SHARP AND OPTIMAL DECAY ESTIMATES FOR SOLUTIONS OF GRADIENT-LIKE SYSTEMS

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Abstract. In this paper we study rate of convergence to equilibrium for solutions of abstract gradient-like systems and second order ODEs with damping. The estimates of rate of convergence are based on Łojasiewicz inequality and its generalizations. We prove sharpness of recently derived decay estimates for gradient-like systems with or without the angle condition and give sufficient conditions for optimality of these estimates for particular problems (in terms of inverse Łojasiewicz type inequalities). We also derive new decay estimates for second order problems with weak damping under additional assumptions on the second gradient of the potential and we discuss optimality of these estimates.

1. Introduction

In this paper we study long-time behavior for solutions of ordinary differential equations of first order with a gradient-like structure

$$\dot{u} + \mathcal{F}(u) = 0 \tag{GLS}$$

and damped second order problems

$$\ddot{u} + g(\dot{u}) + \nabla E(u) = 0. \tag{SOP}$$

In particular, we estimate rate of convergence to equilibrium or rate of energy decay for solutions and we investigate sharpness and optimality of these estimates. By sharpness we mean that the estimates are the best possible in the class of problems satisfying the assumptions (e.g. for gradient systems satisfying the Łojasiewicz gradient inequality with the exponent θ, for gradient-like systems satisfying the Kurdyka–Łojasiewicz gradient inequality with function Θ and the angle and comparability conditions, for second order equations with E and g satisfying certain conditions, etc.). By optimality we mean that the estimates are the best possible for every particular problem which satisfies the assumptions.

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Gradient systems.

In 1962 Łojasiewicz has shown (see [20]) that any analytic function $\mathcal{E} : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies the gradient inequality (now called *Łojasiewicz gradient inequality*)

$$|\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta} \leq C\|\nabla\mathcal{E}(u)\| \tag{LI}$$

on a neighborhood of any stationary point φ with some $\theta \in (0, \frac{1}{2}]$. Since then, this inequality (and its generalizations) was applied to many problems (some classes of ODEs, PDEs, Evolution equations) in order to show convergence of solutions to an equilibrium. See e.g. [9], [10], [12], [15], [16], [17], [21], [22].

Later, it was observed that the convergence proofs based on (LI) allow to estimate the rate of convergence. In 2001, Haraux and Jendoubi [17] proved decay estimate

$$\|u(t) - \varphi\| = \begin{cases} O(e^{-ct}) & \text{if } \theta = \frac{1}{2}, \\ O(t^{-\theta/(1-2\theta)}) & \text{if } \theta < \frac{1}{2} \end{cases} \qquad \text{for } t \to +\infty \tag{1}$$

for solutions of a gradient system in $\mathbb{R}^n$

$$\dot{u} + \nabla\mathcal{E}(u) = 0 \tag{GS}$$

with $\mathcal{E}$ satisfying (LI). In 2006, Chill and Fiorenza [11] generalized the result to energy functions satisfying *Kurdyka–Łojasiewicz gradient inequality*

$$\Theta(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \leq \|\nabla\mathcal{E}(u)\| \tag{KLI}$$

on a neighborhood of an equilibrium φ with a non-negative function Θ such that $\frac{1}{\Theta} \in L^1_{loc}([0, +\infty))$. They proved for the gradient system (GS) the decay estimate

$$|\mathcal{E}(u(t)) - \mathcal{E}(\varphi)| = O(\psi^{-1}(t - t_0)), \qquad \text{for } t \to +\infty \tag{2}$$

$$\|u(t) - \varphi\| = O(\Phi(\psi^{-1}(t - t_0))), \qquad \text{for } t \to +\infty \tag{3}$$

where ψ is a primitive function to $-1/\Theta^2$, ψ^{-1} the inverse function to ψ and $\Phi(s) = \int_0^s \frac{1}{\Theta(s)}\, ds$. If we take $\Theta(s) = s^{1-\theta}$, (KLI) becomes (LI) and (3) becomes (1).

Gradient-like systems, linear damping.

Motivated by damped second order problems (SOP), gradient-like systems have been studied. We call (GLS) *a gradient-like system* on $M \subset \mathbb{R}^n$ if there exists a function $\mathcal{E} \in C^1(M)$ such that $\langle \nabla\mathcal{E}(u), \mathcal{F}(u)\rangle > 0$ on M excluding points where $F(u) = 0$ (where the inequality cannot be strict). Such a function $\mathcal{E}$ is called *a strict Lyapunov function*. Of course, for a gradient system

(GS) the function $\mathcal{E}$ is a strict Lyapunov function. It was shown that for proving convergence to equilibrium, so called *angle condition*

$$\langle \nabla \mathcal{E}(u), \mathcal{F}(u) \rangle \geq \alpha \|\mathcal{F}(u)\| \, \|\nabla \mathcal{E}(u)\|, \tag{AC}$$

(α being a positive constant) plays an important role and for decay estimates so called *comparability condition*

$$c\|\nabla \mathcal{E}(u)\| \leq \|\mathcal{F}(u)\| \leq C\|\nabla \mathcal{E}(u)\| \tag{C}$$

(with positive constants c, C) is important (see [2], [3], [6], [19]). These two conditions together are equivalent to

$$\langle \nabla \mathcal{E}(u), \mathcal{F}(u) \rangle \geq c(\|\nabla \mathcal{E}(u)\|^2 + \|\mathcal{F}(u)\|^2) \tag{AC+C}$$

(with a positive constants c). Condition (AC+C) means that the vectors $\mathcal{F}(u)$ and $\nabla \mathcal{E}(u)$ are of comparable sizes and the angle between them is bounded above by a constant strictly less than $\frac{\pi}{2}$.

A second order problem (SOP) can be rewritten as a first order problem

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \begin{pmatrix} -v \\ g(v) + \nabla E(u) \end{pmatrix} = 0. \tag{4}$$

The function

$$\mathcal{E}_1(u, v) = \frac{1}{2}\|v\|^2 + E(u)$$

is a (not strict) Lyapunov function for this problem (if $\langle g(v), v \rangle \geq 0$) and often it becomes a strict Lyapunov function after adding a small term, e.g. $\varepsilon \langle \nabla E(u), v \rangle$. Morever, if the damping function is linear, i.e. $g(\dot{u}) = c\dot{u}$, then the corresponding first order problem with

$$\mathcal{E}(u, v) = \mathcal{E}_1(u, v) + \varepsilon \langle \nabla E(u), v \rangle$$

satisfies (AC+C).

In 2015, Begout, Bolte and Jendoubi [3] proved that the decay estimates (3) are valid also for gradient-like systems satisfying (KLI), (AC+C) and for second order problems with linear damping with E satisfying (KLI), and that they are sharp in the class of gradient-like systems satisfying (KLI), (AC+C). We show that the decay estimate is valid also for second order problems with *linear-like damping*, i.e. if g satisfies

$$\langle g(v), v \rangle \geq c\|v\|^2, \qquad c_1\|v\| \leq g(v) \leq c_2\|v\| \tag{5}$$

Further, we show that the estimates are sharp in the class of second order problems with linear-like (resp. linear) damping. We also show that the decay estimate is optimal for every gradient-like system with (AC+C) where $\mathcal{E}$ satisfies the inverse inequality to (KLI).

Gradient-like systems without (AC+C), weak damping

The fact, that second order problems (SOP) with *weak damping*, i.e. if $g'(0) = 0$, do not satisfy (AC+C), leads to studying more general gradient-like systems. In particular, conditions (KLI) and (AC+C) were in [6] replaced by *generalized Łojasiewicz inequality*

$$\text{(GLI)} \qquad \Theta(\mathcal{E}(u) - \mathcal{E}(\varphi)) \leq \frac{1}{\|\mathcal{F}(u)\|}\langle \nabla\mathcal{E}(u), \mathcal{F}(u)\rangle.$$

It is easy to see that (KLI) and (AC+C) together imply (GLI). The converse is not true, (GLI) is valid e.g. for second order problems with $g(\dot{u}) = c\|\dot{u}\|^{\alpha}\dot{u}$, $\alpha \in (0, 1)$, where (AC+C) does not hold.

Convergence to equilibrium for gradient-like systems satisfying (GLI) was proved in [6]. In [4] it was shown that the decay estimates (3) are not valid in this case (in general) and some other decay estimates were derived. Here we show that the decay estimate for $\mathcal{E}$ derived in [4] is sharp in the class of gradient-like systems satisfying (GLI) and that it is optimal whenever the gradient-like systems satisfies the inverse inequality to (GLI) with a multiplicative constant C. Sharpness (and optimality) of the estimates of $\|u(t) - \varphi\|$ remains open.

Decay estimates for second order problems with weak damping were derived in [9], [8]. Their sharpness and optimality remain open. Here we show better decay estimates under additional assumptions on $\|\nabla^2 E\|$ and in some cases also their optimality if E satisfies the inverse (KLI). Similar results were obtained by Haraux in [14] under different assumptions.

This paper is organized as follows. In Section 2 we present notations, definitions and settings. Section 3 is devoted to the relation of (KLI) resp. inverse (KLI) and the growth of $\mathcal{E}$ in a neighborhood of a critical point. Section 4 contains results on gradient-like systems with (AC+C) and second order problems with linear-like damping (decay estimate for (SOP) with linear-like damping and its sharpness, and optimality for gradient-like systems with (AC+C) if the inverse (KLI) holds). Section 5 is devoted to gradient-like systems satisfying (GLI) (sharpness for estimates of $\mathcal{E}$ and u and optimality of estimate of $\mathcal{E}$ for systems satisfying inverse (GLI)). In Section 6 we derive decay estimates for weakly damped (SOP) under additional assumptions on $\nabla^2 E$ and their optimality in some special cases. In Section 7 we present two examples of (SOP), one showing that in one-dimensional case we obtain optimality for all analytic functions E, second showing that in higher dimensions there is a large class of nice functions where optimal decay estimates are still unknown.

2. Basic definitions and notations

By $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ we denote the usual norm and scalar product on $\mathbb{R}^d$ and $B(\varphi, r)$ denotes the open ball of radius $r > 0$ centered at $\varphi \in \mathbb{R}^d$. For a differentiable function $E : G \subset \mathbb{R}^d \to \mathbb{R}^k$ we denote by $\mathrm{Cr}(E) = \{x \in G : \nabla E(x) = 0\}$ the set of critical points. For nonnegative functions f, $g : G \subset \mathbb{R}^d \to \mathbb{R}$ and a point $a \in \mathbb{R}^d$ we write $f \sim g$ for $x \to a$, if there exist ε, c, $C > 0$ such that $cf(x) \leq g(x) \leq Cf(x)$ for all $x \in B(a, \varepsilon)$. If $g(x) \leq Cf(x)$ for all $x \in B(a, \varepsilon)$ for some $\varepsilon, C > 0$, we write $g(x) = O(f(x))$ for $x \to a$.

