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Sobolev homeomorphism in $W^{k,p}$ and the Lusin (N) condition

Tomáš Roskovec

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Abstract We characterize when the Luzin (N) condition holds in $W^{k,p}$ in dependence on k, p and dimension. We generalize well-known counterexamples in $W^{1,p}$ both for general mappings and for homeomorphisms.

Keywords Lusin condition, Sobolev space

Mathematics Subject Classification (2000) 46E35

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set. We say that the mapping $f\colon \Omega \to \mathbf{R}^m$ satisfies Lusin (N) condition if for every $A \subset \Omega, |A| = 0$, it holds that $|f(A)| = 0$. We can see that this condition is needed for any natural physical model such as the deformation of a solid body in space. Otherwise we can make new material "from nothing", so the mapping is really unnatural for physical applications. Another important application is the connection between the area formula and the Lusin condition. If the mapping is a Sobolev mapping and satisfies the Lusin (N) condition, the area formula holds, for more see [10].

Marcus and Mizel prove in [9], that the Lusin (N) condition is guaranteed for the mappings in $W^{1,p}(\Omega, \mathbf{R}^n)$ for $p > n$. If we consider Sobolev homeomorphisms, the condition is valid even in $W^{1,n}(\Omega, \mathbf{R}^n)$, see Reshetnyak [16]. These positive results are sharp and the main tool of the proof is the Sobolev Embedding Theorem. Our work is based on two classical counterexamples of the functions violating the (N) condition, which complete the characterization of validity in $W^{1,p}$.

T. Roskovec
Department of Applied Mathematics and Informatics, Faculty of Economics of University of South Bohemia in České Budějovice, Studentská 13, 370 05 České Budějovice, Czech Republic
E-mail: tomas.roskovec@gmail.com

The first counterexample is the Cesari's construction originally written in [2] for $n = 2$, later reminded and improved by Malý and Martio in [8]. That is the continuous mapping in the Sobolev space $W^{1,p}([0,1]^n, [0,1]^m)$, $p \leq n$, which maps the line segment onto a domain with positive m-dimension measure, so the (N) condition is violated.

The second counterexample is the Ponomarev's construction [15]. This is the Sobolev homeomorphism in $W^{1,p}([0,1]^n, [0,1]^n)$, $p < n$, violating the (N) condition. The construction is essentially different from Cesari's construction and cannot be obtain as a simple modification.

The limiting case $W^{1,n}$ is the most important case in both settings. There is no work considering counterexamples for higher derivative spaces $W^{k,p}$, but by Sobolev embedding it is natural to expect that the space $W^{k,\frac{n}{k}}$ is the limiting case. The positive results can be easily obtained, but both counterexamples mentioned above lack the second derivative and cannot be used. In our work we fill this gap by careful smoothing the classical constructions and we show that the space $W^{k,\frac{n}{k}}$ is limiting in both cases. Our results can be summarized by these two theorems.

Theorem 1 *Let $k, n \in \mathbf{N}$, $n \geq k$ let $p \geq \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a homeomorphism $f \in W^{k,p}(\Omega, \mathbf{R}^n)$ satisfies Lusin (N) condition.*

On the other hand for every $k, n \in \mathbf{N}$, $n > k$ and $p \in [1, \frac{n}{k})$ there is a homeomorphism $f \in W^{k,p}((-1,1)^n, \mathbf{R}^n)$ which fails Lusin (N) condition.

If we consider general Sobolev mappings and not only homeomorphisms, then the scale shifts and a mapping in $W^{k,\frac{n}{k}}$ may fail to satisfy the condition. We give a counterexample violating the condition for this case.

Theorem 2 *Let $k, m, n \in \mathbf{N}$, $n > k$, let $p > \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{k,p}(\Omega, \mathbf{R}^m)$ satisfies Lusin (N) condition.*

Moreover, let $m, n \in \mathbf{N}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{n,1}(\Omega, \mathbf{R}^m)$ satisfies Lusin (N) condition.

On the other hand for every $k, m, n \in \mathbf{N}$, $n > k$ and $p \in [1, \frac{n}{k}]$ there is a mapping $f \in W^{k,p}((-1,1)^n, \mathbf{R}^m)$ which fails Lusin (N) condition.

The generalizations of the positive results can be proved by the Sobolev Embedding Theorem. However the counterexamples require a new approach because the classical counterexamples are defined as non-smooth mappings and they lack even the second weak derivative. The special question is the validity of the condition in case $n = k$, $p = 1$. We answer this question by the finer version of Sobolev Embedding Theorem by Peetre [14] for the Lorentz spaces and by the result by Kauhanen, Koskela and Malý [5].

For other results concerning the research of the (N) condition in spaces close to $W^{1,n}$ see [7] and [5]. Although the classical results are not new, there are fresh applications using these constructions as the limiting case for example in the varifold theory [12], [18] or in the field of the metric measure spaces [6]. There are also works concerning Lusin (N) condition using different methods of construction in order to get particular properties such that the

Sobolev homeomorphism with $J_f = 0$ almost everywhere, see [3], or even the homeomorphism satisfying $\operatorname{rank}(Df) < n$, $\operatorname{rank}(Df^{-1}) < n$, see [13].

The paper is divided into two parts. In Section 3 we prove Theorem 1 and we give the example of the homeomorphism in $W^{k,p}$. In in Section 4 we prove Theorem 2 and we give the improved Cesari example of the mapping in $W^{k,\frac{n}{k}}$.

2 Preliminaries

We denote an open cube by

$$Q(x,r) = \{y \in \mathbf{R}^n, \|x-y\|_\infty < r\}.$$

We denote an open ball with the centre at x and radius r as $B(x,r)$. We denote a sign as $\operatorname{sgn}(t)$, i.e. $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$, $\operatorname{sgn}(t) = 0$ for $t = 0$.

By C we denote a generic positive constant whose exact value may change at each occurrence. We write for example $C(a,b,c)$ if C may depend on parameters a, b and c. Since we fix parameters n, k and p, the dependence of C on these parameters would not be mentioned at all.

Let us consider the convolution kernel $\phi : (-1,1) \to \mathbf{R}$ such that

$$\begin{aligned}
&(1)\ \ \phi(t) \geq 0,\\
&(2)\ \ \int_{-1}^{1} \phi(t) = 1,\\
&(3)\ \ |D^k \phi(t)| \leq C(k),\\
&(4)\ \ \int_{-1}^{1} |D^k \phi(t)|\, dt \leq C(k),\\
&(5)\ \ \phi(t) \in C_0^\infty((-1,1)).
\end{aligned}$$

For $r > 0$ we define $\phi_r(t) = r^{-1}\phi(r^{-1}t)$. Function $\phi_r(t)$ satisfies

$$\begin{aligned}
&(1)\ \ \phi_r(t) \geq 0,\\
&(2)\ \ \int_{-r}^{r} \phi_r(t) = 1,\\
&(3)\ \ |D^k \phi_r(t)| \leq r^{-k-1} C(k),\\
&(4)\ \ \int_{-r}^{r} |D^k \phi_r(t)|\, dt \leq r^{-k} C(k),\\
&(5)\ \ \phi_r(t) \in C_0^\infty((-r,r)).
\end{aligned} \tag{1}$$

Now we prepare a function for the smooth partition of unity. We consider $\lambda_{a,c,A} : [0,A] \to [0,1]$ for $0 < a < c < A$ such that

$$\begin{aligned}
&(1)\ \ \lambda(t) \in C^\infty((0,A)),\\
&(2)\ \ \lambda(t) \equiv 1 \text{ for } t \in [0,a),\\
&(3)\ \ \lambda(t) \equiv 0 \text{ for } t \in (c,A],\\
&(4)\ \ |D^k \lambda(t)| \leq (c-a)^{-k} C(k) \text{ for } t \in [a,c], k \in \mathbf{N}.
\end{aligned} \tag{2}$$

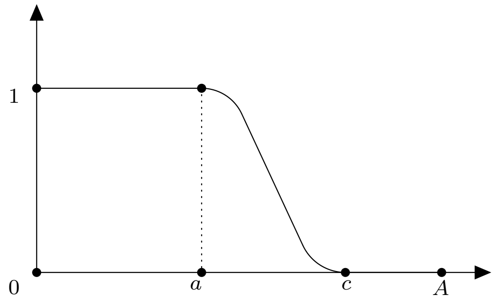


Fig. 1 Graph of $\lambda_{a,c,A}$.

We can construct such $\lambda(t)$ by connecting points $[0,1]$, $[a+\frac{c-a}{3},1]$, $[c-\frac{c-a}{3},0]$ and $[A,0]$ by the lines to form the piecewise affine function and then make this function smooth by convolution with $\phi_{\frac{c-a}{3}}$. The last estimate can be provided by Lemma 2 proven in the next part of the Section. The graph of $\lambda_{a,c,A}$ is sketched in Figure 1.

The n-dimesional Lebesgue measure of a measurable set A in $\mathbf{R}^n$ is denoted by $|A|$ or $\mathcal{L}^n(A)$. We use the formula for the derivative of the product

$$D^k(ab)(t) = \sum_{j=0}^{k} \binom{k}{j} D^j a(t) D^{k-j} b(t), \tag{3}$$

if both $D^j a(t)$, $D^j b(t)$ exist for all $j \in \{0, \dots k\}$ and the right-hand side term makes sense. For a measurable function f belonging to Lebesgue space L^p on some domain Ω we denote the norm

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

Lemma 1 *Let $f : [-1,1]^n \to [-1,1]^n$ be a bijective continuous mapping. Then f is a homeomorphism.*

Proof In order to prove that open sets are mapped to open sets we want to prove that any closed set would be mapped to a closed set. A closed set in $[-1,1]^n$ is bounded and therefore it is a compact set and since f is continuous, the image of the compact set is a compact set or specially closed set. The mapping f is bijective, continuous and maps open sets to open sets and hence f is homeomorphism.

