_j|^{\frac{\beta_j+p_j}{p_j-1}}\left(\sum_{k=1}^n|v_k|^\alpha\right)^{\frac{\beta_j+p_j}{p_j-1}}.$$

Since $\frac{\beta_j+p_j}{p_j-1} \geq \frac{\alpha+2}{\alpha+1}$, the last expression is estimated by

$$\frac{\varepsilon}{4}|u_j|^{\beta_j+p_j} + C\varepsilon|v_j|^{\alpha+2} + C\varepsilon|v_j|^{\frac{\alpha+2}{\alpha+1}}\sum_{k\neq j}|v_k|^{\alpha\frac{\alpha+2}{\alpha+1}}$$

and by the Young inequality this is less than

$$\frac{\varepsilon}{4}|u_j|^{\beta_j+p_j} + C\varepsilon|v_j|^{\alpha+2} + C\varepsilon\sum|v_k|^{\alpha+2}.$$

This term cannot be absorbed into the first line of (15) but after summing over j it can be absorbed and we obtain

$$H' = \sum H_j' \sim -\sum|v_j|^{\alpha+2} - \varepsilon\sum|u_j|^{\beta_j+p_j},$$

i.e.,

$$H' \sim -\left(\|v\|^{\alpha+2} + \sum|u_j|^{\beta_j+p_j}\right).$$

It follows that

$$-\frac{d}{dt}\frac{H'}{H^B} \sim \frac{\|v\|^{\alpha+2} + \sum|u_j|^{\beta_j+p_j}}{\left(\|v\|^2 + \sum|u_j|^{p_j}\right)^B}. \tag{16}$$

The right-hand side is bounded from below by a positive constant if $2B \geq \alpha+2$, $Bp_j \geq \beta_j + p_j$ for all j. For oscilatory coodrinates, i.e. if $\alpha \geq 1 - \frac{2}{p_j}$ these inequalities hold if $B \geq \frac{\alpha+2}{2}$. So, if $\alpha \geq 1 - \frac{2}{p_1}$ (all coordinates are oscilatory), we have

$$\mathcal{E}(t) \sim H(t) \leq Ct^{-\frac{1}{B-1}} = Ct^{-\frac{2}{\alpha}}.$$

For the non-ocsilatory coordinates $\alpha < 1 - \frac{2}{p_j}$ we need to take a larger B, in particular $B \geq (1 - \frac{1}{p_j})\frac{\alpha+2}{\alpha+1}$. Since p_1 is the largest among the non-oscilatory coordinates, number $B = (1-\frac{1}{p_1})\frac{\alpha+2}{\alpha+1}$ is the least suitable and we obtain

$$\mathcal{E}(t) \sim H(t) \leq Ct^{-\frac{1}{B-1}} = Ct^{-\frac{(1+\alpha)p_1}{p_1-2-\alpha}}.$$

On the other hand, the right-hand side of (16) is bounded from above if $2B \leq \alpha+2$, $Bp_j \leq \beta_j + p_j$ for all j. Here, the best choice (largest possible B) is always $B = \frac{\alpha+2}{2}$ (for both oscilatory and non-oscilatory coordinates) and we obtain

$$\mathcal{E} \sim H(t) \geq Ct^{-\frac{2}{\alpha}},$$

which completes the proof. □

From now on, let us assume that u_i are scalar valued. For a solution $u = (u_1, \dots, u_n)$ and any fixed $i \in \{1, 2, \dots, n\}$ let us denote $f_j(t) = \sum_{i\neq j} \dot{u}_i^2(t) \geq 0$. Then u_j solves the nonautonomous problem (4) with

$$g_j(\dot{u}_i, t) = g\left(\sqrt{\dot{u}_j^2 + f_j(t)}\right) \geq c_g\left(\dot{u}_j^2 + f_j(t)\right)^{\frac{\alpha}{2}} \geq c_g|\dot{u}_j|^\alpha,$$

so (Gn) is satisfied. Moreover, by Theorem 14 we know that every solution converges to zero. Now, we can apply the results from the previous section to obtain more gentle properties of solutions. In particular, we show that each solution to (DP) has one of (at most) $n+1$ speeds of convergence to the origin that are given by fast and slow solutions of the equations (12). First of all, by Theorem 12 we have the following.