We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$

- has *property (K)* if for every $K > 0$ there exists $C(K) > 0$ such that $f(Ks) \leq C(K)f(s)$ holds for all $s > 0$.
- is *C-sublinear* if there exists $C > 0$ such that $f(t + s) \leq C(f(t) + f(s))$ holds for all $t, s > 0$.

It is easy to see that for nondecreasing functions these two properties coincide and that every concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies these properties.

In this paper we study two types of equations (GLS) and (SOP). Whenever we consider the equation (GLS), we assume that $\mathcal{F}$ is a continuously differentiable vector field defined on an open connected set $M \subset \mathbb{R}^n$ and with values in $\mathbb{R}^n$ and that there exists a strict Lyapunov function $\mathcal{E} \in C^2(M)$ to (GLS). If we speak about (SOP) we assume that E is a scalar function defined on an open connected set $\Omega \subset \mathbb{R}^m$ and $E \in C^2(\Omega)$ and $g \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. By a solution to (GLS) or (SOP) we always mean a classical solution defined on $[0, +\infty)$.

For a function $u : [0, +\infty) \to \mathbb{R}^d$ we define the *omega-limit set* by

$$\omega(u) = \{\varphi \in \mathbb{R}^d : \exists t_n \nearrow +\infty,\ u(t_n) \to \varphi\}.$$

We say, that $\varphi \in \Omega$ is an asymptotically stable equilibrium for

$$\dot{u} + \nabla E(u) = 0 \tag{GS_E}$$

if $\varphi \in \mathrm{Cr}(E)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in B(\varphi, \delta)$ the solution u to (GS_E) with $u(0) = u_0$ satisfies $u(t) \in B(\varphi, \varepsilon)$ for all $t \geq 0$ and $\lim_{t\to+\infty} u(t) = \varphi$.

Our basic tool are Łojasiewicz type inequalities. In the introduction, we have introduced (LI), (KLI), (GLI). For optimality results, inverse inequalities play a crucial role: inverse Łojasiewicz inequality

$$|\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta_1} \geq c\|\nabla\mathcal{E}(u)\|, \tag{ILI}$$

inverse Kurdyka–Łojasiewicz inequality

$$\Theta_1(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \geq \|\nabla\mathcal{E}(u)\|, \tag{IKLI}$$

and inverse generalized Łojasiewicz inequality

$$\Theta_1(\mathcal{E}(u)-\mathcal{E}(\varphi)) \geq \frac{1}{\|\mathcal{F}(u)\|}\langle \nabla\mathcal{E}(u), \mathcal{F}(u)\rangle. \tag{IGLI}$$

When we say that inequality (LI) (resp. (KLI), (GLI), (ILI), (IKLI), (IGLI)) holds on a set U it means that the inequality holds for all $u \in U$ with a given fixed φ and Θ (resp. Θ_1, θ, θ_1, c).

A continuous function $\Theta : [0,+\infty) \to [0,+\infty)$ is called *a KL-function* if $\Theta(0) = 0$, $\Theta(s) > 0$ for all $s > 0$, and $\frac{1}{\Theta} \in L^1_{loc}([0,+\infty))$. If a KL-function is nondecreasing and satisfies property (K), then we call it *a KLS-function*. When we consider a gradient-like system (GLS), we usually assume that functions Θ, Θ_1 from (KLI), (GLI) are KL-function, for (SOP) we usually assume they are KLS-function. For a KL-function Θ we define $\Phi_\Theta = \int_0^t \frac{1}{\Theta}$. We often write Φ instead of Φ_Θ. We also define $\psi_\Theta = -\int_{1/2}^t \frac{1}{\Theta^2}$ and often write ψ instead of ψ_Θ. These functions appear in the decay estimates (2), (3). Let us mention that due to [3, Theorem 2.8], if a function $E \in C^2(\Omega)$ satisfies (KLI) with a KL-function Θ on a neighborhood of φ, then necessarily $\Theta(s) = O(\sqrt{s})$ as $s \to 0+$. Since (GLI) implies (KLI), we also have $\Theta(s) = O(\sqrt{s})$ whenever (GLI) holds.

By c, C, $\tilde{C}$ we denote generic constants, their values can change from line to line or from expression to expression.

3. Relation of Łojasiewicz type inequalities and growth of $\mathcal{E}$

In this section we derive some relations between (Kurdyka–)Łojasiewicz inequality resp. its inverse and the growth of E. Basic motivation for doing this is that a typical example of a function E satisfying (LI) with $\theta = \frac{1}{p}$ on a neighborhood of zero is $E(u) = \|u\|^p$ and that sharp decay estimates obtained in [14] are formulated for E satisfying certain growth conditions (instead of a Łojasiewicz type inequality). So, the following statements are useful for comparison of the results of the present paper and the results from [14], but also they are of help for deriving some decay estimates and for simplifying the assumptions of some theorems. A characterization of functions satisfying (KLI) in terms of level sets can be found in [7].

Proposition 1. *Let $E \in C^1(\Omega)$, $\varphi \in \mathrm{Cr}(E)$ and u be a solution to* (GS$_E$) *such that $\varphi \in \omega(u)$. Let E satisfy* (KLI) *on $U = \{u(t) : t \geq 0\}$ with a KL-function Θ. Then*

$$\Phi(E(x)-E(\varphi)) \geq \|x-\varphi\| \tag{6}$$

for all $x \in U$.

Proof. By the well known convergence theorem (see [18, Theorem 2]) we have $\lim_{t\to+\infty} u(t) = \varphi$. Then we can compute

$$-\frac{d}{dt}E(u(t)) = -\nabla E(u(t))\dot{u}(t) = \|\nabla E(u(t))\| \cdot \|\dot{u}(t)\| \geq \Theta(E(u(t) - E(\varphi)) \cdot \|\dot{u}(t)\|,$$

hence

$$-\frac{d}{dt}\Phi(E(u(t)) - E(\varphi)) = -\frac{1}{\Theta(E(u(t) - E(\varphi))}\frac{d}{dt}E(u(t)) \geq \|\dot{u}(t)\|$$

and integrating from 0 to $+\infty$ we obtain

$$\begin{aligned}
\Phi(E(x) - E(\varphi)) &= -\int_0^{+\infty} \frac{d}{dt}\Phi(E(u(t)) - E(\varphi))\,dt \\
&\geq \int_0^{+\infty} \|\dot{u}(t)\|\,dt \\
&\geq \left\|\int_0^{+\infty} \dot{u}(t)\,dt\right\| \\
&= \left\|\lim_{t\to+\infty} u(t) - x\right\| \\
&= \|\varphi - x\|.
\end{aligned}$$

□

Corollary 2. *Let $E \in C^2(\Omega)$, $\varphi \in \mathrm{Cr}(E)$ be an asymptotically stable equilibrium for* (GS$_E$). *Let E satisfy* (KLI) *on a neighborhood of φ with a KL-function Θ. Then* (6) *holds on a neighborhood of φ.*

Remark 3. In particular, if φ is an isolated point of $\mathrm{Cr}(E)$ and E has a local minimum in φ, then φ is an asymptotically stable equilibrium for (GS$_E$).

Proposition 4. *Let $\varphi \in \mathbb{R}^m$, $\varepsilon > 0$ and $B = B(\varphi, \varepsilon)$. Assume that $E \in C^2(B)$ satisfy* (IKLI) *on B with a KL-function Θ. Then $\Phi(E(x) - E(\varphi)) \leq \|x - \varphi\|$ for all $x \in B$.*

Proof. For $x \in B$ and $s \in (0, 1)$ we have

$$\frac{d}{ds}E(\varphi+s(x-\varphi)) \leq \|\nabla E(\varphi+s(x-\varphi))\|\cdot\|x-\varphi\| \leq \Theta(E(\varphi+s(x-\varphi))-E(\varphi))\|x-\varphi\|.$$

Hence,

$$\frac{d}{ds}\Phi(E(\varphi+s(x-\varphi))-E(\varphi)) = \frac{1}{\Theta(E(\varphi + s(x - \varphi)) - E(\varphi))}\cdot\frac{d}{ds}E(\varphi+s(x-\varphi)) \leq \|x-\varphi\|.$$

Integrating from 0 to 1 we obtain

$$\Phi(E(x) - E(\varphi)) \leq \int_0^1 \|x - \varphi\|\,dt = \|x - \varphi\|.$$

□

From the previous propositions we immediately have.

Corollary 5. *Let $E \in C^2(\Omega)$ and $\varphi \in \mathrm{Cr}(E)$. Let E satisfy* (KLI) *with a KL-function Θ and* (IKLI) *with a KL-function $C\Theta$ for some $C \geq 1$ on a neighborhood of φ. Assume that φ is an asymptotically stable equilibrium of* (GS_E)*. Then*

$$\Phi(E(x) - E(\varphi)) \sim \|x - \varphi\| \quad \text{as } x \to \varphi.$$

If $\Theta(s) = s^{1-\theta}$, $\theta \in (0, \frac{1}{2}]$ in previous propositions and corollaries, we obtain

Corollary 6. *1. Let $E \in C^2(\Omega)$ satisfies* (LI) *with $\theta = \frac{1}{p} \in (0, \frac{1}{2}]$ on a neighborhood of φ, with φ being an asymptotically stable equilibrium of* (GS_E)*. Then*

$$E(x) - E(\varphi) \geq c\|x - \varphi\|^p$$

on a neighborhood of φ. 2. Let $E \in C^2(\Omega)$ satisfies (ILI) *with $\theta = \frac{1}{p} \in (0, \frac{1}{2}]$ on $B = B(\varphi, \varepsilon)$ for some $\varepsilon > 0$ and $\varphi \in \mathrm{Cr}(E)$. Then*

$$E(x) - E(\varphi) \leq C\|x - \varphi\|^p \qquad \text{for all } x \in B.$$

In the results on second order equations, we employ some estimates of the second gradient of E. The following results will be of help. It follows immediately when we apply Proposition 4 to the function ∇E instead of E.

Corollary 7. *Let $E \in C^2(B)$ with $B = B(\varphi, \varepsilon)$ for some $\varepsilon > 0$ and $\varphi \in \mathrm{Cr}(E)$ and let*

$$\|\nabla^2 E(x)\| \leq \Gamma(\|\nabla E(x)\|) \tag{7}$$

hold on B with a KL-function Γ. Then $\Phi_\Gamma(\|\nabla E(x)\|) \leq \|x - \varphi\|$ on B. In particular, if

$$\|\nabla^2 E(x)\| \leq C\|\nabla E(x)\|^\gamma \tag{8}$$

on B, then $\|\nabla E(x)\| \leq \tilde{C}\|x - \varphi\|^{\frac{1}{1-\gamma}}$ on B.