2.1 Convolution method

We use the convolution on some piecewise smooth function with some points of broken smoothness. We can control the value of its derivatives by the derivatives of the original function and by the radius of the convolution kernel. This control is described by the following Lemma.

Lemma 2 *Let $\{a_i\}_{i=1}^{j+1} \subset \mathbf{R}$ be a finite increasing sequence, let $I = [a_1, a_{j+1}]$, $I_i = [a_i, a_{i+1}]$ be closed intervals. Let $h_i \in W^{k,\infty}(I_i, \mathbf{R}) \cap C^{k-1}(I_i, \mathbf{R})$. We denote*

$$h(t) = h_i(t) \text{ for } t \in I_i.$$

Then for $t \in (a_1 + r, a_{j+1} - r)$ we have

$$\begin{aligned} D^k(h * \phi_r)(t) = &\sum_{i=1}^{j} \int_{B(t,r)\cap I_i} \phi_r(t-s) D^k h_i(s) ds \\ &+ \sum_{i=1}^{j-1} \Big(\sum_{l=0}^{k-1} D^{k-1-l} \phi_r(t - a_{i+1}) \big(D^l h_{i+1}(a_{i+1}) - D^l h_i(a_{i+1}) \big) \Big). \end{aligned} \tag{4}$$

Especially we can estimate by (1) (3) *and* (1) (4)

$$|D^k(h * \phi_r)(t)| \le C \max_{i\in\{1,\dots j\}, l=\{0,\dots k\}} r^{-k+l} \|D^l h_i\|_{L^\infty(I_i)}. \tag{5}$$

Moreover, it is sufficient to consider only i such that $a_i \in (t - r, t + r)$ in (5).

Proof Firstly we consider the case $j = 2$, $k = 2$. For $t \in (a_1 + r, a_3 - r)$ we consider a small positive u such that $t+u \in (a_1+r, a_3-r)$. We split $(t-r, t+r)$ into three (possible empty) subintervals

$$\begin{aligned} J_1(u) &= (t - r, t + r) \cap (a_1, a_2 - u), \\ J_2(u) &= (t - r, t + r) \cap (a_2, a_3), \\ S_2(u) &= (t - r, t + r) \cap (a_2 - u, a_2). \end{aligned}$$

By simple equality $\phi_r(t + u - s - u) = \phi_r(t - s)$ we calculate

$$\begin{aligned} \lim_{u\to 0_+} \frac{(h * \phi_r)(t + u) - (h * \phi_r)(t)}{u} = \\ = \lim_{u\to 0_+} \Big(&\int_{J_1(u)} \frac{h_1(s + u) - h_1(s)}{u} \phi_r(t - s)\, ds \\ &+ \int_{J_2(u)} \frac{h_2(s + u) - h_2(s)}{u} \phi_r(t - s)\, ds \\ &+ \int_{S_2(u)} \frac{h_2(s + u) - h_1(s)}{u} \phi_r(t - s)\, ds \Big). \end{aligned}$$

Firstly we deal with the first and the second term. We can interchange the limiting and the integration process (the integrable dominating function is $\phi_r \|Dh_i\|_\infty$). The interval $J_i(u)$ slightly depends on u, but since we integrate the bounded function we can get rid of this dependence by simple calculation, we get

$$\lim_{u\to 0_+} \int_{J_i(u)} \frac{h_i(s + u) - h_i(s)}{u} \phi_r(t - s)\, ds = \int_{B(t,r)\cap I_i} Dh_i(s) \phi_r(t - s)\, ds.$$

We split the last term into two integrals and since we get the average integral of continuous function, we get

$$\lim_{u\to 0_+}\int_{S_2(u)}\frac{h_2(s+u)\phi_r(t-s)-h_1(s)\phi_r(t-s)}{u}\,ds = \\ = h_2(a_2)\phi_r(t-a_2)-h_1(a_2)\phi_r(t-a_2).$$

This gives us (4) for $k=1$. Now we iterate the previous to get the result for $k=2$. We rewrite the proven formula as

$$D(h*\phi_r)(t)=(Dh*\phi_r)(t)+(h_2-h_1)(a_2)\phi_r(t-a_2).$$

To get $D^2(h*\phi_r)$, we differentiate both terms. The first term gives us the same formula as for $D(h*\phi_r)$, we just replace h with Dh and we get

$$D(Dh*\phi_r)(t)=(D^2h*\phi_r)(t)+(Dh_2-Dh_1)(a_2)\phi_r(t-a_2).$$

The derivative of the second term is

$$D\big((h_2-h_1)(a_2)\phi_r(t-a_2)\big)=(h_2-h_1)(a_2)D\phi_r(t-a_2).$$

Together we have

$$\begin{aligned}D^2(h*\phi_r)=&(D^2h*\phi_r)(t)\\ &+(Dh_2-Dh_1)(a_2)\phi_r(t-a_2)+(h_2-h_1)(a_2)D\phi_r(t-a_2)\\ =&\sum_{i=1}^{2}\int_{B(t,r)\cap I_i}\phi_r(t-s)D^2h_i(s)\,ds\\ &+\sum_{i=1}^{1}\Big(\sum_{l=0}^{1}D^{1-l}\phi_r(t-a_{i+1})\big(D^lh_{i+1}(a_{i+1})-D^lh_i(a_{i+1})\big)\Big).\end{aligned}$$

That is (4) for $k=2$. The process how to get the formula for $k=2$ from $k=1$ can be used in general as the induction step, hence

$$\begin{aligned}&D(D^{k-1}h*\phi_r)(t)=(D^kh*\phi_r)(t)+\phi_r(t-a_2)(D^{k-1}h_2-D^{k-1}h_1)(a_2),\\ &D\big(D^m\phi_r(t-a_2)(D^lh_2-D^lh_1)(a_2)\big)=D^{m+1}\phi_r(t-a_2)(D^lh_2-D^lh_1)(a_2),\end{aligned}\tag{6}$$

as long as the we can interchange limiting and integration process by $\|D^kh_i\|_\infty<\infty$ and $D^{k-1}h_i$ is continuous. We suppose the validity of formula (4) for $k-1$

and by (6) we get

$$\begin{aligned}
D^k(h*\phi_r) =& D(D^{k-1}h*\phi_r)(t)\\
&+ D\sum_{l=0}^{k-2} D^{k-2-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2)\\
=&(D^k h*\phi_r)(t) + \phi_r(t-a_2)(D^{k-1}h_2 - D^{k-1}h_1)(a_2)\\
&+\sum_{l=0}^{k-2} D^{k-1-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2)\\
=&(D^k h*\phi_r)(t) + \sum_{l=0}^{k-1} D^{k-1-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2).
\end{aligned}$$

This is formula (4) proven by induction in case $j=2$. In general case $j>2$ we have to consider more terms inside the sums of (4), but the proof is the same.

The last part (5) follows by (1), we use the estimates $|D^k\phi_r(t)| \le Cr^{-k-1}$, $\int |D^k\phi_r(t)|\,dt \le Cr^{-k}$ and the Hölder's inequality.

For later use we consider the mapping defined by several different smooth mappings on several different sub-domains of $(-1,1)^n$. We formulate an observation how we preserve the smoothness. In general we cannot assume the smoothness since the derivatives on the boundaries of the domains of the mappings does not have to be equal.

Observation 3 *Let $f_1 : \Omega_1 \to \mathbf{R}^n$ be a Sobolev mapping smooth inside Ω_1 and $f_2 : \Omega_2 \to \mathbf{R}^n$ be a Sobolev mappings smooth inside Ω_2, such that $f_1 = f_2$ for $x \in \Omega_1 \cap \Omega_2$. If $\partial\Omega_1 \subset \Omega_2$ (as in Figure 2) then we can define Sobolev mapping $f : \Omega_1 \cup \Omega_2 \to \mathbf{R}^n$ as $f = f_1$ in Ω_1 and $f = f_2$ in Ω_2 and this mapping is smooth.*

3 Proof of Theorem 1

3.1 Proof of the positive part

The proof of positive part can be found in [4, Chapter 4.2, p. 68, Theorem 4.5] for case $k=1$. For a domain $\Omega \subset \mathbf{R}^n$ the theorem claims that homeomorphism $f \in W^{1,n}_{\text{loc}}(\Omega,\mathbf{R}^n)$ satisfies Lusin (N) condition. To prove the general form we just use the Sobolev Embedding Theorem multiple times. Since the Lusin (N) condition is a local property, we can assume without loss of generality that Ω is a ball. We have

$$W^{k,\frac{n}{k}}_{\text{loc}}(\Omega,\mathbf{R}^n) \subset W^{k-1,\frac{n}{k-1}}_{\text{loc}}(\Omega,\mathbf{R}^n) \subset \cdots \subset W^{1,n}_{\text{loc}}(\Omega,\mathbf{R}^n),$$

hence for any $k \le n$ a homeomorphism $f \in W^{k,\frac{n}{k}}_{\text{loc}}(\Omega,\mathbf{R}^n)$ satisfies the assumptions for the well-known version of the theorem.