Corollary 15. *Let $u = (u_1, \dots, u_n)$ be a solution to* (DP)*. If $i \in \{1, \dots, n\}$ is such that $\alpha < 1 - \frac{2}{p_i}$, then u_i does not oscilate. Moreover, for such i, function u_i (as a solution of* (4)*) is either fast and satisfies*

$$|u(t)| \leq Ct^{-\frac{1-\alpha}{\alpha}}, \qquad |v(t)| \leq Ct^{-\frac{1}{\alpha}}$$

or slow and satisfies

$$|u(t)| \geq Ct^{-\frac{1+\alpha}{p-2-\alpha}}.$$

So, we speak about *a non-oscilatory coordinate* if $\alpha < 1 - \frac{2}{p_i}$ and about *oscilatory coordinate* if $\alpha \geq 1 - \frac{2}{p_i}$ (we do not know whether the oscilatory coordinates really oscilate) and a non-oscilatory coordinate of a particular solution can be called *slow coordinate* or *fast coordinate*. We now show that there appear at most $n+1$ different rates of convergence of solutions to (DP), in particular, if there are k non-oscilatory coordinates, then each solution has one of the $k+1$ possible decay rates.

Theorem 16. *Let m be such that $1 - \frac{2}{p_{m+1}} \leq \alpha < 1 - \frac{2}{p_m}$ (set $m = 0$ if $1 - \frac{2}{p_j} \leq \alpha$ for all j and $m = n$ if $\alpha < 1 - \frac{2}{p_j}$ for all j). Then for any solution to* (DP) *its energy* *satisfies*

$$\mathcal{E}(t) \sim t^{-\frac{2}{\alpha}} \quad or \quad \mathcal{E}(t) \sim t^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}} \tag{17}$$

for some $j \in \{1, \ldots, m\}$. Moreover, to each of the $m+1$ decay rates there exists a solution with this decay.

Proof. The moreover part is easy, if all coordinates except u_j are zero, then u_j satisfies (AP). Hence, by [9], it decays as $t^{-\frac{\alpha}{2}}$ if it is an oscilatory coordinate and if it is a non-oscilatory coordinate, then it is a slow solution with $\mathcal{E}_j(t) \sim t^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}$ or a fast solution with $\mathcal{E}_j(t) \sim t^{-\frac{2}{\alpha}}$. Existence of slow solutions follows from Lemma 4, existence of fast solutions was proved in [9, Theorem 3.4] for $g(s) = c|s|^\alpha$, $E(u) = |u|^p$ and the general case can be proved by modifying that proof. It remains to show that no other speeds of convergence appear.

If $m = 0$, i.e. all coordinates are oscilatory, the statement follows from Theorem 14. Let $m \geq 1$. By Corollary 15, any non-oscilatory coodrinate behaves like fast or slow solution. Let $u = (u_1, \ldots, u_n)$ be a solution to (DP). Let us first assume, that u has a slow coordinate and let j be the first coordinate, which is slow, i.e. $\frac{v_j(t)^2}{|u_j(t)|^p} \to 0$ and $\frac{v_i(t)^2}{|u_i(t)|^p} \to +\infty$ for $i = 1, \ldots, j-1$. We show that u satisfies $\mathcal{E}(t) \sim t^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}$.