Proposition 8. *Let a nonconstant function $E \in C^2(\Omega)$ satisfy* (KLI) *with a KLS-function Θ and* (7) *with a KL-function Γ on a neighborhood of $\varphi \in \mathrm{Cr}(E)$. Let φ be an asymptotically stable equilibrium for* (GS_E)*. Then for some $\delta > 0$ we have $\Phi_\Gamma(s) \leq \Phi(\Theta^{-1}(s))$ for all $s \in (0, \delta)$. Moreover, if $\Phi(\Theta^{-1}(s)) \leq C\Phi_\Gamma(s)$ for some $C \geq 1$, then $\Phi(|E(x) - E(\varphi)|) \sim \|x - \varphi\|$ and $\Phi_\Gamma(\|\nabla E(x)\|) \sim \Phi(\Theta^{-1}(\|\nabla E(x)\|)) \sim \|x - \varphi\|$ on a neighborhood of φ (in particular, if Φ^{-1} has property (K), then* (IKLI) *holds with $\tilde{C}\Theta$ for some $\tilde{C} > 1$ on a neighborhood of φ).*

Proof. By Proposition 1 we have

$$\|x - \varphi\| \leq \Phi(E(x) - E(\varphi)) \leq \Phi(\Theta^{-1}(\|\nabla E(x)\|)).$$

By Corollary 7 we have

$$\|x-\varphi\| \geq \Phi_\Gamma(\|\nabla E(x)\|).$$

Since both inequalities hold on a neigborhood of φ, we have $\Phi_\Gamma(s) \leq \Phi(\Theta^{-1}(s))$ for all $s \in (0,\delta)$, $\delta > 0$ chosen appropriately. The asymptotic equivalence follows immediately from the chain of inequalities

$$\|x-\varphi\| \leq \Phi(E(x)-E(\varphi)) \leq \Phi(\Theta^{-1}(\|\nabla E(x)\|)) \leq C\Phi_\Gamma(\|\nabla E(x)\|) \leq C\|x-\varphi\|$$

and (IKLI) follows from

$$\|\nabla E(x)\| \leq \Theta(\Phi^{-1}(C\|x-\varphi\|)) \leq \tilde{C}\Theta(\Phi^{-1}(\|x-\varphi\|)) \leq \tilde{C}\Theta(E(x)-E(\varphi)),$$

where we used property (K) in the second step and inequality $\|x-\varphi\| \leq \Phi(E(x)-E(\varphi))$ in the last step. □

Corollary 9. *Let a nonconstant function $E \in C^2(\Omega)$ satisfy* (LI) *with $\theta \in (0, \frac{1}{2}]$ and* (8) *with $\gamma \geq 0$ on a neighborhood of $\varphi \in \mathrm{Cr}(E)$. Let φ be an asymptotically stable equilibrium for* (GS_E)*. Then $\gamma \leq \frac{1-2\theta}{1-\theta}$. If the equality holds, then $\|E(x)\| \sim \|x-\varphi\|^{\frac{1}{\theta}}$ and $\|\nabla E(x)\| \sim \|x-\varphi\|^{\frac{1-\theta}{\theta}}$ (in particular,* (ILI) *holds with $\theta_1 = \theta$) on a neighborhood of φ.*

Due to the Łojasiewicz's result, every analytic function $E : \mathbb{R}^n \to \mathbb{R}$ satisfies (LI) on a neighborhood of every critical point φ with some $\theta \in (0, \frac{1}{2}]$. As the following proposition shows, if $n = 1$, then (ILI) is satisfied with the same exponent $\theta_1 = \theta$ and the growth of E on the neighborhood of φ is uniquely determined by θ (it follows from Corollary 6 or directly from the proof below). On the other hand, for $n \geq 2$ there are analytic functions with $\theta_1 > \theta$, a simple example is

$$E(x,y) = |x|^p + |y|^q.$$

If $p > q \geq 2$, then (LI) holds with $\theta \leq \frac{1}{p}$ and (ILI) holds with $\theta_1 \geq \frac{1}{q}$ (see Example 26)

Proposition 10. *Let $E : \mathbb{R} \to \mathbb{R}$ be analytic and $\varphi \in \mathrm{Cr}(E)$. Then there exists $\theta \in (0, \frac{1}{2}]$ such that* (LI), (ILI) *with $\theta_1 = \theta$ and* (8) *with $\gamma = \frac{1-2\theta}{1-\theta}$ are satisfied on a neighborhood of φ.*

Proof. We have

$$E(x) = E(\varphi) + \sum_{n=k}^{\infty} a_n (x-\varphi)^n = E(\varphi) + a_k(x-\varphi)^k + o((x-\varphi)^k)$$

on a neigborhood of φ for appropriate $a_n \in \mathbb{R}$, $k \in \mathbb{N}$, $k \geq 2$. It follows that

$$E'(x) = \sum_{n=k}^{\infty} n a_n (x-\varphi)^{n-1} = k a_k (x-\varphi)^{k-1} + o((x-\varphi)^{k-1})$$

on a neighborhood of φ. Hence,

$$|E(x) - E(\varphi)| \le 2|a_k||x-\varphi|^k = C\left(\frac{1}{2}k|a_k||x-\varphi|^{k-1}\right)^{\frac{k}{k-1}} \le C|E'|^{\frac{k}{k-1}}.$$

So, (LI) holds with $1-\theta = \frac{k-1}{k}$, i.e. $\theta = \frac{1}{k}$. Similarly, we have (ILI). Further,

$$E''(x) = \sum_{n=k}^{\infty} n(n-1)a_n(x-\varphi)^{n-2} = k(k-1)a_k(x-\varphi)^{k-2} + o((x-\varphi)^{k-2})$$

and therefore (8) holds with $\gamma = \frac{k-2}{k-1} = \frac{1-2\theta}{1-\theta}$. □

4. Gradient-like systems with (AC+C) and (SOP) with linear damping

In this section we study gradient-like systems with (AC+C). It was shown in [3] that (SOP) with linear damping can be rewritten as a gradient like system satisfying (AC+C) and that these systems satisfy the same decay estimate as gradient systems. We first show that also (SOP) with linear-like damping belongs to the same category. Then we turn our attention to sharpness and optimality of the decay estimates.

Theorem 11. *Let us consider* (SOP) *and let* $\varphi \in \mathrm{Cr}(E)$. *Let* E *satisfy* (KLI) *on a neighborhood of* φ *with a KLS-function* Θ *and let* g *satisfy* (5). *Let* u *be a bounded solution to* (SOP) *with* $\varphi \in \omega(u)$. *Then the decay estimate* (3) *holds.*

Proof. Let us rewrite the equation as a first order system and define

$$\mathcal{E}(u,v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon\langle \nabla E(u), v\rangle$$

where $\varepsilon > 0$ is small. We show that $\mathcal{E}$ is a strict Lyapunov function for the first order system and (KLI), (AC+C) are satisfied. Then (3) follows from [3, Theorem 3.7] and the statement is proved.

Let us denote $v(t) = \dot{u}(t)$ and compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u(t),v(t)) &= \langle v, \dot{v}\rangle + \nabla E(u)\dot{u} + \varepsilon\langle \nabla^2 E(u)\dot{u}, v\rangle + \varepsilon\langle \nabla E(u), \dot{v}\rangle \\
&= -\langle v, g(v)\rangle + \varepsilon\langle \nabla^2 E(u)v, v\rangle - \varepsilon\langle \nabla E(u), g(v)\rangle - \varepsilon\langle \nabla E(u), \nabla E(u)\rangle \\
&\le -c\|v\|^2 + \varepsilon C\|v\|^2 + \varepsilon c_2\|\nabla E(u)\|\|v\| - \varepsilon\|\nabla E(u)\|^2 \\
&\le -(c - \varepsilon C - \frac{\varepsilon}{2}c_2^2)\|v\|^2 - \frac{\varepsilon}{2}\|\nabla E(u)\|^2 \\
&\le -c(\|v\|^2 + \|\nabla E(u)\|^2),
\end{aligned}$$

where we used the definition of $\mathcal{E}$ in the first equality, the equation (SOP) in the second equality, boundedness of $\nabla^2 E$, estimates for g, and Cauchy-Schwarz inequality in the third line, Young inequality in the fourth line and we have taken ε small enough in the last line. In the following let us write $\nabla\mathcal{E}$ instead of $\nabla\mathcal{E}(u,v)$ and similarly for $\mathcal{E}, \mathcal{F}$. Since

$$\nabla\mathcal{E} = \begin{pmatrix} \nabla E(u) + \varepsilon\nabla^2 E(u)v \\ v + \varepsilon\nabla E(u) \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} -v \\ -g(v) - \nabla E(u) \end{pmatrix},$$

we have

$$\|\nabla\mathcal{E}\|^2 \le C(\|v\|^2 + \|\nabla E(u)\|^2), \qquad \|\mathcal{F}\|^2 \le C(\|v\|^2 + \|\nabla E(u)\|^2) \tag{9}$$

and (AC+C) follows from

$$\langle \nabla\mathcal{E}, \mathcal{F}\rangle = -\frac{d}{dt}\mathcal{E}(u(t), v(t)) \ge c(\|v\|^2 + \|\nabla E(u)\|^2) \ge c(\|\nabla E\|^2 + \|\mathcal{F}\|^2). \tag{10}$$

It remains to show (KLI). By Cauchy-Schwarz and Young inequalities, monotonicity and C-sublinearity of Θ, property (K), $\Theta(s) \le C\sqrt{s}$ and $\Theta(E) \le \|\nabla E\|$ we have

$$\begin{aligned} \Theta(\mathcal{E}) &\le \Theta\left(\frac{1}{2}\|v\|^2 + E(u) + \varepsilon\|\nabla E(u)\|^2 + \varepsilon\|v\|^2\right) \\ &\le C\left(\Theta(\frac{1}{2}\|v\|^2) + \Theta(E(u)) + \Theta(\|\nabla E(u)\|^2) + \Theta(\|v\|^2)\right) \\ &\le C\left(\|v\| + \|\nabla E(u)\| + \|\nabla E(u)\| + \|v\|\right). \end{aligned}$$

To complete the proof we show that $\|v\| + \|\nabla E(u)\| \le C\|\nabla\mathcal{E}\|$. Assume that $\varepsilon < \frac{1}{2}$ is so small that $\varepsilon\|\nabla^2 E(u)\| < \frac{1}{2}$ on a neighborhood of φ. If $\|v\| \le \|\nabla E(u)\|$, then we consider the first coordinate of $\nabla\mathcal{E}$ to get

$$\|\nabla\mathcal{E}\| \ge \|\nabla E(u) + \varepsilon\nabla^2 E(u)v\| \ge \|\nabla E(u)\| - \frac{1}{2}\|v\| \ge \frac{1}{2}\|\nabla E(u)\| \ge \frac{1}{4}(\|v\| + \|\nabla E(u)\|).$$

If $\|v\| \ge \|\nabla E(u)\|$, then we take the second coordinate of $\nabla\mathcal{E}$ and we obtain

$$\|\nabla\mathcal{E}\| \ge \|v + \varepsilon\nabla E(u)\| \ge \|v\| - \frac{1}{2}\|\nabla E(u)\| \ge \frac{1}{2}\|v\| \ge \frac{1}{4}(\|v\| + \|\nabla E(u)\|).$$

□

In [3, Section 3.2] it was shown that the decay estimates (3) are sharp in the class of gradient systems (therefore they are sharp in the class of gradient-like systems with (AC+C) as well). In fact, the gradient system on $\mathbb{R}$ with $\mathcal{E}(u) = \Phi^{-1}(|u|)$ satisfies (KLI) with a KL-function Θ and the solutions $u(t) = \pm\Phi(\psi^{-1}(t - t_0))$ have the decay equal to the upper estimate (3). Now, we show that the decay estimates (3) are sharp in the class of (SOP) with linear damping (resp. linear-like damping).