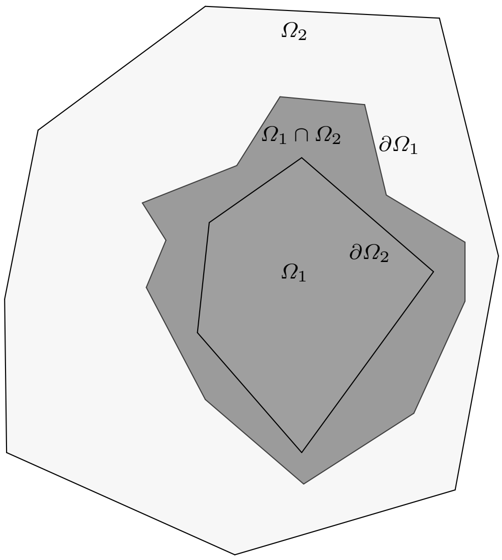


Fig. 2 Position of Ω_2 and $\partial\Omega_1$

3.2 Construction of Cantor sets for the homeomorphism f in $W^{k,p}$

In this subsection we prepare tools for the construction of a homeomorphism $f \in C \cap W^{k,p}([-1,1]^n, [-1,1]^n)$, $p < \frac{n}{k}$, such that f is an identity on the boundary and f does not satisfy Lusin (N) condition. Moreover, f is C^∞ a. e. in $[-1,1]^n$, in fact outside the Cantor type set of 0 measure.

Based on $k, n \in \mathbf{N}$, $p \in [1, \frac{n}{k})$ we choose A, B such that

$$A > \frac{kp}{n-kp} \text{ and } B > 1. \tag{7}$$

We denote

$$\begin{aligned} a_i &:= 2^{-i} 2^{-Ai}, \\ b_i &:= 2^{-i}(2^{-1} + 2^{-Bi-1}). \end{aligned} \tag{8}$$

We recall the construction used in [4, Chapter 4.3, p. 69, Theorem 4.10]. We will first give two Cantor-set constructions in $(-1,1)^n$. Our mapping f will be defined as the limit of the sequence of smooth homeomorphisms $f_i : (-1,1)^n \to (-1,1)^n$, where each f_i maps the i-th step of the first Cantor-set construction onto that of the second. Then the limit mapping f maps the first Cantor set onto the second one.

By $\mathbb{V}$ we denote the set of the 2^n vertices of the cube $[-1,1]^n$, we can index this set $\mathbb{V} = \{\vartheta^1, \vartheta^2, \dots \vartheta^{2^n}\}$. The sets $\mathbb{V}^i = \mathbb{V} \times \ldots \times \mathbb{V}$, $i \in \mathbf{N}$, will serve as the sets of indices for our construction. Let us set $z_0 = \tilde{z}_0 = 0$.

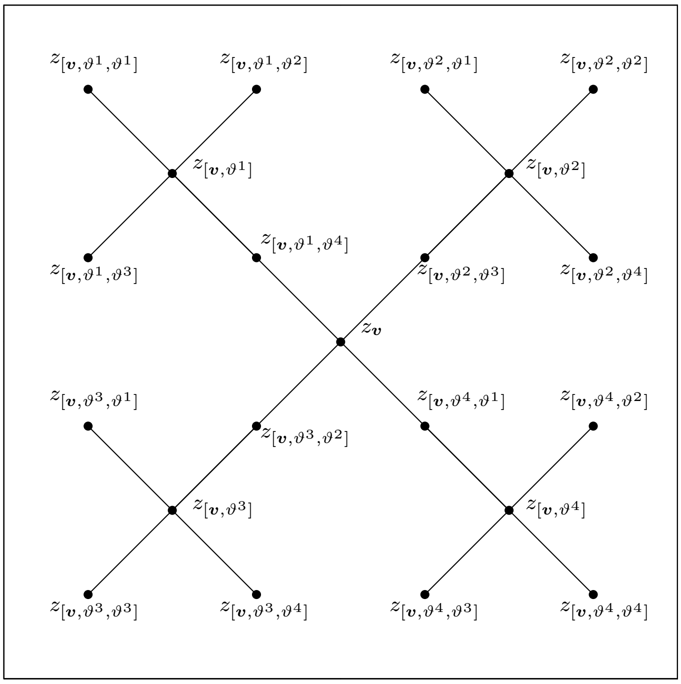


Fig. 3 Structure of centres $z_{\boldsymbol{v}}$

It follows that $(-1,1)^n = Q(z_0, a_0)$ and we proceed by induction. For $\boldsymbol{v} = [v_1, \dots, v_i] \in \mathbb{V}^i$ we denote $\boldsymbol{w} = [v_1, \dots, v_{i-1}]$ and we define

$$z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + \frac{1}{2}(a_{i-1})v_i = z_0 + \sum_{j=1}^{i} \frac{1}{2}(a_{j-1})v_j,$$

$$Q'_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, \frac{1}{2}a_{i-1}) \text{ and } Q_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, a_i).$$

The decomposition of $Q_{\boldsymbol{v}}$ into the cubes with higher index is sketched in Figure 4. Formally we should write $\boldsymbol{w}(\boldsymbol{v})$ instead of $\boldsymbol{w}$ but for simplicity of notation we neglect this. The number of the cubes in $\{Q_{\boldsymbol{v}} : \ \boldsymbol{v} \in \mathbb{V}^i\}$ is 2^{ni}. It is not difficult to show that the resulting Cantor set

$$\bigcap_{i=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^i} Q_{\boldsymbol{v}} =: C_A = C_a \times \ldots \times C_a$$

is a product of n Cantor sets in $\mathbf{R}$. Moreover, $\mathcal{L}_n(C_A) = 0$ since

$$\mathcal{L}_n\big(\bigcup_{\boldsymbol{v} \in \mathbb{V}^i} Q_{\boldsymbol{v}}\big) \geq 2^{ni}(2^{-iA-i})^n \overset{i\to\infty}{\to} 0.$$

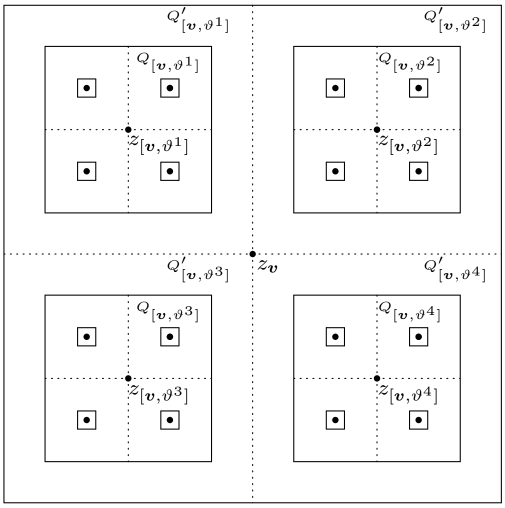


Fig. 4 $Q_{\boldsymbol{v}}$ and its decomposition

Analogously we define

$$\tilde{z}_{\boldsymbol{v}} = \tilde{z}_{\boldsymbol{w}} + \frac{1}{2} b_{i-1} v_i = \tilde{z}_0 + \frac{1}{2} \sum_{j=1}^{i} b_{j-1} v_j,$$

$$\tilde{Q}'_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, \frac{1}{2} b_{i-1}) \text{ and } \tilde{Q}_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, b_i).$$

The resulting Cantor set

$$\bigcap_{i=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^i} \tilde{Q}_{\boldsymbol{v}} =: C_B = C_b \times \ldots \times C_b$$

satisfies $\mathcal{L}_n(C_B) > 0$ since $\lim_{i\to\infty} 2^i b_i > 0$. It remains to find a homeomorphism f which maps C_A onto C_B and satisfies our assumptions. By $f(C_A) = C_B$, f does not satisfy the condition (N) since $\mathcal{L}_n(C_A) = 0$ and $\mathcal{L}_n(C_B) > 0$.

3.3 Basic functions for the construction of the homeomorphism f in $W^{k,p}$

In this part we prepare functions and mappings in order to construct the sequence of the suitable smooth homeomorphisms f_i converging uniformly to f. The desired property is $f_i(Q_{\boldsymbol{v}}) = \tilde{Q}_{\boldsymbol{v}}$ for every $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$.

We denote a constant

$$\alpha_i = \frac{\frac{b_{i-1}}{2} - b_i - \left(\frac{b_{i-1}}{a_{i-1}} + \frac{b_i}{a_i}\right)\frac{a_{i+1}}{2}}{\frac{a_{i-1}}{2} - a_i - a_{i+1}}, \tag{9}$$

in order to define a function

$$l_i(t) := \alpha_i(t - a_i - \frac{a_{i+1}}{2}) + b_i + \frac{a_{i+1}b_i}{2a_i}.$$

The graph of this affine function connects the points $(a_i + \frac{a_{i+1}}{2}, b_i + \frac{a_{i+1}b_i}{2a_i})$ and $(\frac{a_{i-1}}{2} - \frac{a_{i+1}}{2}, \frac{b_{i-1}}{2} - \frac{a_{i+1}b_{i-1}}{2a_{i-1}})$, see Figure 5. We define the sequence of continuous functions $h_i^* : [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}] \to \mathbf{R}$ as

$$h_i^*(t) = \begin{cases} \frac{b_i}{a_i}t & \text{for } t \in [-\frac{a_{i+1}}{2}, a_i + \frac{a_{i+1}}{2}], \\ l_i(t) & \text{for } t \in [a_i + \frac{a_{i+1}}{2}, \frac{a_{i-1}-a_{i+1}}{2}], \\ \frac{b_{i-1}}{a_{i-1}}t & \text{for } t \in [\frac{a_{i-1}-a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}]. \end{cases} \tag{10}$$

We sketch the graph of $h_i^*(t)$ in Figure 5. The important property is the strict monotonicity of $h_i^*(t)$. We define the sequence of smooth functions

$$h_i(t) = (h_i^* * \phi_{\frac{a_{i+1}}{4}})(t) \text{ for } t \in \big[0, \frac{a_{i-1}}{2}\big). \tag{11}$$

We can see that these functions are smooth and strictly monotone, and its derivatives can be calculated and estimated by Lemma 2. Moreover, h_i is linear inside $[0, a_i)$ and it is linear inside $(\frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}, \frac{a_{i-1}}{2})$. All these properties and the relation with $h_i^*(t)$ are sketched in Figure 5.