For $i = 1, \ldots, j-1$ we have by Theorem 12 $|u_i(t)| \leq Ct^{-\frac{1-\alpha}{\alpha}}$, $v_i(t) \leq Ct^{-\frac{1}{\alpha}}$. On the other hand, u_j satisfies $|u_j(t)| \geq Ct^{-\frac{\alpha+1}{p_j-2-\alpha}}$. Since $\beta_i + p_i = (p_i - 1)\frac{2+\alpha}{1+\alpha}$, $\beta_j + p_j = (p_j - 1)\frac{2+\alpha}{1+\alpha}$, and $p_i \geq p_j$, we have

$$|u_i|^{\beta_i+p_i} \leq |u_j|^{\beta_i+p_i} \leq C|u_j|^{\beta_j+p_j}$$

and

$$E_i(u_i) \sim |u_i|^{p_i} \leq |u_j|^{p_i} \leq C|u_j|^{p_j} \sim E_j(u_j)$$

Now, as in the proof of Theorem 14 we obtain (16) and due to

$$\sum_{i=1}^n E_i(u_i) \sim \sum_{i=j}^n E_i(u_i) \quad \text{and} \quad \sum_{i=1}^n |u_i|^{\beta_i+p_i} \sim \sum_{i=j}^n |u_i|^{\beta_i+p_i}$$

we can sum over $i \geq j$ only and obtain

$$-\frac{d}{dt}\frac{H'}{H^B} \sim \frac{\|v\|^{\alpha+2} + \sum_{i=j}^n |u_i|^{\beta_i+p_i}}{\left(\|v\|^2 + \sum_{i=j}^n |u_i|^{p_i}\right)^B}.$$

We can proceed as in the proof of Theorem 14, take $B = (1-\theta_j)\frac{\alpha+2}{\alpha+1}$ and obtain

$$ct^{-\frac{2}{\alpha}} \leq \mathcal{E}(t) \sim H(t) \leq Ct^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}.$$

By Theorem 12 we have $|u_j(t)| \geq ct^{-\frac{1+\alpha}{p_j-2-\alpha}}$. Hence, $|\mathcal{E}_j(t)| \geq ct^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}$ and therefore $\mathcal{E}(t) \sim t^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}$.

It remains to discuss the case when all non-oscilatory coordinates are fast. We show that in this case $\mathcal{E}(t) \sim t^{-\frac{2}{\alpha}}$. If there are no oscilatory coordinates, we are done, since fast coordinates satisfy $\mathcal{E}_i(t) \sim v_i(t)^2 \leq Ct^{-\frac{2}{\alpha}}$ by Theorem 12.

Let us now assume that coordinates $1, \dots, j-1$ are fast non-oscilatory and coordinates $j, \dots, n$ are oscilatory. We show that the fast coordinates $i = 1, \dots, j-1$ satisfy

$$E_i(u_i(t)) \leq Cv_i^2(t) \quad \text{and} \quad |u_i(t)|^{\beta_i+p_i} \leq Cv_i(t)^{2+\alpha}, \tag{18}$$

and therefore we can sum over $i \geq j$ again, i.e.

$$-\frac{d}{dt}\frac{H'}{H^B} \sim \frac{\|v\|^{\alpha+2} + \sum_{i=1}^n |u_i|^{\beta_i+p_i}}{\left(\|v\|^2 + \sum_{i=1}^n E_i(u_i)\right)^B} \sim \frac{\|v\|^{\alpha+2} + \sum_{i=j}^n |u_i|^{\beta_i+p_i}}{\left(\|v\|^2 + \sum_{i=j}^n |u_i|^{p_i}\right)^B}. \tag{19}$$

In fact, the first inequality in (18) follows immediately from the definition of fast solutions and the second inequality in (18) follows from

$$|u_i|^{\beta_i+p_i} = |u_i(t)|^{\frac{2+\alpha}{1+\alpha}(p_i-1)} \leq Cv_i(t)^{\frac{2+\alpha}{1+\alpha}(p_i-1)(1-\alpha)} \leq Cv_i(t)^{2+\alpha}$$

since $\frac{1-\alpha}{1+\alpha}(p_i-1) \geq 1$. Now, we can again proceed as in the proof of Theorem 14, take $B = \frac{2+\alpha}{2}$ and obtain that the right-hand side in (19) is larger than a positive constant, which yields $\mathcal{E}(t) \sim H(t) \leq Ct^{-\frac{2}{\alpha}}$. Hence, $\mathcal{E}(t) \sim t^{-\frac{2}{\alpha}}$. □

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