Proposition 12 (Sharpness for (SOP) with linear damping)**.** *For every differentiable KLS-function Θ satisfying $\Theta(s) = O(\sqrt{s})$ as $s \to 0+$ there exists*

$B = B(0,\varepsilon) \subset \mathbb{R}^m$, $\alpha > 0$ and $E \in C^2(B)$ satisfying (KLI) *on B such that* (SOP) *with linear damping $g(v) = \alpha v$ has a solution u satisfying*

$$\lim_{t\to+\infty} \frac{\|u(t)\|}{\Phi(\psi^{-1}(t))} > 0.$$

Proof. We find such an equation for $n = 1$, for $n \geq 1$ it is enough to consider the corresponding radially symmetric system. For $n = 1$ let us find α and E such that $u(t) = \Phi(\psi^{-1}(t))$ be a solution to (SOP). We have

$$\dot{u}(t) = \Phi'(\psi^{-1}(t)) \cdot \frac{1}{\psi'(\psi^{-1}(t))} = \frac{1}{\Theta(\psi^{-1}(t))} \cdot -\Theta^2(\psi^{-1}(t)) = -\Theta(\psi^{-1}(t))$$

and

$$\ddot{u}(t) = -\Theta'(\psi^{-1}(t))\Theta^2(\psi^{-1}(t)).$$

Set $\alpha = 1$ and

$$E_1(u) = \Phi^{-1}(u), \quad \text{then} \quad E_1'(u) = \frac{1}{\Phi'(\Phi^{-1}(u))} = \Theta(\Phi^{-1}(u)) = \Theta(\psi^{-1}(t)).$$

Then we have (setting $z = \psi^{-1}(t)$)

$$\ddot{u} + \alpha\dot{u} + \nabla E_1(u) = -\Theta'(z)\Theta^2(z) - \Theta(z) + \Theta(z) = -\Theta'(z)\Theta^2(z).$$

It is enough to take $E = E_1 + E_2$ with $E_2(u)$ satisfying $\nabla E_2(u(t)) = \nabla E_2(\Phi(\psi^{-1}(t))) = \Theta'(\psi^{-1}(t))\Theta^2(\psi^{-1}(t))$, i. e.,

$$\nabla E_2(w) = \Theta'(\Phi^{-1}(w))\Theta^2(\Phi^{-1}(w)) = \Theta'(\Phi^{-1}(w))\Theta^2(\Phi^{-1}(w))\frac{\Phi'(\Phi^{-1}(w))}{\Phi'(\Phi^{-1}(w))}.$$

Since $\Phi' = 1/\Theta$, we have

$$\nabla E_2(w) = \frac{\Theta'(\Phi^{-1}(w))\Theta(\Phi^{-1}(w))}{\Phi'(\Phi^{-1}(w))} = \frac{d}{dw}\frac{1}{2}\Theta^2(\Phi^{-1}(w)),$$

so

$$E_2(w) = \frac{1}{2}\Theta^2(\Phi^{-1}(w)).$$

It remains to show that $E(u) = \Phi^{-1}(|u|) + \frac{1}{2}\Theta^2(\Phi^{-1}(|u|))$ satisfies (KLI). Since Θ is small on a neighborhood of zero, we have $E(u) \leq C\Phi^{-1}(|u|)$. By property (K), we have $\Theta(E(u)) \leq \tilde{C}\Theta(\Phi^{-1}(|u|))$. On the other hand, it holds that

$$\begin{aligned}
|\nabla E(u)| &= \Theta(\Phi^{-1}(|u|)) + \frac{\Theta'(\Phi^{-1}(|u|))\Theta(\Phi^{-1}(|u|))}{\Phi'(\Phi^{-1}(|u|))} \\
&= \Theta(\Phi^{-1}(|u|))(1 + \Theta\Theta') \\
&\geq \Theta(\Phi^{-1}(|u|))
\end{aligned}$$

since Θ and Θ' are non-negative. So, we have $\Theta(E(u)) \leq \tilde{C}|\nabla E(u)|$ and the proof is complete. □

Theorem 13 (Optimality for (GLS) with (AC+C))**.** *Let us consider a gradient-like system* (GLS) *with a strict Lyapunov function* $\mathcal{E} \in C^2(M)$*. Let* $\varphi \in \mathrm{Cr}(\nabla\mathcal{E})$ *and let* $\mathcal{E}$ *satisfy* (AC+C) *and* (KLI) *with a KL-function* Θ *on a neighborhood of* φ*. If, moreover,* $\mathcal{E}$ *satisfies* (IKLI) *with* $\Theta_1 = C\Theta$ *for some* $C > 0$ *on a neighborhood of* φ*, then the decay estimates* (3), (2) *are optimal, i.e. any solution* u *to* (GLS) *with* $\varphi \in \omega(u)$ *satisfies*

$$\psi^{-1}(ct) \le \mathcal{E}(u(t)) - \mathcal{E}(\varphi) \le \psi^{-1}(t)$$

$$\tilde{c}\Phi(\psi^{-1}(ct)) \le \|u(t) - \varphi\| \le \Phi(\psi^{-1}(t))$$

as $t \to +\infty$ *for appropriate constants* $c, \tilde{c} > 0$*.*

Remark 14. Here we cannot say $\mathcal{E}(u(t)) - \mathcal{E}(\varphi) \sim \psi^{-1}(t)$ if φ does not have property (K). This can happen, e.g. if $\Theta(s) = s^{1/2}$, then $\psi^{-1}(s) = e^{-s}$.

Proof. The upper estimates are known ([3, Theorem 3.7 and its proof]), it remains to show the lower estimates. Let t_0 be so large that $u(t)$ is in the neighborhood of φ where (IKLI) holds for all $t \ge t_0$. Then we can compute

$$\begin{aligned}
\frac{d}{dt}\psi(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) &= -\frac{1}{\Theta^2(\mathcal{E}(u(t)) - \mathcal{E}(\varphi))}\langle \nabla\mathcal{E}(u(t)), \dot{u}(t)\rangle \\
&= \frac{1}{\Theta^2(\mathcal{E}(u(t)) - \mathcal{E}(\varphi))}\langle \nabla\mathcal{E}(u(t)), F(u(t))\rangle \\
&\le \frac{1}{\Theta^2(\mathcal{E}(u(t)) - \mathcal{E}(\varphi))} C\|\nabla\mathcal{E}(u(t))\|^2 \\
&\le C.
\end{aligned}$$

Integrating this inequality from t_0 to $t > t_0$ we obtain

$$\psi(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) \le C(t - t_0) + \psi(\mathcal{E}(u(t_0)) - \mathcal{E}(\varphi)) \le Ct$$

for large t. Hence, $\mathcal{E}(u(t)) - \mathcal{E}(\varphi) \ge \psi^{-1}(Ct)$. Further, by Proposition 4 we have

$$\|u(t) - \varphi\| \ge \Phi_{\Theta_1}(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) = C\Phi(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) \ge C\Phi(\psi^{-1}(\tilde{C}t)).$$

□

Similar optimality result for (SOP) with linear or linear-like damping does not hold. It is due to the fact, that $\mathcal{E}$ in general does not satisfy (IKLI) with the same function as E. In fact, since by (IKLI) we have

$$\mathcal{E}(u, v) \ge (1 - \varepsilon)\|v\|^2 + E(u) - C\varepsilon\theta(E(u))^2 \ge (1 - \varepsilon)\|v\|^2 + (1 - \varepsilon\tilde{C})E(u)$$

it follows that

$$\Theta(\mathcal{E}(u, v)) \ge c\left(\Theta(\|v\|^2) + \|\nabla E(u)\|\right)$$

instead of $\Theta(\mathcal{E}) \geq c\,(\|v\| + \|\nabla E(u)\|)$ that we need for optimality. So, the decay estimates are optimal for (SOP) if $\Theta(s) = C\sqrt{s}$. On the other hand, we always have

$$\sqrt{\mathcal{E}(u,v)} \geq c\,(\|v\| + \|\nabla E(u)\|)\,,$$

i.e. $\mathcal{E}$ satisfies (IKLI) with $\Theta_1(s) = C\sqrt{s}$, which implies

$$\|u(t) - \varphi\| \geq \Phi_{\Theta_1}(\psi^{-1}(Ct)) = ce^{-Ct}.$$

Haraux has proved in [13] that there always exist solutions to $\ddot{u}+\dot{u}+\nabla E(u) = 0$ decaying to zero exponentially. Question, whether the decay estimate (3) is optimal at least for some solutions of any (SOP) with E satisfying (KLI), (IKLI), remains open. In [13] optimality for $\nabla E(u) = u^3$ is proved.

5. Optimality for gradient-like systems without (AC+C)

Motivated by second order problems with weak damping we have introduced in [6] a new sufficient condition for convergence of solutions to gradient-like systems, condition (GLI). A convergence result was proved in [6], decay estimates were derived in [4]. In this section we first show that the decay estimate for $\mathcal{E}$ derived in [4] hold under more general assumptions, even in the case when $\frac{1}{\Theta}$ is not integrable at zero (this means that $\omega(u)$ may contain more than one point). Then we show that the estimates of $\mathcal{E}$ and the estimates of the length of the trajectory are sharp and that they are optimal if (IGLI) and an inverse to condition (11) hold.

Let us first formulate the result on decay estimates.

Theorem 15 (Decay for (GLS) without (AC+C))**.** *Let $\mathcal{E} \in C^1(M)$ be a strict Lyapunov function to* (GLS) *and let $u : \mathbb{R}_+ \to M$ be a nonconstant solution of* (GLS) *and $\varphi \in \omega(u)$. Assume that $\mathcal{E}$ satisfies* (GLI) *on a neighborhood of $\omega(u)$ with a continuous function $\Theta : [0,+\infty) \to [0,+\infty)$ satisfying $\Theta(0) = 0$ and $\Theta(s) > 0$ for all $s > 0$. Assume that $\alpha : (0,1) \to (0,+\infty)$ is nondecreasing and satisfies*

$$\alpha(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) \geq \|F(u(t))\| \quad \textit{for all } t \textit{ large enough.} \tag{11}$$

Then we have for some $c > 0$

$$\|\mathcal{E}(u(t)) - \mathcal{E}(\varphi)\| = O(\tilde{\psi}^{-1}(ct)) \quad \textit{as } t \to +\infty \tag{12}$$

with

$$\tilde{\psi}(s) := \int_s^{\frac{1}{2}} \frac{1}{\Theta(r)\alpha(r)}dr. \tag{13}$$

Moreover, if $\frac{1}{\Theta} \in L^1_{loc}([0,+\infty))$ (i.e. Θ is a KL-function), then $\lim\limits_{t\to+\infty} u(t) = \varphi$ and

$$\|u(t) - \varphi\| \leq \int_t^{+\infty} \|\dot{u}\| = O(\Phi(\tilde{\psi}^{-1}(ct))) \quad \textit{for } t \to +\infty. \tag{14}$$

We can see that the decay estimates are similar to the (AC+C) case, the only difference is that in the definition of $\tilde{\psi}$ we have $\Theta\alpha$ in contrast to ψ_Θ where we had Θ^2.