We use this function to define mapping $g_i^* : Q(0, \frac{a_{i-1}}{2}) \to Q(0, \frac{b_{i-1}}{2})$ by coordinates as

$$(g_i^*(x))_j = \operatorname{sgn}(x_j)h_i(|x_j|) \text{ for } j \in \{1, \dots n\}. \tag{12}$$

This mapping maps $Q(0, \frac{a_{i-1}}{2})$ onto $Q(0, \frac{b_{i-1}}{2})$ and its shifted version can map $Q'_{\boldsymbol{v}}$ onto $\tilde{Q}'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i$. It is clearly continuous, smooth and it is strictly monotone in every direction and therefore it is one-to-one and homeomorphism. Moreover, the j-th coordinate of $g_i(x)$ depends only on the j-th coordinate of x, so the only non-zero coordinate of partial derivatives of $D_j^k g_i$ may be the diagonal ones for $j \in \{1, \dots n\}$, $k \in \mathbf{N}$. We have to modify this mapping once more in order to get $g_i(x) = \frac{b_{i-1}}{a_{i-1}}x$ near the boundary of the cube $\|x\|_\infty = \frac{a_{i-1}}{2}$.

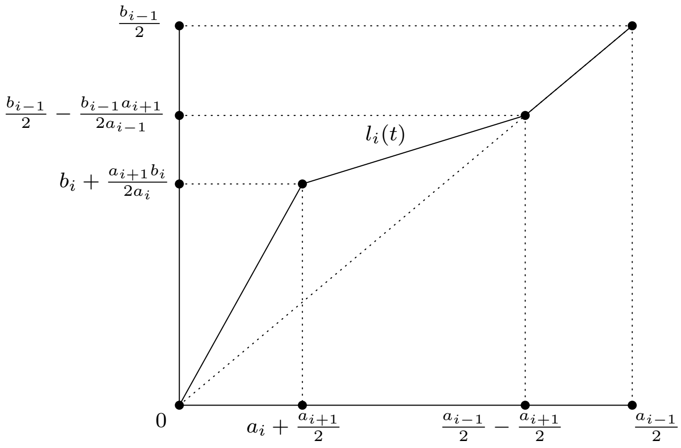
We recall (2) and we denote

$$\lambda(t) = \lambda_{\frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}, \frac{a_{i-1}}{2}}(t).$$

For $x \in Q(0, \frac{a_{i-1}}{2})$ we define

$$g_i(x) = \prod_{j=1}^n \lambda(|x_j|) g_i^*(x) + \big(1 - \prod_{j=1}^n \lambda(|x_j|)\big)\frac{b_{i-1}}{a_{i-1}}x. \tag{13}$$

This mapping still maps $Q(0, \frac{a_{i-1}}{2})$ onto $Q(0, \frac{b_{i-1}}{2})$. Since both $g_i^*(x)$ and $\lambda(t)$ are smooth, $g_i(x)$ is also smooth. Moreover, g_i is equal to $\frac{b_{i-1}}{a_{i-1}}x$ near the



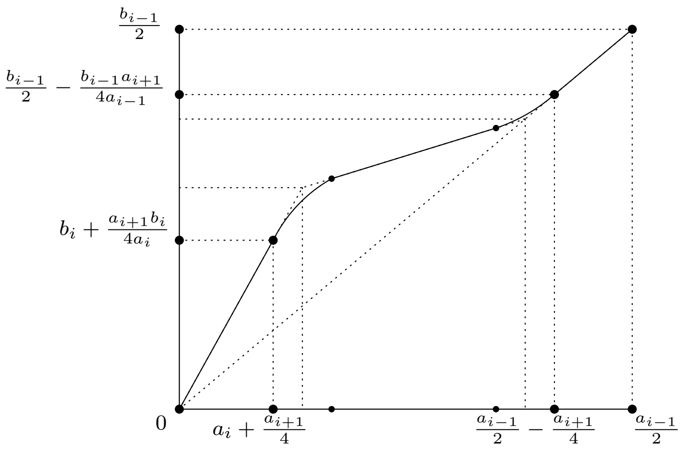


Fig. 5 Functions $h_i^*(t)$ and $h_i(t)$

boundary of $Q(0, \frac{a_{i-1}}{2})$, since $\prod_{j=1}^n \lambda(|x_j|) = 0$ if any $|x_j| > \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}$. We note that $\prod_{j=1}^n \lambda(|x_j|) \in (0,1)$ only in the set

$$T_i = Q\Big(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}\Big) \setminus Q\Big(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}\Big). \tag{14}$$

Both $g_i^*(x)$ and $\frac{b_{i-1}}{a_{i-1}}x$ are strictly increasing in every coordinate and therefore bijective, we claim that $g_i(x)$ is also bijective. To prove this we have to focus only on the set T_i, elsewhere g_i is equal to one of the bijective mappings $g_i^*(x)$

or $\frac{b_{i-1}}{a_{i-1}}x$. We should also show that

$$g_i(T_i) = Q\Big(0, \frac{b_{i-1}}{2} - \frac{b_{i-1}a_{i+1}}{8a_{i-1}}\Big) \setminus Q\Big(0, \frac{b_{i-1}}{2} - \frac{b_{i-1}a_{i+1}}{4a_{i-1}}\Big),$$

we do not do the calculations here, it is straightforward after we check the bijectivity.

Remark 1 We can avoid the following discussion by the estimate of the Jacobian $|Jg_i| > 0$. Since this holds in whole $Q(0, \frac{a_{i-1}}{2})$, we can use advanced topological degree theory to claim the bijectivity, see [17, p. 17, Proposition 4.4]. We present the discussion for convenience of the reader not familiar with this theory.

We discuss if it is possible to have $g_i(x) = g_i(y)$ for $x \neq y$, $x, y \in T_i$ to get a contradiction. We split the indices of coordinates of $x \in T_i$ into two sets S_x and R_x such that

$$\begin{aligned}
&S_x \cup R_x = \{1, \dots n\}, \\
&x_s \in \big(-\tfrac{a_{i-1}}{2} + \tfrac{a_{i+1}}{8}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{8}\big) \setminus \big(-\tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}\big) \qquad \text{for } s \in S_x, \\
&x_r \in \big(-\tfrac{a_{i-1}}{2} + \tfrac{a_{i+1}}{4}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}\big) \qquad \text{for } r \in R_x.
\end{aligned}$$

This sets can provide us the decomposition of T_i based on the belonging of indices of $x \in T_i$ to S_x and R_x. This decomposition in case $n = 2$ is sketched in Figure 6. In general, there are n-dimensional cubes by the vertices of $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, where $R_x = \emptyset$ and $S_x = \{1, \dots n\}$. The rest of T_i is split into n-dimensional hyper-rectangles with both R_x and S_x non-empty. We intend to prove that $g(x)$ is a homeomorphism in each component of decomposed T_i and maps this component onto the corresponding component in the image. This imply $g(x)$ is a homeomorphism in whole T_i. Further we consider only $x, y \in T_i, x \neq y$ lying inside the same component, it means $R_x = R_y, S_x = S_y$.

For x_s, $s \in S_x$, we have by (13), (12), (11) and (10) that $h_i(|x_s|) = \frac{b_{i-1}}{a_{i-1}}|x_s|$ and hence

$$\begin{aligned}
(g_i(x))_s &= \prod_{j=1}^n \lambda(|x_j|)(g_i^*(x))_s + (1 - \prod_{j=1}^n \lambda(|x_j|))\frac{b_{i-1}}{a_{i-1}}x_s \\
&= \prod_{j=1}^n \lambda(|x_j|)\operatorname{sgn}(x_s)h_i(|x_s|) + (1 - \prod_{j=1}^n \lambda(|x_j|))\frac{b_{i-1}}{a_{i-1}}x_s = \frac{b_{i-1}}{a_{i-1}}x_s.
\end{aligned} \tag{15}$$

This imply that $g_i(x) = \frac{b_{i-1}}{a_{i-1}}x$ in parts where $R_x = \emptyset$ and it is clearly a homeomorphism. Let us consider a part where $R_x \neq \emptyset$. Let $s \in S_x$, then we get $(g_i(x))_s = \frac{b_{i-1}}{a_{i-1}}x_s$ so we can suppose that $x_s = y_s$ for all $s \in S_x = S_y$. Now we discuss if $g(x) = g(y)$ is possible for some x and y belonging to the same $|R_x|$-dimensional rectangle defined by

$$\{z \in T_i : R_z = R_x; z_s = x_s \text{ for } s \in S_x\}. \tag{16}$$

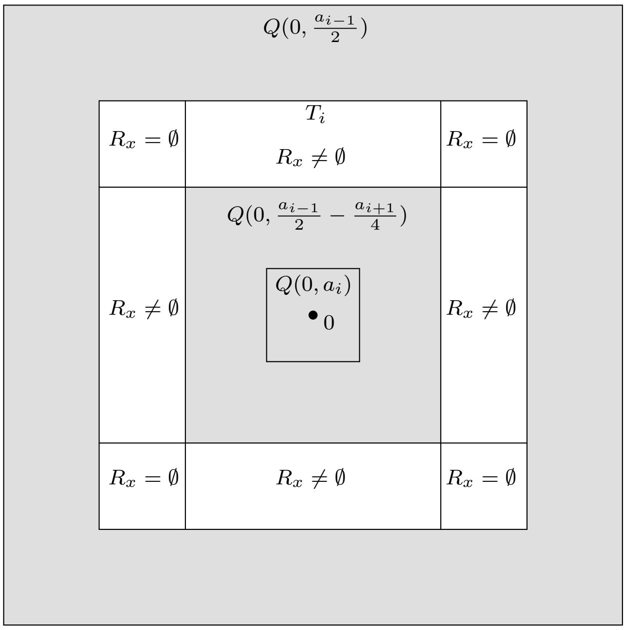


Fig. 6 Set T_i and its splitting

Specially for $n = 2$, we get line segments since both S_x and R_x are sets of one element.