Proof. The proof is the same as in [4, Theorem 1]. In fact, we may assume $\mathcal{E}(\varphi) = 0$ to shorten the formulas and compute

$$\begin{aligned}
\frac{d}{dt}\tilde{\psi}(\mathcal{E}(u(t))) &= \tilde{\psi}'(\mathcal{E}(u(t)))\langle\nabla\mathcal{E}(u(t)), \dot{u}(t)\rangle \\
&= -\tilde{\psi}'(\mathcal{E}(u(t)))\langle\nabla\mathcal{E}(u(t)), F(u(t))\rangle \\
&\geq -\tilde{\psi}'(\mathcal{E}(u(t)))\Theta(\mathcal{E}(u(t)))\|F(u(t))\| \\
&\geq -\tilde{\psi}'(\mathcal{E}(u(t)))\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t))) \\
&= 1.
\end{aligned} \tag{15}$$

Integrating this inequality from t_0 to $t > t_0$ we obtain

$$\tilde{\psi}(\mathcal{E}(u(t))) \geq (t - t_0) + \tilde{\psi}(\mathcal{E}(u(t_0))) = t - t_0 + c.$$

Since $\tilde{\psi}$ is decreasing, we have $\mathcal{E}(u(t)) \leq \tilde{\psi}^{-1}(t - t_0 + c) \leq \tilde{\psi}^{-1}(ct)$. The estimate (14) follows from

$$\begin{aligned}
\|u(t) - \varphi\| &\leq \int_t^{+\infty} \|\dot{u}\| \\
&= \int_t^{+\infty} \|\mathcal{F}(u)\| \\
&\leq \int_t^{+\infty} \frac{\langle\nabla\mathcal{E}(u), \mathcal{F}(u)\rangle}{\Theta(\mathcal{E}(u))} \\
&= -\int_t^{+\infty} \frac{\langle\nabla\mathcal{E}(u), \dot{u}\rangle}{\Theta(\mathcal{E}(u))} \\
&= \Phi(\mathcal{E}(u(t))) - \lim_{t\to+\infty} \Phi(\mathcal{E}(u(s))) \\
&= \Phi(\mathcal{E}(u(t))) \\
&\leq \Phi(\tilde{\psi}^{-1}(t - t_0 + c),
\end{aligned} \tag{16}$$

if $\mathcal{E}(u(s)) > 0$ for all $s \geq t_0$. If $\mathcal{E}(u(s)) = 0$, (14) is trivial. Since $\Phi(0) = 0$ and $\lim_{s\to 0+} \tilde{\psi}(s) = +\infty$, (14) implies $\lim_{t\to+\infty} u(t) = \varphi$. □

In the following theorem we show that the estimates of the decay of $\mathcal{E}$ and the length of the trajectory $\int_t^{+\infty} \|\dot{u}\|$ are optimal if inverse inequalities to (GLI) and (11) hold.

Theorem 16 (Optimality for (GLS) without (AC+C))**.** *Let the assumptions of Theorem 15 are satisfied. Further, we assume that there exists $c > 0$ such that*

(IGLI) *holds on a neighborhood of* $\omega(u)$ *with* $\Theta_1 = \frac{1}{c}\Theta$ *and*

$$c\alpha(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) \le \|F(u(t))\| \tag{17}$$

holds for all t large enough. Then

$$\mathcal{E}(u(t)) - \mathcal{E}(\varphi) \ge \tilde{\psi}^{-1}(Ct) \qquad \text{and} \qquad \int_t^{+\infty} \|\dot{u}\| \ge \frac{1}{C}\Phi(\tilde{\psi}^{-1}(Ct))) \tag{18}$$

for all t large enough with an appropriate constant $C > 0$ *and* $\tilde{\psi}$ *defined in* (13).

Proof. Let us again assume that $\mathcal{E}(\varphi) = 0$. We can replace the inequalities in (15) with the oposite ones and obtain

$$\frac{d}{dt}\tilde{\psi}(\mathcal{E}(u(t))) \le C,$$

so $\mathcal{E}(u(t)) \ge \tilde{\psi}^{-1}(C(t - t_0) - \tilde{\psi}(\mathcal{E}(u(t_0)))) \ge \tilde{\psi}^{-1}(Ct)$. Reversing the inequalities in (16) (except the first one) we prove the second inequality in (18). □

To show that the estimates for $\mathcal{E}$ and $\int_t^{+\infty} \|\dot{u}\|$ are sharp, it is enough to find a problem satisfying the assumptions of Theorem 16.

Corollary 17 (Sharpness for (GLS) without (AC+C))**.** *Let* Θ *be a KL-function satisfying* $\Theta(s) = O(\sqrt{s})$ *as* $s \to 0+$ *and let* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous nondecreasing with* $\alpha(0) = 0$ *and* $\alpha(s) > 0$ *for* $s > 0$*. There exists* $\mathcal{F} \in C(\mathbb{R}^2)$ *and a strict Lyapunov function* $\mathcal{E} \in C^2(\mathbb{R}^2)$ *satisfying* (11) *and* (GLI) *with* $\varphi = 0$ *and there exists a solution* $u : \mathbb{R}_+ \to M$ *to* (GLS) *such that* (18) *holds.*

Proof. For given Θ and α let us define $\mathcal{E}(u) = \|u\|^2$ for $u \in \mathbb{R}^2$ and

$$\mathcal{F}(u) = \beta(\|u\|)\left(\frac{\Theta(\|u\|^2)}{\|u\|}u_1 - u_2, u_1 + \frac{\Theta(\|u\|^2)}{\|u\|}u_2\right)$$

with $\beta(s) = \frac{\alpha(s^2)}{s}$ for $s > 0$ and $\beta(0) = 0$. Then

$$\|\mathcal{F}(u)\|^2 = \beta^2(\|u\|)\|u\|^2\left(1 + \frac{\Theta^2(\|u\|^2)}{\|u\|^2}\right)$$

and due to $\Theta(s) \le C\sqrt{s}$ we have

$$\beta(\|u\|)\|u\| \le \|\mathcal{F}(u)\| \le \beta(\|u\|)\|u\|\sqrt{1 + C}.$$

Since

$$\langle \nabla\mathcal{E}(u), \mathcal{F}(u)\rangle = 2\beta(\|u\|)\Theta(\|u\|^2)\|u\|,$$

(GLI) and (IGLI) hold with Θ and $\Theta_1 = C\Theta$. Further, we have

$$\alpha(\mathcal{E}(u)) = \sqrt{\mathcal{E}(u)}\beta(\sqrt{\mathcal{E}(u)}) = \|u\|\beta(\|u\|),$$

so (11) and (17) hold. Since any solution converges to the origin (transformation to polar coordinates yields $r' = -\beta(r)\Theta(r^2)$), the lower bounds (18) follow from Theorem 16. □

It is not clear, whether the estimate for $\|u(t) - \varphi\|$ in (14) is sharp (i.e. optimal for some problems). Obviously, for the problem from the proof of Corollary 17 we have $\|u(t)\| = \sqrt{\mathcal{E}(u(t))} \leq \sqrt{\psi^{-1}(Ct)}$, i.e. the estimate (14) is optimal (for this particular problem) only if $\Phi(s) \sim \sqrt{s}$, i.e. $\Theta(s) \sim C\sqrt{s}$ which corresponds to the Łojasiewicz exponent $\theta = \frac{1}{2}$ (like in the case of linearly damped second order problems).

Obviously, if (GLI) is satisfied with a function Θ, then (KLI) holds with the same Θ (we assume $\varphi = 0$, $\mathcal{E}(\varphi) = 0$ in this paragraph):

$$\Theta(\mathcal{E}) \leq \frac{1}{\|\mathcal{F}\|}\langle F, \nabla\mathcal{E}\rangle \leq \|\nabla\mathcal{E}\|.$$

But sometimes (and it seems to be a typical case for equations without (AC)) $\mathcal{E}$ can satisfy (KLI) with a better (it means bigger) function $\tilde{\Theta}$. In fact, if the angle condition (AC) holds, then

$$\tilde{\Theta}(\mathcal{E}) \leq \|\nabla\mathcal{E}\| \qquad \Rightarrow \qquad \Theta(\mathcal{E}) \leq \frac{1}{c}\frac{1}{\|\mathcal{F}\|}\langle\mathcal{F}, \nabla\mathcal{E}\rangle.$$

But if the angle $\rho(u)$ between $\nabla\mathcal{E}(u)$ and $\mathcal{F}(u)$ tends to $\frac{\pi}{2}$ as $u \to 0$, we have

$$\|\nabla\mathcal{E}(u)\| = \frac{1}{\cos(\rho(u))}\frac{1}{\|\mathcal{F}(u)\|}\langle\mathcal{F}(u), \nabla\mathcal{E}(u)\rangle \geq \frac{1}{\cos(\rho(u))}\Theta(\mathcal{E}(u)) =: \tilde{\Theta}(\mathcal{E}(u)).$$

If $\mathcal{E}$ satisfies (KLI) with $\tilde{\Theta}$ on a neighborhood of zero, then the growth of $\mathcal{E}$ on the neighborhood of zero is typically determined by $\tilde{\Theta}$ (see Proposition 1 and its corollaries). This yields the following corollary.

Corollary 18. *Let the assumptions of Theorem 15 hold. Let $\tilde{\Phi}(\mathcal{E}(u) - \mathcal{E}(\varphi)) \geq \|u - \varphi\|$ hold on a neighborhood of φ. Then $\|u(t) - \varphi\| \leq C\tilde{\Phi}(\psi^{-1}(Ct))$ for t large. If, moreover, assumptions of Theorem 16 holds and $\tilde{\Phi}(\mathcal{E}(u) - \mathcal{E}(\varphi)) \leq C\|u - \varphi\|$, then the decay estimate $C\tilde{\Phi}(\psi^{-1}(Ct))$ is optimal. In particular, if $\mathcal{E}$ satisfies* (KLI) *with $\tilde{\Theta}$ and* (IKLI) *with $C\tilde{\Theta}$ and φ is asymptoticaly stable for* (GS_E)*, then these decay estimates hold and are optimal.*

6. Optimal decay estimates for second order problem with weak damping

In this section we consider the second order problem (SOP) with weak damping, i.e. $g'(0) = 0$. For E satisfying (LI) with $\theta \in (0, \frac{1}{2}]$ and $g(s) = |s|^\alpha s$ convergence to a stationary point was proved by Chergui in [9] for $\alpha \in (0, \frac{\theta}{1-\theta})$ accompanied by decay estimates

$$\|u(t) - \varphi\| + \|\dot{u}\| \leq Ct^{-\frac{\theta-\alpha(1-\theta)}{1-2\theta+\alpha(1-\theta)}}. \tag{19}$$

This decay estimate was generalized to more general damping functions

$$C\|v\|^{\alpha+2} \geq \langle g(v), v\rangle \geq c\|v\|^{\alpha+2} \tag{20}$$

in [8] and for E satisfying (KLI) and even more general damping functions $\langle g(v), v\rangle \geq h(\|v\|)\|v\|^2$ in [4]. The generalized decay estimates read

$$\|u(t) - \varphi\| + \|\dot{u}\| \leq C\Phi_h(\psi_h^{-1}(Ct)), \tag{21}$$

$$\mathcal{E}(u, v) - \mathcal{E}(\varphi, 0) \leq \psi_h^{-1}(Ct), \tag{22}$$

where $\mathcal{E}(u, v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon h(\|v\|)\langle \nabla E(u), v\rangle$,

$$\Phi_h(t) = C_1 \int_0^t \frac{1}{\Theta(s)h(\Theta(s))} ds, \quad \psi_h(t) = C_2 \int_t^{\frac{1}{2}} \frac{1}{\Theta^2(s)h(\Theta(s))} ds. \tag{23}$$

If $h(s) = s^\alpha$, $\Theta(s) = s^{1-\theta}$, then (21) becomes (19). It is an open problem, whether these estimates are sharp.