By the definition of R_x we have $\lambda(|x_r|) = 1$ for $r \in R_x$, hence

$$(g_i(x))_r = \prod_{s \in S_x} \lambda(|x_s|)(g_i^*(x))_r + \Big(1 - \prod_{s \in S_x} \lambda(|x_s|)\Big)\frac{b_{i-1}}{a_{i-1}} x_r.$$

For $m, r \in R_x$, $m \neq r$ we get

$$\begin{aligned} D_r(g_i(x))_r &= \prod_{s \in S_x} \lambda(|x_s|) D h_i(|x_r|) + \Big(1 - \prod_{s \in S_x} \lambda(|x_s|)\Big)\frac{b_{i-1}}{a_{i-1}}, \\ D_m(g_i(x))_r &= 0. \end{aligned} \tag{17}$$

The derivative $D_r(g_i(x))_r$ is positive, since it is a convex combination of two positive numbers. This observation gives us the contradiction for case $|R_x| = 1$ as $(g_i(x))_r$ is increasing and $(g_i(x))_r = (g_i(y))_r$ implies $x_r = y_r$. Hence the proof is finished in case $n = 2$. In general case we study the rectangle defined by (16). Restriction of g_i on this rectangle has a diagonal Jacobi matrix with positive numbers on diagonal positions and hence it is a strictly monotone mapping in every direction and therefore it is a bijection. This gives us the contradiction and g_i is bijective in every part of T_i, therefore in whole T_i and in whole $Q(0, \frac{a_{i-1}}{2})$.

For later use we estimate the derivatives of $g_i(x)$ inside the set T_i defined in (14). We recall (8) to get the comparability of a_i and a_{i+1} by estimates $a_{i+1} < a_i$ and $a_{i+1} > Ca_i$. For $x \in T_i$ we estimate by (2)(4) and (3)

$$\Big|D^m\Big(\prod_{s\in S}\lambda(|x_s|)\Big)\Big| \leq C\max\Big\{\prod_{s\in S}\big|D^{\gamma_s}\lambda(|x_s|)\big|; \gamma_s\in\mathbf{N}_0, \sum_{s\in S}\gamma_s = m)\Big\} \leq Ca_i^{-m}.$$

By (15) and (17) we estimate $|D^k(g_i(x))|$ for $k>1$, for $x\in T_i$ as

$$\begin{aligned}
|D^k(g_i(x))| \leq & \max_{r\in R_x, d\in S_x}\Big\{\big|D^k((g_i(x))_d)\big|, \big|D^k((g_i(x))_r)\big|\Big\}\\
\leq & \max_{r\in R_x, d\in S_x}\Big\{\big|D^k(\frac{b_{i-1}}{a_{i-1}}x_d)\big|,\\
& \qquad \Big|D^k\Big(\prod_{s\in S}\lambda(|x_s|)h_i(|x_r|) + \big(1-\prod_{s\in S}\lambda(|x_s|)\big)\frac{b_{i-1}}{a_{i-1}}x_r\Big)\Big|\Big\}\\
\leq & C\max_{r\in R_x, 0\leq m\leq k}\Big\{0, \big|D^m\big(\prod_{s\in S}\lambda(|x_s|)\big)D^{k-m}h_i(|x_r|)\big|, \Big|D^{k-1}\prod_{s\in S}\lambda(|x_s|)\frac{b_{i-1}}{a_{i-1}}\Big|,\\
& \qquad \Big|D^k\big(\prod_{s\in S}\lambda(|x_s|)\big)\frac{b_{i-1}}{a_{i-1}}x_r\Big|\Big\}\\
\leq & C\max_{r\in R_x, 0\leq m\leq k}\Big\{|a_i^{-m}D^{k-m}h_i(|x_r|)|, \big|a_i^{-k+1}\frac{b_{i-1}}{a_{i-1}}\big|, |a_i^{-k}b_{i-1}|\Big\}.
\end{aligned} \tag{18}$$

3.4 Construction of the homeomorphism f in $W^{k,p}$

We will give the sequence of smooth homeomorphisms $f_i : (-1,1)^n \to (-1,1)^n$. We set $f_0(x) = x$ and we proceed by induction. Firstly we give a mapping f_1 which stretches each cube Q_v, $v\in\mathbb{V}^1$, homogeneously so that $f_1(Q_v) = \tilde{Q}_v$. We also need $f_1(Q'_v\setminus Q_v) = \tilde{Q}'_v\setminus\tilde{Q}_v$. We define

$$f_1(x) = \begin{cases} f_0(x) = x & \text{for } x\in(-1,1)^n\setminus\bigcup_{\boldsymbol{v}\in\mathbb{V}}Q'_{\boldsymbol{v}},\\ z_{\boldsymbol{v}} + g_1(x - z_{\boldsymbol{v}}), & \text{for } x\in Q'_{\boldsymbol{v}},\ \boldsymbol{v}\in\mathbb{V}. \end{cases}$$

We check the desired properties, f_1 is bijective and smooth inside both parts of the domain of function. The bijectivity inside $Q'_{\boldsymbol{v}}$ is given by the bijectivity of g_1 proven in previous Subsection 3.3, the property $f_1(Q'_{\boldsymbol{v}}) = \tilde{Q}'_{\boldsymbol{v}}$ follows from (13). In order to prove the smoothness we check the assumptions of Observation 3. We have $h_1^*(t) = \frac{a_0}{b_0}t = t$ for $t\in(\frac{a_0-a_2}{2}, \frac{a_0+a_2}{2})$, $h_1(t) = t$ for $t\in(\frac{a_0}{2}-\frac{a_2}{4}, \frac{a_0}{2})$. We have $z_v = \tilde{z}_v$ for $v\in\mathbb{V}^1$ and therefore we get $f_1(x) = f_0(x) = x$ inside annuli $Q(z_v, \frac{a_0}{2})\setminus Q(z_v, \frac{a_0}{2}-\frac{a_2}{8})$. So we can extend the set where $f_1(x) = f_0(x) = x$ to the set $(-1,1)^n\setminus\bigcup_{\boldsymbol{v}\in\mathbb{V}}Q(z_{\boldsymbol{v}}, \frac{a_0}{2}-\frac{a_2}{4})$ as is required in Observation 3 and hence f_1 is smooth in whole $(-1,1)^n$.

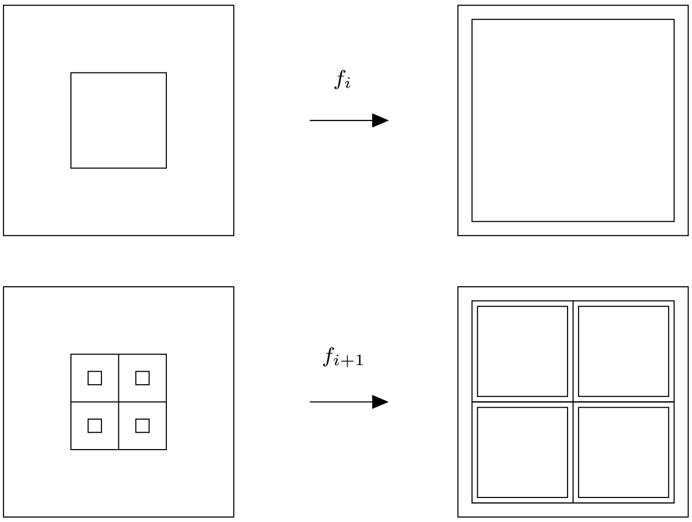


Fig. 7 Modifying f_i into f_{i+1}

This first step also shows us the idea of the induction process. We define

$$f_i(x) = \begin{cases} f_{i-1}(x) & \text{for } x \in (-1,1)^n \setminus \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q'_{\boldsymbol{v}}, \\ f_{i-1}(z_{\boldsymbol{v}}) + g_i(x - z_{\boldsymbol{v}}) & \text{for } x \in Q'_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^i. \end{cases} \tag{19}$$

In the general step we change the $f_i(x)$ only inside the cubes $Q'_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^i$. This f_i stretches each cube $Q_{\boldsymbol{v}}$ homogeneously onto $\tilde{Q}_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i$. Moreover our definition of f_i will coincide with f_{i-1} on some neighbourhood of the boundary of $\bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q'_{\boldsymbol{v}}$ and hence f_i is smooth in whole $(-1,1)^n$ by Observation 3, as we have showed for f_1.