One can show that for many special cases (most common in applications), these estimates are not optimal. In fact, Haraux has shown in [14] better decay estimates if E satisfies

$$\langle \nabla E(u), u\rangle \geq c\|u\|^p, \qquad \|E(u)\| \leq C\|u\|^p.$$

For scalar equation

$$\ddot{u} + c|\dot{u}|^\alpha \dot{u} + d|u|^{p-2}u = 0$$

he also showed that if $\alpha > 1 - \frac{2}{p}$ (which corresponds to the case when the damping force is small and the solution oscilates), then every solution has the same decay $\mathcal{E}(t) = |v(t)|^2 + E(u(t)) \sim Ct^{-\frac{2}{\alpha}}$. If $\alpha < 1 - \frac{2}{p}$ (the damping force is large and the solution slows down very quickly and converges to an equilibrium without oscilations), then there exist exactly two types of solutions: 'slow solutions' with $\mathcal{E}(t) \sim Ct^{-\frac{p(\alpha+1)}{p-2-\alpha}}$ and 'fast solutions' with $\mathcal{E}(t) \sim Ct^{-\frac{2}{\alpha}}$. By [1] the same is true for the vector equation

$$\ddot{u} + c\|\dot{u}\|^\alpha \dot{u} + d\|A^{1/2}u\|^{p-2}A^{1/2}u = 0$$

with A being a symmetric, positive definite matrix.

We study (SOP) with E satisfying (LI) and (8) with a constant $\gamma \geq 0$. We show that if $\gamma > 0$ one obtains better estimates than (21) and if $\gamma = \frac{1-2\theta}{1-\theta}$ we obtain the same decay estimates as in [14] and in fact, also the assumptions formulated in terms of (LI), (ILI) are very close to those in [14].

Let us consider (SOP) and its solution u with a nonempty omega-limit set and $\varphi \in \omega(u)$. In order to shorten the formulas we shall assume that $\varphi = 0$, $E(\varphi) = 0$. We assume that E satisfies (LI) and (8) on a neighborhood of $\omega(u)$ and that g satisfies

$$\langle g(v), v\rangle \geq c\|v\|^{\alpha+2} \quad \text{and} \quad |g(v)| \leq C\|v\|^\delta \tag{24}$$

on bounded sets. Then neccessarily $\delta \leq \alpha+1$ (in the special case $g(v) = \|v\|^\alpha v$ this holds with $\delta = \alpha+1$). Moreover, for lower bounds we shall assume that (ILI) holds with θ_1 on a neighborhood of $\omega(u)$ (obviously, we neccessarily have $\theta_1 \geq \theta$).

Let us define

$$H(t) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon\|\nabla E(u)\|^\beta\langle\nabla E(u), v\rangle$$

for a small $\varepsilon > 0$ and an appropriate $\beta > -1$ (to be specified later). If $\beta \leq 0$, we define $\|\nabla E(u)\|^\beta\langle\nabla E(u), v\rangle$ to be zero for $\nabla E(u) = 0$. Function H plays the role of $\mathcal{E}(u, v)$ if we rewrite (SOP) as a first order problem.

Before we start with some estimates, let us formulate the following lemma.

Lemma 19. *Let a positive decreasing function $H : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$-\frac{H'(t)}{H(t)^B} \geq c \quad \left(\textit{resp. } -\frac{H'(t)}{H(t)^B} \leq c\right) \qquad \textit{for all } t \geq t_0$$

with some $B > 1$ and c, $t_0 > 0$. Then the following holds for some $C > 0$

$$H(t) \leq Ct^{-\frac{1}{B-1}} \quad \left(\textit{resp. } H(t) \geq Ct^{-\frac{1}{B-1}}\right) \qquad \textit{for all } t \geq 2t_0.$$

Proof. Integrate the inequality in the Lemma from t_0 to t and obtain

$$\frac{1}{B-1}(H(t)^{1-B} - H(t_0)^{1-B}) \geq c(t - t_0),$$

hence

$$H(t) \leq ((B-1)c(t-t_0) + H(t_0)^{1-B})^{\frac{1}{1-B}} \leq Ct^{-\frac{1}{B-1}} \qquad \text{for all } t \geq 2t_0$$

with $C = (c(B-1)/2)^{\frac{1}{1-B}}$ since $H(t_0) \geq 0$ and $t - t_0 \geq \frac{t}{2}$. For the other estimate we only replace the inequalities with the opposite ones and take $C^{1-B} = (B-1)c + H(t_0)^{1-B}t_0^{-1}$ since then $H(t_0)^{1-B} = t_0(C^{1-B} - (B-1)c) \leq (t-t_0)(C^{1-B} - (B-1)c) \leq tC^{1-B} - (t-t_0)(B-1)c$. ◻

It follows that we need $-\frac{H'}{H^B} \geq c$ with as small B as possible to get the best upper estimate for the decay of H. On the contrary, if we have $-\frac{H'}{H^B} \leq c$ with as large B as possible we get the best lower estimate for the decay of H.

First of all, let us estimate

$$\begin{aligned} H(t) &\leq \frac{1}{2}\|v\|^2 + C\|\nabla E\|^{\frac{1}{1-\theta}} + \varepsilon\|\nabla E\|^{\beta+1}\|v\| \\ &\leq \frac{1}{2}\|v\|^2 + C\|\nabla E\|^{\frac{1}{1-\theta}} + \varepsilon\|\nabla E\|^{2\beta+2} + \varepsilon\|v\|^2 \qquad (25) \\ &\leq C(\|v\|^2 + \|\nabla E\|^{\frac{1}{1-\theta}}), \end{aligned}$$

where the last inequality holds if

$$\beta \geq -1 + \frac{1}{2(1-\theta)}. \tag{$\beta 1$}$$

Similarly, we get a lower bound using (ILI)

$$\begin{aligned} H(t) &\geq \frac{1}{2}\|v\|^2 + c\|\nabla E\|^{\frac{1}{1-\theta_1}} - \varepsilon\|\nabla E\|^{\beta+1}\|v\| \\ &\geq \frac{1}{2}\|v\|^2 + c\|\nabla E\|^{\frac{1}{1-\theta_1}} + \varepsilon\|\nabla E\|^{2\beta+2} + \varepsilon\|v\|^2 \\ &\geq c(\|v\|^2 + \|\nabla E\|^{\frac{1}{1-\theta_1}}), \end{aligned} \tag{26}$$

provided

$$\beta \geq -1 + \frac{1}{2(1-\theta_1)}. \tag{$\beta I1$}$$

Let us now estimate $H'(t)$

$$\begin{aligned} H'(t) = &-\langle g(v), v\rangle - \varepsilon\|\nabla E(u)\|^{\beta+2} \\ &+ \varepsilon\beta\|\nabla E(u)\|^{\beta-2}\langle \nabla E(u), \nabla^2 E(u)v\rangle\langle \nabla E(u), v\rangle \\ &+ \varepsilon\|\nabla E(u)\|^{\beta}\langle \nabla^2 E(u)v, v\rangle \\ &+ \varepsilon\|\nabla E(u)\|^{\beta}\langle \nabla E(u), -g(v)\rangle| \end{aligned} \tag{27}$$

Here the first line is less than

$$-\|v\|^{\alpha+2} - \varepsilon\|\nabla E(u)\|^{\beta+2}.$$

The second and third lines are less than

$$\varepsilon c\|\nabla E(u)\|^{\beta+\gamma}\|v\|^2$$

and applying the Young inequality we obtain

$$\varepsilon c\|\nabla E(u)\|^{\beta+\gamma}\|v\|^2 \leq c\|v\|^{\alpha+2} + c(\varepsilon)\|\nabla E(u)\|^q$$

with $q = \frac{\alpha+2}{\alpha}(\beta+\gamma)$. We need $q \geq \beta+2$, which means

$$\beta \geq \alpha - \frac{1}{2}\gamma(2+\alpha). \tag{$\beta 2$}$$

The last line in (27) is less than

$$\varepsilon c\|\nabla E(u)\|^{\beta+1}\|v\|^{\delta}$$

and applying the Young inequality we get

$$\varepsilon c\|\nabla E(u)\|^{\beta+1}\|v\|^{\delta} \leq c\|v\|^{\alpha+2} + c(\varepsilon)\|\nabla E(u)\|^q$$

with $q = (\beta+1)\frac{\alpha+2}{\alpha+2-\delta}$. Again, we would like $q \geq \beta+2$, which means

$$\delta \geq \frac{\alpha+2}{\beta+2}. \tag{δ}$$

Since $\delta \leq \alpha + 1$, this condition yields

$$(\beta 3) \qquad \beta \geq -\frac{\alpha}{1+\alpha}.$$

So, we have

$$(28) \qquad -H'(t) \geq c(\|v\|^{\alpha+2} + \|\nabla E(u)\|^{\beta+2}),$$

provided (δ) and ($\beta 2$).

Let us derive upper estimates for $-H'(t)$. Starting with (27) we obtain

(29)

$$\begin{aligned} -H'(t) &\leq \|v\|^{\delta+1} + \varepsilon\|\nabla E\|^{\beta+2} + \varepsilon\beta\|\nabla E\|^{\beta+\gamma}\|v\|^2 + \varepsilon\|\nabla E\|^{\beta+\gamma}\|v\|^2 + \varepsilon\|\nabla E\|^{\beta+1}\|v\|^{\delta} \\ &\leq C(\|v\|^{\delta+1} + \|\nabla E\|^{\beta+2}) \end{aligned}$$

provided $\delta \geq \frac{1}{\beta+1}$ and

$$(\beta I2) \qquad \beta \geq \delta - 1 - \frac{\gamma}{2}(1+\delta).$$

These conditions are weaker than $\delta \geq \frac{\alpha+2}{\beta+2}$ and ($\beta 2$) since $\delta \leq \alpha + 1$.

Puting together (25) and (28) we obtain for $B > 0$

$$-\frac{H'(t)}{H(t)^B} \geq c\frac{\|v\|^{\alpha+2} + \|\nabla E(u)\|^{\beta+2}}{(\|v\|^2 + \|\nabla E\|^{\frac{1}{1-\theta}})^B} \geq c\frac{\|v\|^{\alpha+2} + \|\nabla E(u)\|^{\beta+2}}{\|v\|^{2B} + \|\nabla E(u)\|^{\frac{B}{1-\theta}}}.$$

The last expression is larger than a constant (on bounded sets) if $2B \geq \alpha + 2$ and $\frac{B}{1-\theta} \geq \beta+2$, i.e. $B \geq \max\{1+\frac{\alpha}{2}, (\beta+2)(1-\theta)\}$. To get the best possible result, we take B as small as possible, so if $1 + \frac{\alpha}{2} \geq (\beta+2)(1-\theta)$, i.e. $\beta \leq \frac{\alpha+2}{2(1-\theta)} - 2$, we take $B = 1 + \frac{\alpha}{2}$ and obtain by Lemma 19

$$(30) \qquad H(t) \leq C(t-t_0)^{-\frac{1}{B-1}} = C(t-t_0)^{-\frac{2}{\alpha}}.$$

On the other hand, if $\beta > \frac{\alpha+2}{2(1-\theta)} - 2$, we take $B = (\beta+2)(1-\theta)$ and obtain

$$(31) \qquad H(t) \leq C(t-t_0)^{-\frac{1}{1-2\theta+\beta(1-\theta)}}.$$

Let us now derive lower estimates for H. Using (26) and (29) we get

$$-\frac{H'(t)}{H(t)^{B_1}} \leq C\frac{\|v\|^{\delta+1} + \|\nabla E(u)\|^{\beta+2}}{(\|v\|^2 + \|\nabla E(u)\|^{\frac{1}{1-\theta_1}})^{B_1}} \leq C\frac{\|v\|^{\delta+1} + \|\nabla E(u)\|^{\beta+2}}{\|v\|^{2B_1} + \|\nabla E(u)\|^{\frac{B_1}{1-\theta_1}}}$$

The last expression is less than a constant if $B_1 \leq \frac{1}{2}(\delta+1)$ and $B_1 \leq (1-\theta_1)(\beta+2)$. Due to ($\beta I2$) we have $\frac{1}{2}(\delta+1) \leq (1-\theta_1)(\beta+2)$ whenever $\gamma + \frac{1}{1-\theta_1} \leq 2$, $\delta \geq -1$. Then the best possible estimate is

$$(32) \qquad H(t) \geq C(t-t_0)^{-\frac{1}{B-1}} = C(t-t_0)^{-\frac{2}{\delta-1}}$$

and it is valid for any β large enough and also for 'the limit case $\beta = +\infty$', i.e. for $H(t) = \frac{1}{2}\|v\|^2 + E(u)$.

If the upper bound is equal to the lower bound, we can say that the estimate is optimal. It happens if and only if

$$B = \max\left\{1 + \frac{\alpha}{2}, (\beta+2)(1-\theta)\right\} = \min\left\{\frac{1}{2}(\delta+1), (1-\theta_1)(\beta+2)\right\} = B_1.$$

Since we always have $\theta_1 \geq \theta$, we have $B \geq (\beta+2)(1-\theta) \geq (1-\theta_1)(\beta+2) \geq B_1$, hence $\theta_1 = \theta$. Also, it holds that $B \geq \frac{1}{2}(2+\alpha) \geq \frac{1}{2}(1+\delta) \geq B_1$, hence $\delta = \alpha+1$. Moreover, it must hold

$$1 + \frac{\alpha}{2} = (\beta+2)(1-\theta), \text{ i.e. } \quad \beta = \frac{\alpha+2}{2(1-\theta)} - 2$$

and we have

$$c(t-t_0)^{-\frac{2}{\alpha}} \leq H(t) \leq C(t-t_0)^{-\frac{2}{\alpha}}. \tag{33}$$

So, this is the only case when we can say that the obtained estimates are optimal. It remains to check the conditions (β1), (β2), (β3) (conditions (βI1), (βI2) follow automatically). From (β3) we obtain $\alpha \geq 1-2\theta$ and from (β2) $\gamma \geq \frac{1-2\theta}{1-\theta}$ which by Corollary 9 yields $\gamma = \frac{1-2\theta}{1-\theta}$ if φ is asymptotically stable in (GS$_E$). We have proved the following Theorem.

Theorem 20 (Optimality for (SOP) with weak damping)**.** *Let $\alpha \in (0,1)$ and $\theta \in (0, \frac{1}{2}]$ be such that $\alpha \geq 1-2\theta$. Let E satisfies* (LI) *and* (8) *with $\gamma = \frac{1-2\theta}{1-\theta}$ on a neighborhood of φ, φ being an asymptotically stable equilibrium for* (GS$_E$). *Let g satisfies* (24) *with $\delta = \alpha+1$ on a neighborhood of zero. Then we have* (33) *and*

$$ct^{-\frac{2}{\alpha}} \leq \|\dot{u}\|^2 + \|u(t) - \varphi\|^{\frac{1}{\theta}} \leq Ct^{-\frac{2}{\alpha}} \tag{34}$$

for some $c, C > 0$, $t_0 > 0$ and all $t \geq t_0$.

Proof. By Corollary 9, (ILI) holds with $\theta_1 = \theta$. Hence, the estimate (33) follows from the derivation above. The estimate (34) follows from $H(t) \sim \|v\|^2 + \|\nabla E(u)\|^{\frac{1}{1-\theta}}$ (estimates (25), (26)) and $\|\nabla E(u)\|^{\frac{1}{1-\theta}} \sim E(u) - E(\varphi) \sim \|u - \varphi\|^{\frac{1}{\theta}}$ ((LI), (ILI), Corollary 9). □

Remark 21. This result yields the same optimal decay estimate as in [14], where the same assumptions on g and $\alpha \geq 1-2\theta$ appear but (LI) and (8) are replaced by

$$\langle \nabla E(u), u \rangle \geq c\|u\|^{\frac{1}{\theta}}, \quad E(u) \leq C\|u\|^{\frac{1}{\theta}} \tag{35}$$

Conditions (LI) and (8) with $\gamma = \frac{1-2\theta}{1-\theta}$ do not imply the first inequality in (35), so Theorem 20 is not contained in Haraux's results [14]. However, examples where (35) does not hold are rather artificial: let us consider the energy function on $\mathbb{R}^2$ given in polar coordinates by the formula

$$E(r, \varphi) = r^p c(\varphi), \quad p = \frac{1}{\theta}$$

for a smooth positive function c with small first and second derivatives and with positive derivative on $(0, \frac{\pi}{2})$. Then $\nabla_{x,y} E \sim r^{p-1}$ and $\nabla^2_{x,y} E \sim r^{p-2}$, so (LI) and (8) are satisfied with $\gamma = \frac{1-2\theta}{1-\theta}$. If we change E slightly on neigborhoods of the points $(\frac{1}{k}, \frac{1}{k})$ in such a way, that E becomes constant on small segments of the line $y = x$ but the derivative of E in these points in the direction $(-1, 1)$ stays large, we obtain a function satisfying (LI) and (8) with $\gamma = \frac{1-2\theta}{1-\theta}$ and such that $\langle \nabla E((x, y)), (x, y) \rangle = 0$ in some points near origin.

On the other hand, conditions (35) imply (LI) but $\nabla^2 E$ can be very large on small sets, so (8) and (ILI) do not follow from (35).

In the cases not covered by Theorem 20, i.e. $\gamma < \frac{1-2\theta}{1-\theta}$ or $\alpha < 1 - 2\theta$ or $\delta < \alpha + 1$ or $\theta_1 > \theta$, the upper estimates are different from the lower estimates and it is not clear, whether they are optimal or sharp. Let us collect the estimates in the following two theorems.

Theorem 22. *Let $\alpha \in (0, 1)$ and $\theta \in (0, \frac{1}{2}]$ and E satisfy* (LI) *and* (8) *on a neighborhood of φ (in case $\alpha < \frac{\theta}{1-\theta}$ or φ is asymptotically stable for* (GS_E)*), resp. on a neighborhood of $\omega(u)$ (otherwise). Let g satisfy* (24) *on bounded sets. Then we have*

$$H(t) \leq Ct^{-S}$$

for some $C > 0$, $t_0 > 0$ and all $t \geq t_0$ with S and β being as follows.

(1) *If $\alpha \geq 1 - 2\theta$, $\gamma \geq \frac{1-2\theta}{1-\theta}$, $\delta \geq 2(1-\theta)$, then $\beta = \frac{\alpha+2}{2(1-\theta)} - 2$, $S = \frac{2}{\alpha}$.*
(2) *If $\gamma < \frac{1-2\theta}{1-\theta}$, $\delta \geq \frac{2}{2-\gamma}$, then $\beta = \alpha - \frac{1}{2}\gamma(2+\alpha)$, $S = \frac{1}{1-2\theta-\gamma(1-\theta)+\alpha(1-\gamma/2)(1-\theta)}$.*
(3) *If $\delta < 2(1-\theta)$, $\delta \leq \frac{2}{2-\gamma}$, then $\beta = \frac{\alpha+2}{\delta} - 2$, $S = \frac{\delta}{2-\delta-2\theta+\alpha(1-\theta)}$.*

Moreover, if $\alpha < \frac{\theta}{1-\theta}$ we have

$$\|u(t) - \varphi\| \leq \int_t^{+\infty} \|\dot{u}\| \leq Ct^{-(\theta-\alpha(1-\theta))S}. \tag{36}$$

Moreover, if (ILI) *holds on a neighborhood of φ (resp. $\omega(u)$), then*

$$\|v\| \leq Ct^{-\frac{S}{2}}, \qquad E(u) - E(\varphi) \leq C(t - t_0)^{\frac{1-\theta_1}{1-\theta} \cdot S} \tag{37}$$

and if φ is asymptotically stable for (GS_E)*, then*

$$\|u(t) - \varphi\| \leq Ct^{-\theta S \frac{1-\theta_1}{1-\theta}}. \tag{38}$$

Proof. The cases (1), (2), (3) follow from estimates (30), (31). Estimate (36) follows by the same computations as (16). Estimates (37) follow from (26) and (38) follows from Corollary 6, part 1. □

Theorem 23. *Let u be a solution to* (SOP) *with* $\lim_{t\to+\infty} u(t) = \varphi$. *Let E satisfy* (ILI) *and* (8) *on a neighborhood of φ and* $\frac{1}{1-\theta_1} + \gamma < 2$. *Let g satisfies* (24) *with* $\delta > 1$ *on bounded sets. Then we have*

$$H(t) \geq Ct^{-\frac{2}{\delta-1}}$$

for some $C > 0$, $t_0 > 0$ and all $t \geq t_0$ with any β satisfying ($\beta I1$), ($\beta I2$). *Moreover, if E satisfies* (LI), *then*

$$\|v\|^2 + \|\nabla E\|^{\frac{1}{1-\theta}} \geq Ct^{-\frac{2}{\delta-1}}.$$

Proof. The estimate for H follows from (32) and the estimates for v and ∇E from (25). □

One can see that if $\gamma = 0$, then case (2) in Theorem 22 yields the decay estimate from [9], [8]. However, if (8) holds with $\gamma > 0$, then we have a better estimate. Also if $\gamma = 0$ and (ILI) holds with $\theta_1 < \theta + \frac{\alpha}{\theta}(1-\theta)^2$ and φ is asymptotically stable for (GS_E), then (38) gives a better estimate for $\|u(t) - \varphi\|$.