It is not difficult to check that each f_i is a homeomorphism by the bijectivity of g_i. Moreover, f_i satisfies

$$f_i\Big(\bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q_{\boldsymbol{v}}\Big) = \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} \tilde{Q}_{\boldsymbol{v}}.$$

We illustrate the induction step in Figure 7, we sketch how the squares $Q_{\boldsymbol{v}}$ are homogeneously mapped and how we place the new generations of $Q_{\boldsymbol{v},v_{i+1}}$ and $Q'_{\boldsymbol{v},v_{i+1}}$ inside them.

We define

$$f(x) = \lim_{i\to\infty} f_i(x).$$

The mapping satisfies $f(C_A) = C_B$ from previous step and from the definition of these Cantor sets. Moreover, it is continuous as the uniform limit of the continuous mappings. Mapping $f(x)$ is clearly one-to-one inside these Cantor sets. To check the one-to-one property outside these Cantor sets, we may consider every set $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$ for the same index $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$ separately. These sets cover whole $(-1,1)^n \setminus C_A$. We know that $f = f_i$ is homeomorphism inside $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$. Since f is bijective in every part we consider and the images of these sets do not collide and fill $(-1,1)^n$, f is bijective.

The limit mapping f is continuous and one-to-one, therefore it is a homeomorphism by Lemma 1. It remains to show that the norm $\|D^k f\|_p$ is finite.

3.5 Finiteness of the norm of the homeomorphism f in $W^{k,p}$

Before the proof, we estimate the derivatives of h_i in $[0, \frac{a_{i-1}}{2}]$, $i \in \mathbf{N}$. By Lemma 2 and (11) we have

$$|D^k h_i| \leq C \max_{l=\{0,\dots k\}} (a_{i+1})^{-k+l} \|D^l h_i^*\|_\infty.$$

We consider only the case $k > 1$. Since h_i^* is defined as the continuous piecewise affine function, the only non-zero derivatives would be the zeroth and the first, all others would be zero. We estimate the pointwise values of h_i^* and Dh_i^* by definition (10). By definition of α_i (9) we can see that $\alpha_i \leq \min\{\frac{b_{i-1}}{a_{i-1}}, \frac{b_i}{a_i}\}$ from the meaning of α_i in the graph as is shown in Figure 5. We get

$$\begin{aligned}
|h_i^*(t)| &\leq Cb_{i-1} \text{ for } t \in [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}],\\
|Dh_i^*(t)| &\leq \frac{b_i}{a_i} \text{ for } t \in [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}].
\end{aligned}$$

Together with previous estimates and (8) we get

$$|D^k h_i| \leq C \max_{l=\{0,1\}} (a_{i+1})^{-k+l} \|D^l h_i^*\|_\infty \leq C(A) a_i^{-k} b_i \leq C 2^{(A+1)ik-i}. \tag{20}$$

We estimate the norm of $D^k f$ by the sum

$$\|D^k f\|_p^p \leq \|D^k f_0\|_p^p + \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p = \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p, \tag{21}$$

since f_0 is identity. We consider one member of the sum and we observe by (19), that the set where $f_i \neq f_{i-1}$ can be covered by 2^{ni} cubes $Q'_{\boldsymbol{v}}, \boldsymbol{v} \in \mathbb{V}^i$ of measure

$$|Q'_{\boldsymbol{v}}| = (a_{i-1})^n = 2^{-n(i-1)-An(i-1)}. \tag{22}$$

For $y \in Q'_{\boldsymbol{v}}$ by (19) we have $f_i(y) = f_{i-1}(z_{\boldsymbol{v}}) + g_i(y - z_{\boldsymbol{v}})$. We know by (19) that $f_{i-1}(y)$ is an affine function inside $Q'_{\boldsymbol{v}}$, so its higher derivatives are 0. We denote $x = y - z_{\boldsymbol{v}} \in Q(0, \frac{a_{i-1}}{2})$, we get

$$|D^k f_i(y) - D^k f_{i-1}(y)| = |D^k f_i(y) - 0| = |D^k f_i(x + z_{\boldsymbol{v}})| = |D^k g_i(x)|. \tag{23}$$

Now we discuss the possible values of $D^k g_i$ inside $Q(0, \frac{a_{i-1}}{2})$. By (13), we split $Q(0, \frac{a_{i-1}}{2})$ into three subsets, inner cube $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, middle part T_i and outer annulus $Q(0, \frac{a_{i-1}}{2}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8})$. Inside the inner part $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, all $\lambda(|x_j|) \equiv 1$ and we have

$$g_i(x) = g_i^*(x) = \big(\mathrm{sgn}(\mathrm{x}_1)\, h_i(|x_1|), \mathrm{sgn}(\mathrm{x}_2)\, h_i(|x_2|), \dots \mathrm{sgn}(\mathrm{x}_\mathrm{n})\, h_i(|x_n|)\big).$$

Since every coordinate $(g_i(x))_j$ depends only on the coordinate x_j, the only nonzero partial derivatives would be the diagonal ones. We have to differentiate multiple times in the same direction only so we get

$$|D^k g_i(x)| = \max_{j=1,\dots n} \{|D^k (g_i^*(x))_j|\} = \max_{j=1,\dots n} \{|D^k h_i(|x_j|)|\}.$$

We use (20) to estimate this term

$$|D^k g_i(x)| \leq C 2^{(A+1)ik-i}.$$

The middle part of $Q(0, \frac{a_{i-1}}{2})$ is the annulus $T_i = Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$ described by (14) and we have prepared the estimate (18). We use (20) and (8) and we get

$$\begin{aligned} |D^k(g_i(x))| \leq & C \max_{r\in R_x, 0\leq m\leq k} \big\{|a_i^{-m} D^{k-m} h_i(|x_r|)|, \big|a_i^{-k+1} \frac{b_{i-1}}{a_{i-1}}\big|, |a_i^{-k} b_{i-1}|\big\}. \\ \leq & C \max_{0\leq m\leq k} \big\{|2^{(A+1)im} 2^{(A+1)i(k-m)-i}, 2^{(A+1)ik} 2^{-(i-1)}\big\} \\ \leq & C 2^{(A+1)ik-i}. \end{aligned}$$

The outer part of $Q(0, \frac{a_{i-1}}{2})$ is annulus $Q(0, \frac{a_{i-1}}{2}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8})$, where at least one $\lambda(|x_j|)$ from formula (13) is zero. So we get

$$g_i(x) = 0 g_i^*(x) + 1 \frac{b_{i-1}}{a_{i-1}} x = \frac{b_{i-1}}{a_{i-1}} x,$$

and $D^k(g_i(x)) = 0$ for any $k > 1$. We combine these pointwise estimates for all three parts of $Q(0, \frac{a_{i-1}}{2})$ and we get for any $x \in Q(0, \frac{a_{i-1}}{2})$

$$|D^k g_i(x)| \leq C \max\{2^{(A+1)ik-i}, 2^{(A+1)ik-i}, 0\} \leq C 2^{(A+1)ik-i}.$$

Together with (22) and (23) we get

$$\begin{aligned} \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p &\leq \sum_{i=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^i} \int_{Q'_{\boldsymbol{v}}} |D^k f_i(y) - D^k f_{i-1}(y)|^p \, dy \\ &\leq C \sum_{i=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^i} \int_{Q'_{\boldsymbol{v}}} (2^{Aik+ik-i})^p \, dy \\ &\leq C \sum_{i=1}^{\infty} 2^{ni} 2^{-n(i-1)-An(i-1)} (2^{Aik+ik-i})^p \\ &\leq C \sum_{i=1}^{\infty} 2^{(A(kp-n)+kp)i-ip}. \end{aligned}$$

As we apply the condition for A (7), we see that the term $A(kp-n)+kp$ is negative and the sum is finite. Together with (21) we get

$$\|D^k f\|_p^p \leq \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p \leq C \sum_{i=1}^{\infty} 2^{(A(kp-n)+kp)i-ip} < \infty.$$

4 Proof of Theorem 2

4.1 Proof of the positive part

Let us remind, that the Lusin (N) condition is guaranteed for the continuous mappings in $W^{1,p}(\Omega,\mathbf{R}^n)$ for $p>n$ by the result of Marcus and Mizel [9]. To prove the general form we just use the Sobolev Embedding Theorem multiple times as we have done in Subsection 3.1. As we study the local property, once again we suppose without the loss of generality that Ω is a ball. We have

$$W^{k,\frac{p}{k}}(\Omega,\mathbf{R}^n) \subset W^{k-1,\frac{p}{k-1}}(\Omega,\mathbf{R}^n) \subset \cdots \subset W^{1,p}(\Omega,\mathbf{R}^n),$$

hence for any $k<n$, $p>n$ a mapping $f \in W^{k,\frac{p}{k}}(\Omega,\mathbf{R}^n)$ satisfies the assumptions for the well-known version of the theorem.

Now we consider the special case $n=k$, $p=1$. We consider domain $\Omega \subset \mathbf{R}^n$. We recall the result by Peetre [14] that Sobolev and Lorentz spaces are embedded as

$$W^1 L^{p,q}(\Omega) \subset L^{p^*,q}(\Omega).$$

We repeat this argument $n-1$ times to get

$$W^n L^{1,1}(\Omega) \subset W^{n-1} L^{1^*,1}(\Omega) \subset \cdots \subset W^1 L^{n,1}(\Omega).$$

It follows that

$$D^n f \in L^1 \Rightarrow f \in WL^{n,1}.$$

By the paper [5], mappings in $WL^{n,1}$ have continuous representative and even satisfy the Lusin (N) condition.