If $\alpha < 1 - 2\theta$, $\delta = \alpha + 1$, $\gamma = \frac{1-2\theta}{1-\theta}$, previous Theorems give the same upper and lower estimates as [14]. Due to [14], for $\delta = \alpha + 1$, $\gamma = \frac{1-2\theta}{1-\theta}$ these estimates are sharp in the class of equations satisfying the assumptions of the above theorems. If $\delta < \alpha + 1$ or $\gamma < \frac{1-2\theta}{1-\theta}$, sharpness is an open problem.

7. Examples of second order equations with weak damping

In this section we present two examples. We first focus on the scalar case, where we are typically able to obtain optimal decay (namely, for all analytic functions E), at least in the oscilatory case. On the other hand, for vector problems there is a large class of 'nice' functions (namely functions with different growth in different directions like $E(x, y) = |x|^p + |y|^q$, $p > q > 2$) that do not satisfy (LI) and (ILI) with $\theta = \theta_1$ and optimality of the decay estimates is open.

Example 24. Let us consider a scalar (SOP) with an analytic function $E : I \subset \mathbb{R} \to \mathbb{R}$. Let $\varphi \in \mathrm{Cr}(E)$. Then we have $E(x-\varphi) - E(\varphi) = a_k(x-\varphi)^k + o((x-\varphi)^k)$ for x in a neighborhood of φ and due to Proposition 10 (LI), (ILI) holds with $\theta = \theta_1 = \frac{1}{k}$ and (8) holds with $\gamma = \frac{1-2\theta}{1-\theta}$. Let g satisfy (24) with $\delta = \alpha + 1$ (e.g. any function of the form $g(v) = |v|^\alpha v h(v)$ with h continuous, $h(0) \neq 0$).

In the oscilatory case $\alpha \geq 1 - 2\theta$ the results of the previous section say that any solution u with $\lim_{t\to+\infty} u(t) = \varphi$ satisfies $H(t) \sim Ct^{-\frac{2}{\alpha}}$, i.e., $|v|^2 + |u(t) - \varphi|^{\frac{1}{\theta}} \sim Ct^{-\frac{2}{\alpha}}$ (we do not need asymptotic stability of φ in Theorem 20 since we know the growth of E). In the non-oscilatory case $\alpha < 1 - 2\theta$ every solution u with $\lim_{t\to+\infty} u(t) = \varphi$ satisfies

$$ct^{-\frac{2}{\alpha}} \leq H(t) \leq Ct^{-\frac{\theta(\alpha+1)}{1-2\theta-\alpha\theta}},$$

i.e.,

$$ct^{-\frac{2}{\alpha}} \leq |v|^2 + |u(t) - \varphi|^{\frac{1}{\theta}} \leq Ct^{-\frac{\theta(\alpha+1)}{1-2\theta-\alpha\theta}}.$$

Moreover, in the oscilatory case we can show that

$$\limsup_{t\to+\infty} |u(t) - \varphi| t^{\frac{2\theta}{\alpha}} \in (0, +\infty) \quad \text{and} \quad \limsup_{t\to+\infty} |v(t)| t^{\frac{1}{\alpha}} \in (0, +\infty).$$

In fact, by the upper bound for $|v|^2 + |u(t) - \varphi|^{\frac{1}{\theta}}$ it follows that both limsups are finite. We show that they are positive.

Let us assume for contradiction that for every $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that $|u(t) - \varphi| \leq \varepsilon t^{-\frac{2\theta}{\alpha}}$ for all $t > t_\varepsilon$. Then $|E'(u(t))|^{\frac{1}{1-\theta}} \leq \varepsilon C t^{-\frac{2}{\alpha}}$ for $t > t_\varepsilon$. If ε is small then due to (25) we have $|v(t)|^2 \geq c(1-\varepsilon)t^{-\frac{2}{\alpha}}$ for $t > t_\varepsilon$. Hence, v does not change sign on $(t_\varepsilon, +\infty)$. We can assume without loss of generality that $u(t_\varepsilon) - \varphi > 0$. Then v cannot be positive for all $t > t_\varepsilon$, so v si negative and $-v(t) \geq c(1-\varepsilon)t^{-\frac{1}{\alpha}}$ on $(t_\varepsilon, +\infty)$. Then

$$u(t) - \varphi = -\int_t^{+\infty} v(s)ds \geq \int_t^{+\infty} c(1-\varepsilon)s^{-\frac{1}{\alpha}}ds = ct^{\frac{\alpha-1}{\alpha}} \geq ct^{\frac{-2\theta}{\alpha}}$$

on $(t_\varepsilon, +\infty)$, which is a contradiction with $|u(t) - \varphi| \leq \varepsilon t^{-\frac{2\theta}{\alpha}}$.

Similarly, we prove the lower bound for v. If $|v| \leq \varepsilon t^{-\frac{1}{\alpha}}$ on $(t_\varepsilon, +\infty)$, then $g(v(t)) \leq C\varepsilon t^{-\frac{1+\alpha}{\alpha}}$ and due to (25) $|E'(u(t))| \geq ct^{-\frac{2}{\alpha}(1-\theta)} \geq ct^{-\frac{1+\alpha}{\alpha}}$. We have (we may assume $E'(u(t)) > 0$)

$$v(t) = \int_t^{+\infty} g(v(s)) + E'(u(s))ds \geq \int_t^{+\infty} (c - \varepsilon)s^{-\frac{1+\alpha}{\alpha}}ds = ct^{-\frac{1}{\alpha}},$$

contradiction.

Remark 25. In the previous example we can take any function $E \in C^2(\mathbb{R})$ satisfying $E''(x) = a(x-\varphi)^{p-2} + o((x-\varphi)^{p-2})$ on a neighborhood of φ (not neccessarilly analytic, also $p \notin \mathbb{N}$, $p > 2$ is possible).

In contrast to the scalar case, in higher dimensions there is a large class of functions where θ_1 in (ILI) must be strictly larger than θ from (LI) and $\gamma < \frac{1-2\theta}{1-\theta}$. In this case, the results from [14] do not apply. Theorem 22 gives a decay estimate which is better than the one from [9], [4]. However, it is not clear, whether this estimate is optimal (it rather seems that it is not optimal).

Example 26. Let $E : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$E(x, y) = |x|^p + |y|^q, \qquad p > q \geq 2.$$

Then

$$\nabla E(x, y) = \begin{pmatrix} p|x|^{p-2}x \\ q|y|^{q-2}y \end{pmatrix} \qquad \text{and} \qquad \nabla^2 E(x, y) = \begin{pmatrix} p(p-1)|x|^{p-2} & 0 \\ 0 & q(q-1)|y|^{q-2} \end{pmatrix}.$$

Hence,

$$\|\nabla E(x,y)\| \sim |x|^{p-1} + |y|^{q-1} \qquad \text{and} \qquad \|\nabla^2 E(x,y)\| \sim |x|^{p-2} + |y|^{q-2}$$

on any bounded set. It follows that

$$|E(x,y)|^{\frac{p-1}{p}} \le C\left(|x|^{p-1} + |y|^{q\frac{p-1}{p}}\right) = C\left(|x|^{p-1} + |y|^{q-\frac{q}{p}}\right) \le C\left(|x|^{p-1} + |y|^{q-1}\right),$$

i.e. (LI) holds with (best possible, i.e. largest) $\theta = \frac{1}{p}$. On the other hand,

$$|E(x,y)|^{\frac{q-1}{q}} \ge c\left(|x|^{p\frac{q-1}{q}} + |y|^{q}\right) = C\left(|x|^{p-\frac{p}{q}} + |y|^{q}\right) \ge C\left(|x|^{p-1} + |y|^{q-1}\right),$$

i.e. (ILI) holds with (best possible, i.e. smallest) $\theta_1 = \frac{1}{q} > \frac{1}{p}$. Further, we have on bounded sets

$$\|\nabla^2 E(x,y)\| \le C\left(|x|^{p-2} + |y|^{q-2}\right) \le C\left(|x|^{p-1} + |y|^{q-1}\right)^{\frac{q-2}{q-1}},$$

since $p-2 \ge (p-1) - \frac{p-1}{q-1} = (p-1)\frac{q-2}{q-1}$. It means that the best possible $\gamma = \frac{q-2}{q-1} < \frac{p-2}{p-1} = \frac{1-2\theta}{1-\theta}$ (if $q = 2$ we have $\gamma = 0$).

If we take g satisfying (24) with $\delta = \alpha + 1$, then Theorem 22 case (2) yields

$$H(t) \le C(t-t_0)^{-S} \quad \text{with } S = \frac{1}{\frac{\alpha}{2} + (p-q)\frac{2+\alpha}{2p(q-1)}}. \tag{39}$$

Moreover, since (ILI) holds with $\theta_1 = \frac{1}{q}$ and zero is asymptotically stable for (GS_E) (due to the shape of E), we have

$$\|u(t)\| \le C(t-t_0)^{-\theta S \frac{1-\theta_1}{1-\theta}} = C(t-t_0)^{-S\frac{q-1}{q(p-1)}}. \tag{40}$$

So, if $q > 2$ then (39) is a better estimate than in [9], [4]. If $q = 2$ and α is such that $\frac{1}{q} < \frac{1}{p} + \frac{\alpha}{p}(p-1)^2$, then estimate (40) is better than the one in [9].

On the other hand, if we fix $x = 0$ or $y = 0$ we obtain one-dimensional problems studied in [14]. Hence, all solutions on the x-axis satisfy $H(t) \sim (t-t_0)^{-\frac{2}{\alpha}}$ (or $H(t) \sim (t-t_0)^{-\frac{\alpha+1}{p-2-\alpha}}$ if $\alpha < 1 - \frac{2}{p}$) and all solutions on the y-axis satisfy $H(t) \sim (t-t_0)^{-\frac{2}{\alpha}}$ (or $H(t) \sim (t-t_0)^{-\frac{\alpha+1}{q-2-\alpha}}$ if $\alpha < 1 - \frac{2}{q}$). So, for $q > 2$, $\alpha < 1 - \frac{2}{q}$ we have solutions with three different rates of convergence. The problem, whether there exist solutions with a forth rate of convergence and whether their convergence can be even slower (as slow as (39)), remains open. Even in the case $\alpha \ge 1 - \frac{2}{p}$ it is not clear whether all solutions behave like $(t-t_0)^{-\frac{2}{\alpha}}$ or whether there exist some solutions with slower convergence.

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