4.2 Classical counterexample in $W^{1,n}$

We recall the classic counterexample by Malý and Martio [8] on $W^{1,n}$, the continuous mapping that maps a line segment $[-1,1]\times\{0\}^{n-1}$ onto the whole $[-1,1]^m$. We briefly remind this construction and then we improve the properties of the mapping. The key step is finding the function such that

$$\begin{aligned}&\forall\varepsilon>0\,\forall r>0\,\exists\varrho>0\,\exists h \in W_0^{1,n}(B(0,r)) \cap \overline{C(B(0,r))},\\ &\text{such that } h(x)\equiv 1 \text{ on } B(0,\varrho) \text{ and } \|Dh\|_n^n < \varepsilon.\end{aligned}$$

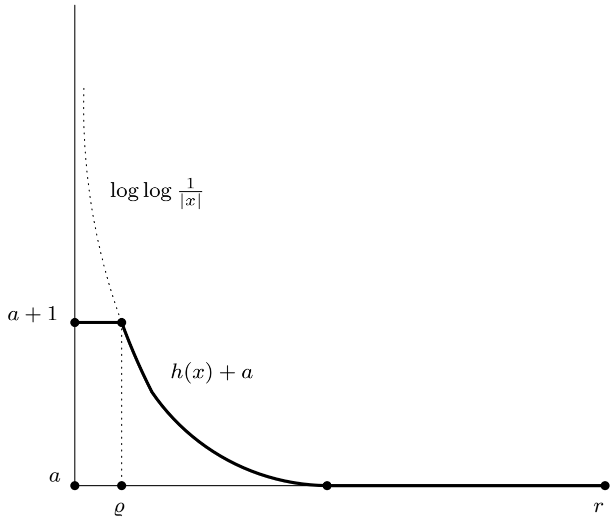


Fig. 8 Function $h(x)$

There is such a function, for a big parameter $a > \log\log\frac{1}{r}$ we set

$$h(x) = \min\Big\{1, \Big(\log\big(\log(\frac{1}{|x|})\big) - a\Big)^+\Big\}.$$

We sketch the graph of $h(x)$ at Figure 8, it is obvious that the function is continuous but not smooth. We estimate its norm as

$$\begin{aligned}\|Dh\|_n^n = \int_{B(0,r)} |Dh(x)|^n \, dx &\leq C \int_0^r t^{n-1} \Big| \big(\log\log\frac{1}{|t|}\big)' \Big|^n \, dt \\ &\leq C \int_0^r t^{-1} \frac{1}{\log^n(\frac{1}{t})} \, dt.\end{aligned}$$

The term on the right hand side tends to zero as r tends to zero and hence it is smaller then ε for r small enough.

Let us consider a sequence of such mappings $\{h_i\}_{i=1}^\infty$ so that corresponding parameters satisfy

- $r_1 < 2^{-m}$,
- $r_i < 2^{-m}\varrho_{i-1}$,
- $\varepsilon_i < 4^{-mi}$.

Let $m \in \mathbf{N}$ and define $\mathbb{V}$ the vertices of the cube $[-1,1]^m$ similarly to the beginning of Subsection 3.2. For $\boldsymbol{v} \in \mathbb{V}^i$, $\boldsymbol{w} \in \mathbb{V}^{i-1}$ such that $\boldsymbol{v} = [\boldsymbol{w}, v_i]$ we redefine

$$z_0 = 0 \text{ and } z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + 2^{-i}v_i = z_0 + \sum_{j=1}^{i} 2^{-j} v_j.$$

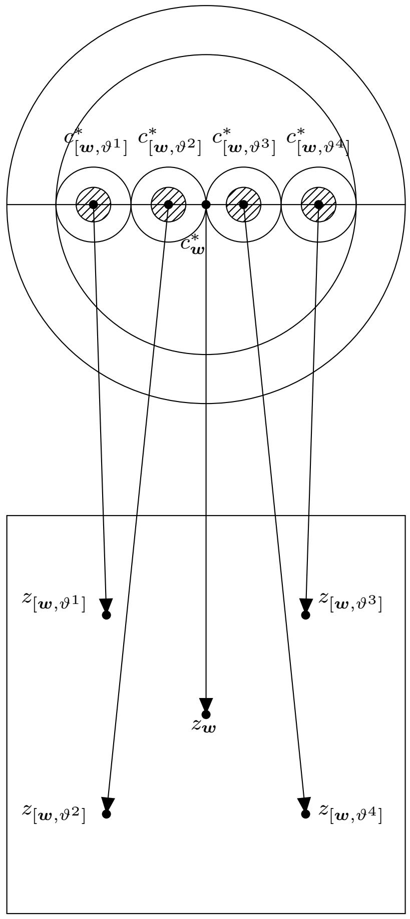


Fig. 9 Step in mapping a line into the structure $z_{\boldsymbol{v}}$

The set $\bigcup_{i=1}^{\infty}\bigcup_{\boldsymbol{v}\in\mathbb{V}^i} z_{\boldsymbol{v}}$ is dense in $[-1,1]^m$ as it is sketched in Figure 3. We will construct the continuous mapping such that it maps the line segment $[-1,1]\times\{0\}^{n-1}$ onto the set which contains $\{z_{\boldsymbol{v}}\}_{i\in\mathbf{N},\boldsymbol{v}\in\mathbb{V}^i}$. Since the continuous image of the compact set is a compact set, the image of $[-1,1]\times\{0\}^{n-1}$ has to be at least $[-1,1]^m$.

We define the set of points $\{c_{\boldsymbol{v}}\}_{i\in\mathbf{N},\boldsymbol{v}\in\mathbb{V}^i}\in(-1,1)$ by induction. We set $c_0=0$. By induction, in every interval $B(c_w,\varrho_{i-1})$ we choose non-overlapping intervals $\{B(c_{[\boldsymbol{w},v]},r_i)\}_{v\in\mathbb{V}}$ around 2^m chosen centres $\{c_{[\boldsymbol{w},v]}\}_{v\in\mathbb{V}}$. We use the inner interval $B(c_{[\boldsymbol{w},v]},\varrho_i)$ of every of these intervals in the next step. This process generate 2^{mi} centres $\{c_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\mathbb{V}^i}$ in i-th step (see Figure 9). We define

$$c^*_{\boldsymbol{v}}=c_{\boldsymbol{v}}\times\{0\}^{n-1}.$$

Now we define the sequence of continuous mapping $f_i : [-1,1]^n \to [-1,1]^m$. First step is $f_0(x) \equiv z_0 = 0^n$, then by induction we define

$$f_i(x) = \begin{cases} f_{i-1}(x) + h_i(x - c^*_{\boldsymbol{v}})(z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) & \text{for } \boldsymbol{v} = [\boldsymbol{w}, v_i] \in \mathbb{V}^i, |x - c_{\boldsymbol{v}}| \leq r_i, \\ f_{i-1}(x) & \text{otherwise.} \end{cases}$$

By this construction we get the uniformly converging sequence of continuous functions. We claim, that for f_i, $\boldsymbol{v} \in \mathbb{V}^i$ the small subinterval of $x_1 \times \{0\}^{n-1}$ around the point $c_{\boldsymbol{v}} \times \{0\}^{n-1}$ is mapped to the point $z_{\boldsymbol{v}}$ as we sketch in Figure 9 (even a small ball around $c^*_{\boldsymbol{v}}$ is mapped to the point $z_{\boldsymbol{v}}$). We use $h_i \equiv 1$ in some small neighbourhood of $c^*_{\boldsymbol{v}}$ and the fact that this small neighbourhood $B(c^*_{\boldsymbol{v}}, \varrho_i)$ is also the part of neighbourhoods in previous step $B(c^*_{\boldsymbol{w}}, \varrho_{i-1})$. We get for $x \in B(c^*_{\boldsymbol{v}}, \varrho_i) \cap (-1,1) \times \{0\}^{n-1}$ the claim

$$f_i(x) = f_{i-1}(x) + h_i(x - c^*_{\boldsymbol{v}})(z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) = z_{\boldsymbol{w}} + (z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) = z_{\boldsymbol{v}}.$$

Altogether we get

$$f_i\Big(\bigcup_{\boldsymbol{v} \in \mathbb{V}^i} B(c_{\boldsymbol{v}}, \varrho_i) \times \{0\}^{n-1}\Big) = \{z_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \mathbb{V}^i}.$$

In order to finish the proof, we check that $f = \lim_{i \to \infty} f_i$ is still a continuous mapping and we verify that its $W^{1,n}$ norm is finite.

Let us remark, that the presented result can cover even some finer scale of spaces and we can improve the $W^{1,n}$ upto $WL^n \log^{n-1} L$. We can also see this result as the corollary of the capacity theory, since we can consider any space where points have zero capacity.

4.3 The improved result in $W^{k,\frac{n}{k}}$

Before we begin, we prepare two estimates. We claim that, for any $F : \mathbf{R} \to \mathbf{R}$ smooth enough and $x \neq 0$ we have

$$|D^k(F(|x|))| \leq C \max_{j=\{0,\dots k-1\}} \{|x|^{-j} |D^{k-j} F(|x|)|\}. \tag{24}$$

We can find a small positive $T_k > 0$, such that for $t \in (0, T_k)$ the derivatives of $\log\log\frac{1}{t}$ of order up to k can be estimated as

$$|D^j(\log\log\tfrac{1}{t})| \leq C \frac{1}{t^j \log\frac{1}{t}} \leq C|D^j(\log\log\tfrac{1}{t})| \text{ for } 0 \leq j \leq k. \tag{25}$$

We do not prove these two estimates, the proofs are straightforward and elementary.

Now we improve the classical construction. We reuse all steps from the classical case, we only have to find some finer function instead of h. Precisely we search for a function g which satisfies

$$\begin{aligned} &\forall \varepsilon > 0\, \forall R > 0\, \exists \varrho > 0\, \exists g \in W_0^{k,\frac{n}{k}}(B(0,R)) \cap \overline{C(B(0,R))} \\ &\text{such that } g(B(0,\varrho)) \equiv 1 \text{ and } \|D^k g\|_{\frac{n}{k}}^{\frac{n}{k}} < \varepsilon. \end{aligned} \tag{26}$$

We use the previous function $h(t)$ as the one dimensional function. We choose the key parameter a big enough so that $h(t) = \min\{1, \big(\log(\log(\frac{1}{t})) - a\big)^{+}\}$ satisfies following

- $h(B(0, 2\varrho)) \equiv 1$ for some $\varrho > 0$,
- $a = \log\log\frac{1}{r}$ for some $r < \frac{1}{2}R$, $r < 12T_k$,
-
$$C\int_0^{2r} \frac{1}{t\log^{\frac{n}{k}}\frac{1}{t}}\,dt < \varepsilon$$
  for given $C > 0$ depending only on p, k and the dimension n.

We can find a formula for C, but we just present the estimates leading to it.

We make $h(t)$ smooth by convolution, but we have to use two different radii, because ϱ and r are incomparable. Because of that we use (2), the partition of unity,

$$\lambda(t) = \lambda_{\frac{r}{4},\frac{3r}{4},R}(t), \tag{27}$$

as it is introduced in the preliminaries. We denote one dimensional function

$$f(t) = \lambda(t)(\phi_{\frac{\varrho}{2}} * h)(t) + (1-\lambda(t))(\phi_{\frac{r}{8}} * h)(t).$$

We define

$$g(x) = f(|x|).$$

Our claim is, that $g(x) = f(|x|)$ satisfies all conditions of (26). Obviously, $g(x)$ is smooth, $\mathrm{spt}(g) \subset B(0,R)$, $g(x) \equiv 1$ on $B(0,\varrho)$. The only remaining and the most important part is the smallness of the norm.

We calculate the derivatives of $f(t)$ in order to estimate them. The support of the function $f(t)$ is $[0, r+\frac{r}{8}]$ and we have by (3) the derivative of the product formula

$$D^k f(t) = \sum_{j=0}^{k}\binom{k}{j}D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}} * h)(t) + \sum_{j=0}^{k}\binom{k}{j}D^j(1-\lambda(t))D^{k-j}(\phi_{\frac{r}{8}} * h)(t). \tag{28}$$

Firstly, we consider only the member with $D^j\lambda(t)$ for $j > 0$. This derivative is non-zero only inside $(\frac{r}{4}, \frac{3r}{4})$ by (27). For such $t \in (\frac{r}{4}, \frac{3r}{4})$ by (2) we estimate

$$\sum_{j=1}^{k}\big|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}} * h)\big| \leq \sum_{j=1}^{k} Cr^{-j}\big|D^{k-j}(\phi_{\frac{\varrho}{2}} * h)(t)\big|. \tag{29}$$

We apply Lemma 2 for $I = I_1 = \{\frac{r}{4} - \frac{\varrho}{2}, \frac{3r}{4} + \frac{\varrho}{2}\}$ on $\phi_{\frac{\varrho}{2}} * h$. There is no point of broken smoothness of $h(t)$ since neither 2ϱ nor r lies inside $(\frac{r}{4} - \frac{\varrho}{2}, \frac{3r}{4} + \frac{\varrho}{2})$, so we get

$$D^{k-j}\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \phi_{\frac{\varrho}{2}}(t-s)h(s)\,ds = \int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \phi_{\frac{\varrho}{2}}(t-s)D^{k-j}h(s)\,ds.$$

Since the estimates of the derivatives of $\log\log\frac{1}{t}$ are bigger for smaller t by (25), we estimate by the Hölder's inequality

$$\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \left|\phi_{\frac{\varrho}{2}}(t-s)D^{k-j}h(s)\right| ds \le |D^{k-j}h(t-\tfrac{\varrho}{2})| \le C\frac{1}{(t-\frac{\varrho}{2})^{k-j}\log\frac{1}{t-\frac{\varrho}{2}}}.$$

We apply this to the estimate (29) and by (25) we get

$$\begin{aligned}\sum_{j=1}^{k}\left|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}}*h)\right| &\le \sum_{j=1}^{k} Cr^{-j}|D^{k-j}h(t-\tfrac{\varrho}{2})| \\ &\le C\sum_{j=1}^{k}\left|r^{-j}\frac{1}{(t-\frac{\varrho}{2})^{k-j}\log\frac{1}{t-\frac{\varrho}{2}}}\right|.\end{aligned}$$

Since we consider only $t\in(\frac{r}{4},\frac{3r}{4})$, there exist some C such that

$$\sum_{j=1}^{k}\left|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}}*h)\right| \le \sum_{j=1}^{k}C\left|t^{-j}\frac{1}{t^{k-j}\log\frac{1}{t}}\right| \le C\left|D^k(\log\log\frac{1}{t})\right|. \tag{30}$$

Analogously for $t\in(\frac{r}{4},\frac{3r}{4})$ we get

$$\sum_{j=1}^{k}\left|D^j(1-\lambda(t))D^{k-j}(\phi_{\frac{r}{8}}*h)(t)\right| \le C\left|D^k(\log\log\frac{1}{t})\right|. \tag{31}$$

Secondly, we estimate the members of the sums in (28) for $j=0$. We consider $t\in[0,\frac{3r}{4}]$, we estimate $\lambda(t)\le 1$ inside this interval, otherwise we have $\lambda(t)=0$. Inside this interval lies 2ϱ, the point of broken smoothnes of $h(t)$. By Lemma 2 and triangle inequality we have

$$\begin{aligned}\left|\lambda(t)D^k(\phi_{\frac{\varrho}{2}}*h)\right| \le &\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}}\left|\phi_{\frac{\varrho}{2}}(t-s)D^k h(s)\right| ds \\ &+\sum_{l=1}^{k-1}\left|D^{k-1-l}(\phi_{\frac{\varrho}{2}})(t-2\varrho)\right|\left|D^l 1 - D^l(\log\log\tfrac{1}{2\varrho})\right|.\end{aligned} \tag{32}$$

In the second term we missed the member of the sum for $l=0$, but this member is zero, since $h(t)$ is continuous at 2ϱ. For $t\in(2\varrho-\frac{\varrho}{2},2\varrho+\frac{\varrho}{2})$ we estimate the second member by (1)(3) as

$$\begin{aligned}\sum_{l=1}^{k}\left|D^{k-1-l}(\phi_{\frac{\varrho}{2}})(t-2\varrho)\right|\left|D^l(\log\log\tfrac{1}{2\varrho})\right| &\le \sum_{l=1}^{k}C\left|\varrho^{-k+l}\right|\left|\frac{1}{\varrho^l\log\frac{1}{\varrho}}\right| \\ &\le C\left|D^k(\log\log\frac{1}{\varrho})\right| \le C\left|D^k(\log\log\frac{1}{t})\right|,\end{aligned} \tag{33}$$

anywhere else this member is zero. We estimate the first member of (32) for $t \in [2\varrho - \frac{\varrho}{2}, \frac{3r}{4}]$ as

$$\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \left|\phi_{\frac{\varrho}{2}}(t-s)D^k h(s)\right| ds \le \left|D^k(\log\log\frac{1}{t-\frac{\varrho}{2}})\right| \le C\left|D^k(\log\log\frac{1}{t})\right|. \quad (34)$$

Analogously to these two estimates (33) and (34) we estimate for $t \in [\frac{r}{4}, r+\frac{r}{8}]$

$$\sum_{j=0}^{k}\left|D^j(1-\lambda(t))D^{k-j}(\phi_{\frac{r}{8}} * h)(t)\right| \le C\left|D^k(\log\log\frac{1}{t})\right|. \quad (35)$$

By estimating members of formula for $D^k f$ written in (28), we get a pointwise estimate for every member anywhere on its support. Altogether by (30), (31), (33), (34) and (35) we get

$$|D^k(f(t))| \le C\left|D^k(\log\log\frac{1}{t})\right|.$$

By (24), (25), spherical coordinates and the condition for r we get

$$\begin{aligned}\|D^k g(x)\|_{\frac{n}{k}}^{\frac{n}{k}} &\le \int_{B(0,2r)} |D^k f(|x|)|^{\frac{n}{k}}\, dx\\ &\le C\int_0^{2r} t^{n-1}\Big(\max_{i=0,\dots,k-1}\big\{t^{-i}|D^{k-i}(\log\log\frac{1}{t})|\big\}\Big)^{\frac{n}{k}}\, dt\\ &\le C\int_0^{2r}\frac{1}{t^{-n+1+n}\log^{\frac{n}{k}}\frac{1}{t}}\, dt < \varepsilon.\end{aligned}$$

All the properties of $g(x)$ and checked. We use it the same way as $h(x)$ is used in the classical case and get the counterexample in $W^{k,\frac{n}{k}}$.

Remark 2 There was a partial result on smoothing of Cesari counterexample, Matějka has proven the case for $k = 2$ in [11]. He smoothed $h(x)$ by redefining it explicitly near the points of discontinuity and his example is C^1 but not C^2.

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